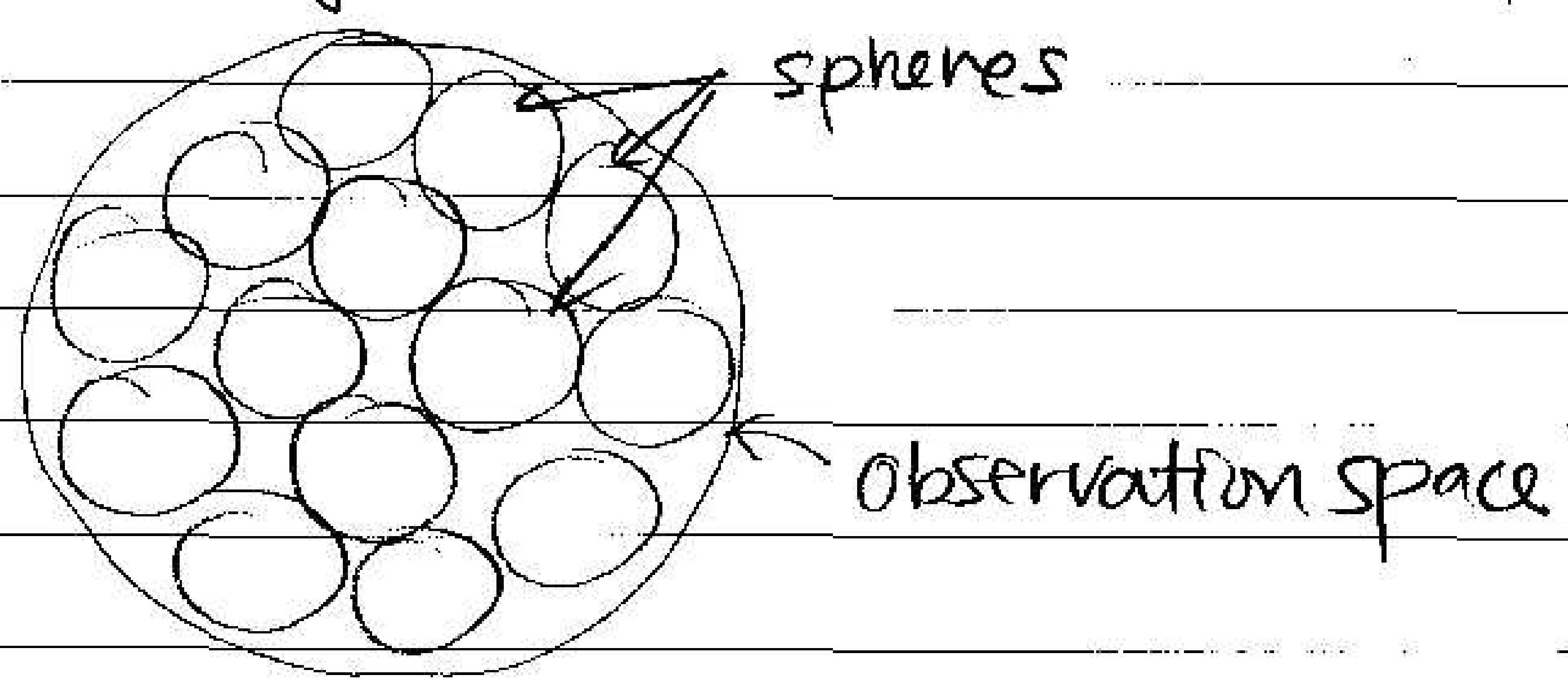
○ Sphere packing argument

- In some proofs for channel coding, the sphere packing argument appears. Usually, a sphere represent the set of channel outputs that are mapped to the same message. The radius of a sphere is directly related to the error probability.



