# CM 17

2018년 2월 19일 월요일　　오후 7:54

☐ Announcements

- Midterm #2 on this Thursday, LEE10~~4~~2 19:30-21:00
- mixture distribution

☐ Review

§17-1

§17-2

§17-3 MS Estimation (EEE645) → Ch.8

§17-4 Stochastic Convergence

§17-5 Random numbers: meaning & generation

# Page 02

2018년 2월 19일 월요일 오후 7:54

## §7-3 MSE

Def.

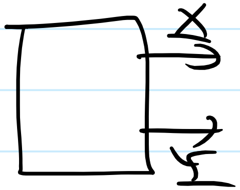
□ Given two random vectors $\underline{X}$ & $\underline{Y}$,

observable

unobservable

$$\underbrace{\hat{\underline{X}}_{MMSE}(\underline{y})}_{\text{MMSE estimator for } X} = \arg\min_{\underline{c}(\underline{y})} E\{\|\underbrace{\underline{X}-\underline{c}(\underline{Y})}_{\text{error}}\|^2\}$$

cf) $f_{\underline{X}|\underline{Y}}(\underline{x}|\underline{y}) \neq f_{\underline{X}}(\underline{x}),\ \forall(\underline{x},\underline{y})$ In general



□ Theorem.

$$\hat{X}_{MMSE}(\underline{y}) = E\{\underline{X}\,|\,\underline{Y}=\underline{y}\}$$

"The MMSE estimator is the conditional mean estimator."

□ ?

(i) $\arg\min_{c} E\{(X-c)^2\} = \mu_X$ ☺ $E\{X\}$

(ii) $\arg\min_{c} E\{(X-c)^2|Y=y\} = \mu_{X|Y=y}$ ☺ $E\{X|Y=y\}$

(iii) $\arg\min_{c(y)} \underbrace{E\{(X-c(Y))^2\}}_{E[E\{(X-c(Y))^2|Y\}]} = \mu_{X|Y=y}$ $c(y)=E\{X|Y=y\}$

☐ Affine MMSE estimator

$$\hat{\underline{X}}_{AMMSE}(\underline{y}) = \underline{A}\,\underline{y} + \underline{b}$$

$$(A_{opt}, b_{opt}) = \underset{A,b}{\mathrm{argmin}}\, E\{\|\underline{X} - (A\underline{Y} + b)\|^2\}$$

$$= \mu_{\underline{X}} + C_{XY}C_{YY}^{-1}(\underline{y} - \mu_{\underline{Y}})$$

☐ LMMSE estimator

$$\hat{\underline{X}}_{LMMSE}(\underline{y}) = A\underline{y} \qquad A_{opt} = \mathrm{argmin}\, E\{\|\underline{X} - AY\|^2\}$$

☐ Orthogonality Principle.

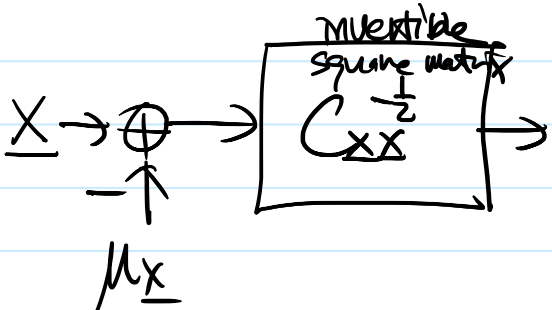
(Exam#2).

Iterated
Nested
> Expection

# Page 04

2018년 2월 19일 월요일 오후 7:54

## ☐ Orthonormal Data Transformation



$\underline{Z} = C_{\underline{xx}}^{-\frac{1}{2}}(\underline{X} - \underline{\mu_x})$

$E\{\underline{X}\} = \underline{\mu_x}$

$C_{\underline{xx}} \succ 0$. positive definite.

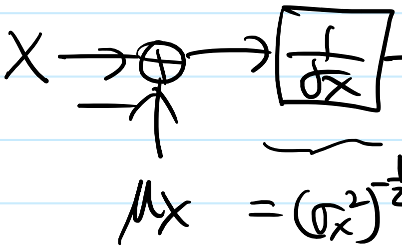
$E\{\underline{Z}\} = \underline{0}$

$$C_{\underline{zz}} = I = \begin{bmatrix} 1 & & & \underline{0} \\ & 1 & & \\ & & 1 & \\ \underline{0} & & & 1 \end{bmatrix} = \begin{bmatrix} C_{z_1z_1} & C_{z_1z_2} & \cdots & C_{z_1z_N} \\ & \vdots & & \\ & C_{z_iz_j} & & \\ & \vdots & & \end{bmatrix}$$

"Whitening"

$N \times N$ ↙ $0$ ← matrix

### Mental Representation



$Z = \frac{X - \mu_x}{\sigma_x}$

$= (\sigma_x^2)^{-\frac{1}{2}}$

$E\{Z\} = 0$

$\mathrm{Var}\{Z\} = 1$

## ☐ $R_{\underline{xx}} = E\{\underline{X}\underline{X}^T\}$ $\qquad R \succ 0_N$

Def. ( R is positive definite if $\underline{a}^T R \underline{a} > 0, \ \forall \underline{a} \neq \underline{0}$.
( R is " semi-definite if $\underline{a}^T R \underline{a} \geq 0, \ \forall \underline{a}$
$R \succeq 0_N$

Lemma. Q $R_{\underline{xx}} \succeq 0_N$ ?