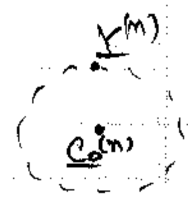
○ Sphere Packing Argument for AWGN NCCT

- Channel

$E\{X^2\} \le P, \quad Z \sim N(0, \sigma^2)$

- Proof using joint typical set
- Sphere packing argument

  - The objective is to find codewords $\underline{x}_i^{(n)}$ s.t $\|\underline{x}_i^{(n)}\|^2 \le nP$.
  - Consider the search space $\{\underline{x}^{(n)} \mid \|\underline{x}^{(n)}\|^2 \le nP\}$ of which surface is a hypersphere with radius $\sqrt{nP}$.
  - If we choose $\underline{c}_0^{(n)}$ as a codeword then the received signal $\underline{c}_0^{(n)} + \underline{z}^{(n)}$ is on the small sphere with center $\underline{c}_0^{(n)}$ and radius $\sqrt{n\sigma^2}$ by the LLN. ($\because$ $\Pr(\,|\|\underline{z}^{(n)}\| - \sqrt{n\sigma^2}| > \varepsilon) \to 0$)



  - We want to choose as many codewords as possible without overlapping among the small spheres
  - If a codeword is on the sphere $\{\underline{x}^{(n)} \mid \|\underline{x}^{(n)}\| \le nP\}$ then the received signal can be on the sphere $\{\underline{y}^{(n)} \mid \|\underline{y}^{(n)}\|^2 \le n(P+\sigma^2)\}$.
  - Thus, we want to pack as many small spheres w/ radius $\sqrt{n\sigma^2}$ as possible inside the sphere w/ radius $\sqrt{n(P+\sigma^2)}$

$$\#\text{ of distinguishable codewords} = \left(\frac{\sqrt{n(P+\sigma^2)}}{\sqrt{n\sigma^2}}\right)^n$$

$$\Rightarrow \quad \log_2 M = \frac{n}{2}\log_2\left(1+\frac{P}{\sigma^2}\right)$$

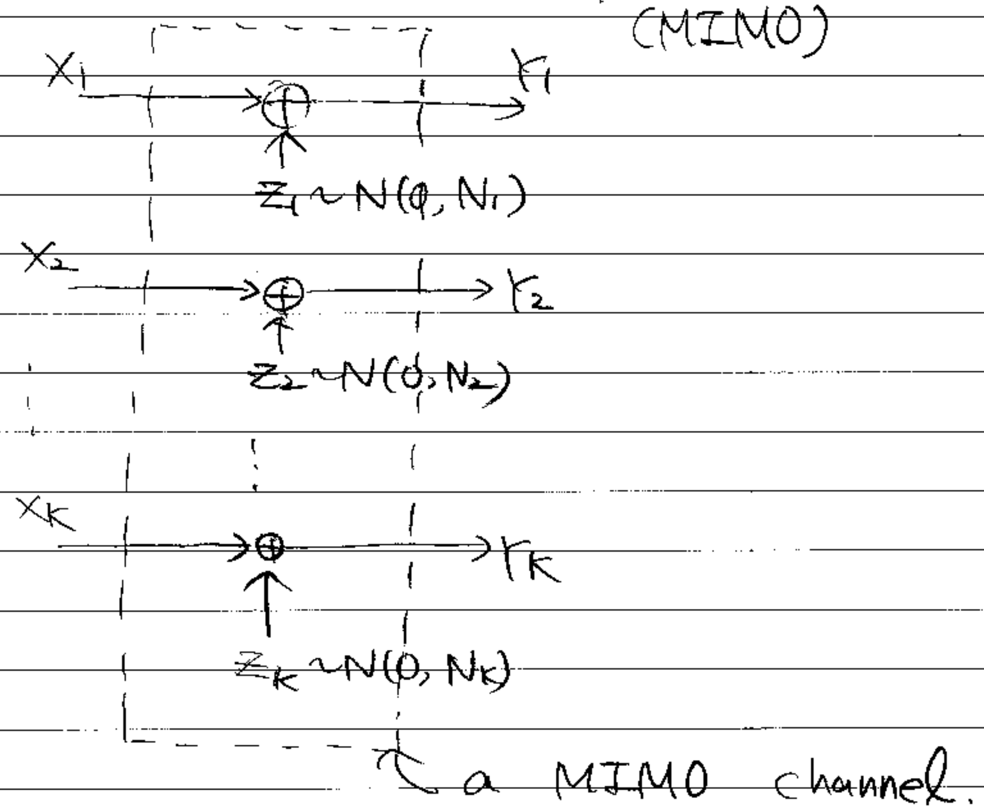
$$\Rightarrow \quad R = \frac{1}{2}\log_2\left(1+\frac{P}{\sigma^2}\right) \quad [\text{bits/channel use}]$$

$$\therefore \quad C = \frac{1}{2}\log_2\left(1+\frac{P}{\sigma^2}\right)$$

Parallel Gaussian Channels

- So far (SISO) we have considered a single-input single-output AGN channel and communication over memoryless extension of it.

Now, let's consider parallel gaussian channels as a multi-input multi-output (MIMO) channel given by



The question is what is the capacity of this MIMO channel under the average power constraint.

We only examine the informational channel capacity of this channel. The equivalence of the operational channel capacity can be similarly proved as SISO AGN channel.

- The informational channel capacity is defined as

$$C = \max_{\sum_{j=1}^{k} E[X_j^2] \le P} I(X_1, X_2, \cdots, X_k ; Y_1, Y_2, \cdots, Y_k)$$

(Sol)

Since $Z_1, \cdots, Z_k$ are independent,

$$\begin{aligned}
& I(X_1, X_2, \cdots, X_k ; Y_1, Y_2, \cdots, Y_k) \\
&= h(Y_1, Y_2, \cdots, Y_k) - h(Y_1, Y_2, \cdots, Y_k | X_1, X_2, \cdots, X_k) \\
&= h(Y_1, Y_2, \cdots, Y_k) - h(Z_1, Z_2, \cdots, Z_k) \\
&= h(Y_1, Y_2, \cdots, Y_k) - \sum_{i=1}^{k} h(Z_i) && \text{independence!} \\
&\le \sum_{i=1}^{k} h(Y_i) - \sum_{i=1}^{k} h(Z_i) && \text{independence bound} \\
&= \sum_{i=1}^{k} I(X_i ; Y_i)
\end{aligned}$$

Suppose that $P_i \triangleq E\{X_i^2\}$. Then,

$$C = \max_{P_i} \sum_{i=1}^{k} I(X_i ; Y_i) = \sum_{i=1}^{k} \frac{1}{2} \log_2 \left(1 + \frac{P_i}{N_i}\right)$$

Hence, we just need to find $\{P_i\}_{i=1}^{k}$ that maximize $\sum_{i=1}^{k} \frac{1}{2} \log_2 \left(1 + \frac{P_i}{N_i}\right)$ subject to $P_i \ge 0$ and $\sum_{i=1}^{k} P_i = P (>0)$.

We form the reduced Lagrangian function

$$\mathcal{L}(P_1, P_2, \cdots, P_k, \lambda) \triangleq \sum_{i=1}^{k} \frac{1}{2} \log_2 \left(1 + \frac{P_i}{N_i}\right) + \lambda \left(\sum_{i=1}^{k} P_i - P\right)$$

then differentiate with respect to P_i's to obtain

$$\frac{1}{2} \log_2 e \, \frac{\frac{1}{N_i}}{1 + \frac{P_i}{N_i}} + \lambda = 0$$

$$\Rightarrow P_i = \nu - N_i, \text{ where } \nu \triangleq -\frac{\log_2 e}{2\lambda}.$$

Since $P_i \geq 0$ and $\sum_{i=1}^{K} P_i = P$, we set

$$P_i^* = [\nu^* - N_i]^+ \quad w/ \quad [x]^+ = \frac{x + |x|}{2}$$

and find ν^* (≥ 0) that is the solution to

$$\sum_{i=1}^{K} [\nu - N_i]^+ = P.$$

Finally, we verify that the solution satisfies the Kuhn-Tucker condition:

" For i with $P_i^* > 0$, $\left.\frac{\partial \mathcal{L}}{\partial P_i}\right|_{P_i^*, \nu^*} = 0$. "