A. Yes.

$$\underline{a}^T R_{\underline{xx}} \underline{a} = \underline{a}^T E\{\underline{X}\underline{X}^T\} \underline{a} = E\{(\underline{a}^T\underline{X})(\underline{X}^T\underline{a})\}$$
$$= E\{(\underline{a}^T\underline{X})^2\} \geq 0, \ \forall \underline{a}$$

# Page 05

2018년 2월 19일 월요일 오후 7:54

$\Rightarrow \quad C_{\underline{xx}} \succeq 0_N$

□ $C_{\underline{xx}} = C_{\underline{xx}}^T$ $\quad \left[E\{(\underline{x}-\underline{\mu}_x)(\underline{x}-\underline{\mu}_x)^T\}\right]^T = \cdots$

symmetric

positive semi-definite matrix

$\Rightarrow \quad C_{\underline{xx}} = V\Lambda V^T \quad \Lambda = \text{diag}\{\lambda_1, \lambda_2 \cdots, \lambda_N\}, \quad \lambda_n \geq 0.$

□. Assume $C_{\underline{xx}} \succ 0_N$.

$\Rightarrow \quad C_{\underline{xx}} = V\Lambda V^T \quad \lambda_i > 0,$

$C_{\underline{xx}}^{\frac{1}{2}} = V\Lambda^{\frac{1}{2}}V^T, = (C_{\underline{xx}}^{\frac{1}{2}})^T \quad C_{\underline{xx}}^{-\frac{1}{2}} = V\Lambda^{-\frac{1}{2}}V^T = (C_{\underline{xx}}^{-\frac{1}{2}})^T$

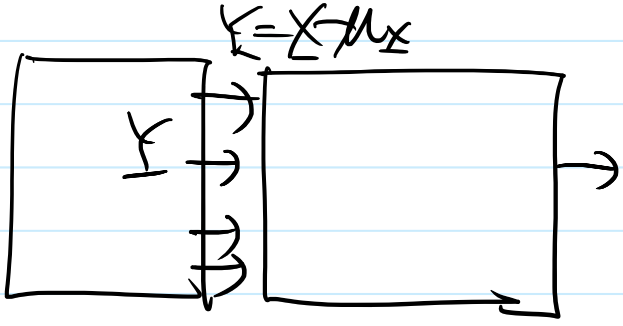
$\underline{z} = C_{\underline{xx}}^{-\frac{1}{2}}(\underline{x}-\underline{\mu}_x)$

$C_{\underline{zz}} = E\{\underline{z}\underline{z}^T\} = E\left[C_{\underline{xx}}^{-\frac{1}{2}}(\underline{x}-\underline{\mu}_x)(\underline{x}-\underline{\mu}_x)^T C_{\underline{xx}}^{-\frac{1}{2}}\right] = C_{\underline{xx}}^{-\frac{1}{2}} C_{\underline{xx}} C_{\underline{xx}}^{-\frac{1}{2}} = I.$

□ $\underline{z} \rightarrow \boxed{U} \rightarrow \underline{w}$ $\quad E\{\underline{w}\} = E\{U\underline{z}\} = UE\{\underline{z}\} = \underline{0}$

NOT UNIQUE! $\quad C_{\underline{ww}} = E\{(U\underline{z})(U\underline{z})^T\} = UE\{\underline{z}\underline{z}^T\}U^T = UIU^T = I.$

# ☐ Gram-Schmidt Orthogonalization (Cholesky factorization)



$$\rightarrow \underline{Z} = \begin{bmatrix} a_1Y_1 \\ a_2Y_1+b_2Y_2 \\ a_3Y_1+b_3Y_2+c_3Y_3 \\ \vdots \\ a_NY_1+b_NY_2+c_NY_3+\cdots \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & 0 & \cdots & 0 \\ a_N & b_N & c_N & \cdots & \end{bmatrix} \underline{Y}$$

$$= L\,\underline{Y}$$

↑ lower

$$C_{\underline{Z}\underline{Z}} = E\{(L\underline{Y})(L\underline{Y})^T\}$$

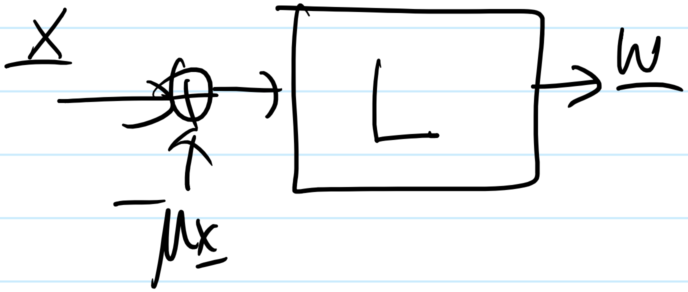
$$= E\{L\underline{Y}\underline{Y}^TL^T\} = LC_{\underline{X}\underline{X}}L^T = (LC_{\underline{X}\underline{X}}L^T = I$$

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}^{-1} = \frac{1}{ac}\begin{bmatrix} c & 0 \\ -b & a \end{bmatrix}$$

$$L^{-1} = \tilde{L}$$

$$C_{\underline{X}\underline{X}} = \tilde{L}\tilde{L}^T$$

$$= \boxed{\triangleright}\,\boxed{\llcorner}^T$$ (Cholesky factorization.)



$$E\{\underline{W}\}=\underline{0}$$

$$C_{\underline{W}\underline{W}} = E\{L(\underline{X}-\mu_x)(\underline{X}-\mu_x)^TL^T\} = LC_{\underline{X}\underline{X}}L^T = I.$$

# Page 07

2018년 2월 19일 월요일 오후 7:54

§7-4. Next.