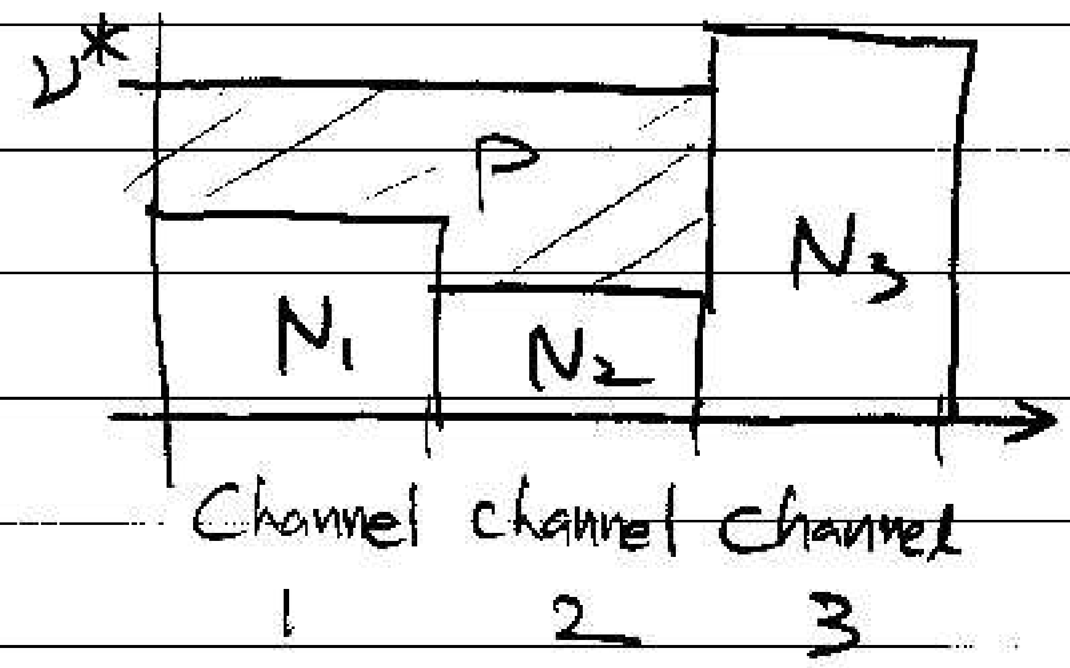
It can be easily verified in this problem. Hence,

$$P_i^* = [\nu^* - N_i]$$

where ν^*(≥ 0) satisfies $\sum_{i=1}^{K} [\nu^* - N_i]^+ = P$ is the solution

- Remark

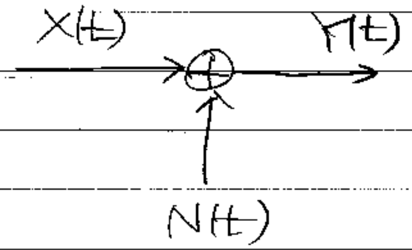
Finding ν^* is called **water-filling** because, e.g.,



Capacity of Bandlimited White Gaussian Noise Channel

- Among many important channels in digital communications, a waveform channel called a **bandlimited AWGN** (additive white Gaussian noise) **channel** is one of the most important ones in theory and practice.

A waveform channel is a probabilistic mapping rule from a set of admissible waveform inputs to a set of feasible waveform outputs. The input-output relation of a bandlimited AWGN channel with **bandwidth** W [Hz] is given by



where

$X(t)$ is the channel input, which is bandlimited to $|f| \le W$.

$N(t)$ is a bandlimited white Gaussian noise process independent of $X(t)$ and bandlimited to $|f| \le W$, and

$Y(t)$ is the summation of the input $X(t)$ and the additive noise $N(t)$.

- Def.

$N(t)$

A **bandlimited** white Gaussian noise (**WGN**) process with bandwidth W [Hz] and two-sided power spectral density (**PSD**) $N_0/2$ is a random waveform such that, $\forall n \in \mathbb{N}$ and $\forall t_1, t_2, \cdots, t_n$, the vector

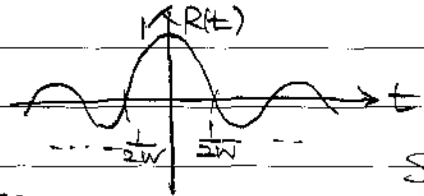
$\begin{bmatrix} N(t_1) \\ N(t_2) \\ \vdots \\ N(t_n) \end{bmatrix}$ is a Gaussian random vector with

mean $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and covariance matrix $E\left\{ \begin{bmatrix} N(t_1) \\ N(t_2) \\ \vdots \\ N(t_n) \end{bmatrix} \begin{bmatrix} N(t_1) \\ N(t_2) \\ \vdots \\ N(t_n) \end{bmatrix}^T \right\}$

$$= N_0 W \begin{bmatrix} R(t_1-t_1) & R(t_1-t_2) & \cdots & R(t_1-t_n) \\ R(t_2-t_1) & R(t_2-t_2) & \cdots & R(t_2-t_n) \\ \vdots & & & \\ R(t_n-t_1) & R(t_n-t_2) & \cdots & R(t_n-t_n) \end{bmatrix}$$

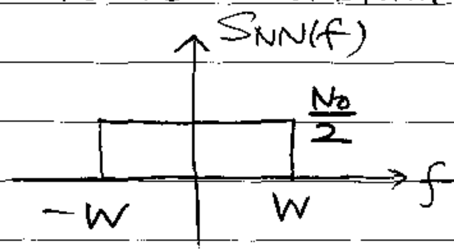
where

$$R(t) = \mathrm{sinc}(2Wt) \triangleq \begin{cases} \dfrac{\sin(2\pi W t)}{2\pi W t}, & \text{for } t \neq 0 \\ 1, & \text{for } t = 0. \end{cases}$$



- Remarks

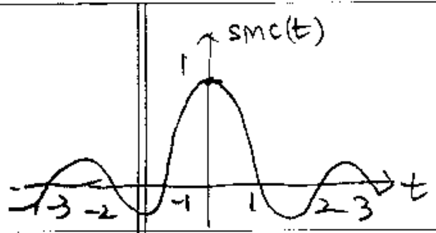
(i) The Fourier transform $S_{NN}(f)$ of $N_0 W R(\tau)$ is sketched as



where the PSD $S_{NN}(f)$ is flat in $f \in [-W, W]$ with spectral height $\frac{N_0}{2}$. Moreover

$$\int_{-W}^{W} S_{NN}(f)\,df = 2W \frac{N_0}{2} = N_0 W = E\{N(t)^2\}.$$

(ii) Suppose that $\{N_i\}_{i=-\infty}^{\infty}$ is an i.i.d. sequence of Gaussian random variables with mean zero and variance $N_0 W$. Then, the random waveform $N(t)$



$$\mathrm{sinc}(x) \triangleq \begin{cases} \frac{\sin \pi x}{\pi x}, & x \neq 0. \\ 1, & x = 0. \end{cases}$$

defined as

$$N(t) = \sum_{i=-\infty}^{\infty} N_i \,\mathrm{sinc}\left(2W\left(t - \frac{i}{2W}\right)\right) \cdots\cdots (*)$$

is a bandlimited WGN process with bandwidth W and two-sided PSD $N_0/2$. Moreover, $N(\frac{i}{2W}) = N_i$, $\forall i$.

- Theorem. (Nyquist's sampling theorem)
  Suppose that a finite energy function $f(t)$ is band-limited to W [Hz]. Then, the function is completely determined by the samples of $f(t)$ at the rate $2W' \geq 2W$. Hence,

$$f(t) = \sum_{i=-\infty}^{\infty} f\left(\frac{i}{2W'}\right) \mathrm{sinc}\left(2W'\left(t - \frac{i}{2W'}\right)\right) \cdots (**)$$

- Note that in $(**)$ $f(t)$ is the lowpass filtered version of $f(t)$, so that it is continuous. Note also that

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \sum_{i=-\infty}^{\infty} \left(f\left(\frac{i}{2W}\right)\right)^2.$$

- Now, combining $(*)$ and $(**)$, we obtain the following result.

The capacity in [bits/sec] of a bandlimited AWGN channel w/ bandwidth W and two-sided PSD $N_0/2$ under the average power constraint P on the input is

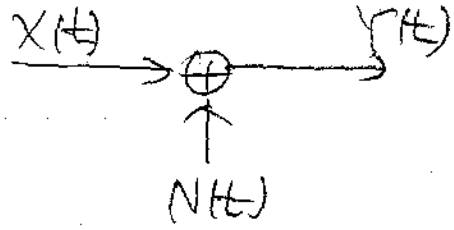
$$C = \underbrace{\frac{1}{2}\log_2\left(1 + \frac{P}{N_0 W}\right)}_{\text{capacity in bits/sample}} \times \underbrace{2W}_{\text{\# of samples/sec}}$$

$$= W \log_2\left(1 + \frac{P}{N_0 W}\right).$$

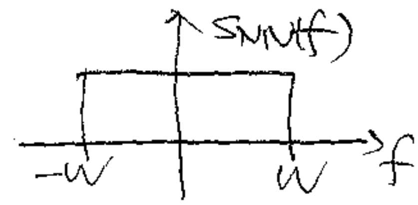
Since $\log(1+x)$ is convex cap, we've chosen to have $E\{X(\frac{i}{2W})^2\} = P$, $\forall i$.

○ AWGN channel vs. ACGN channel

- We have just studied the capacity of band-limited additive white Gaussian noise channel;



w/

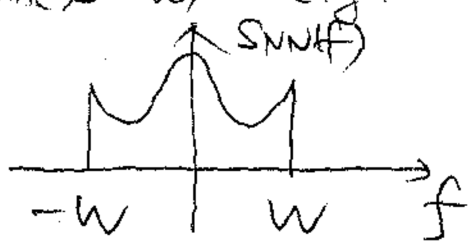


X(t) is band-limited to ±W and average power-limited to P.

and found that

$$C = W \log_2 \left(1 + \frac{P}{N_0 W}\right) \quad [\text{bps}].$$

- Now, the question is what is the capacity of band-limited additive colored Gaussian noise channel w/ e.g. ?

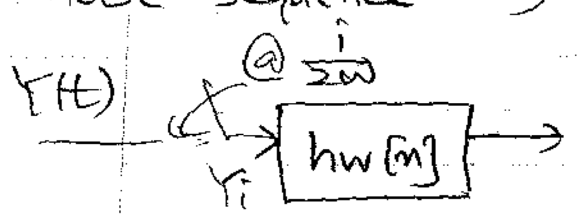


- As in AWGN case, we may sample Y(t) at Nyquist's rate to obtain equivalent discrete-time sequence.

$$Y(t) \xrightarrow{@\, t = \frac{i}{2W}} Y_i = X_i + N_i$$

However, the random sequence $\{N_i\}_{i=-\infty}^{\infty}$ does not have i.i.d. (even though identically Gaussian distributed) elements. Thus, we cannot use the result from Gaussian CMC capacity.

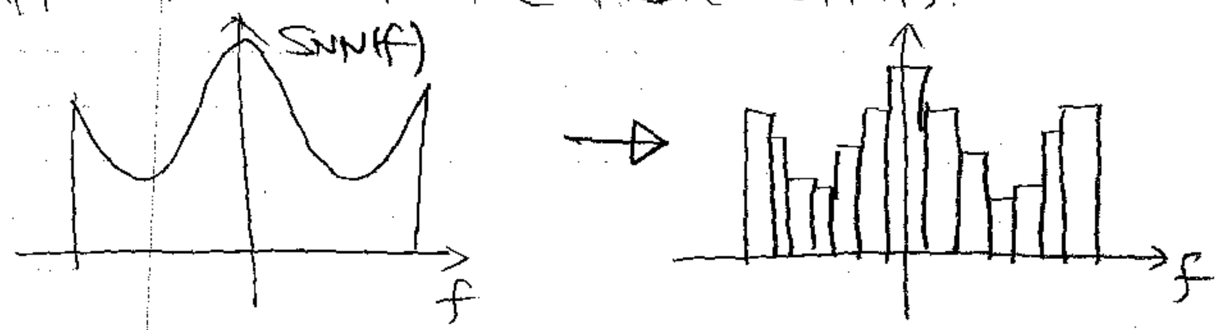
- The noise sequence may be whitened as



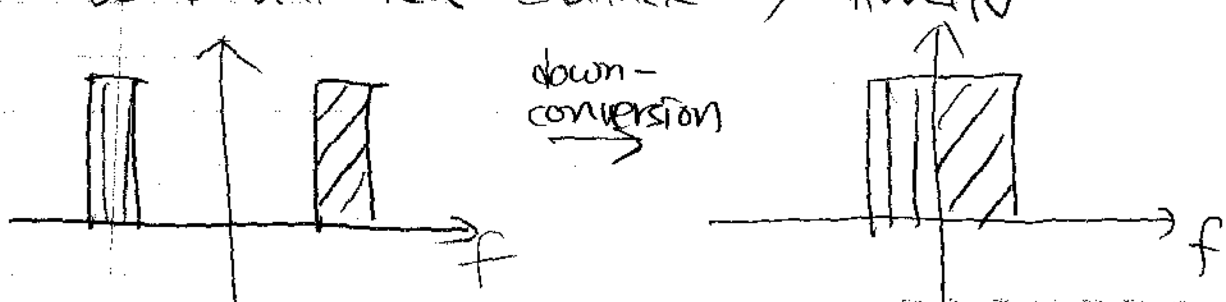
$\tilde{Y}_i = \tilde{X}_i + \tilde{N}_i$

by using a DT LTI filter. However, the input power constraint does not simply convert to the input constraint on $\tilde{X}_i$.

- The key trick starts from piecewise-constant approximation of the PSD $S_{NN}(f)$.

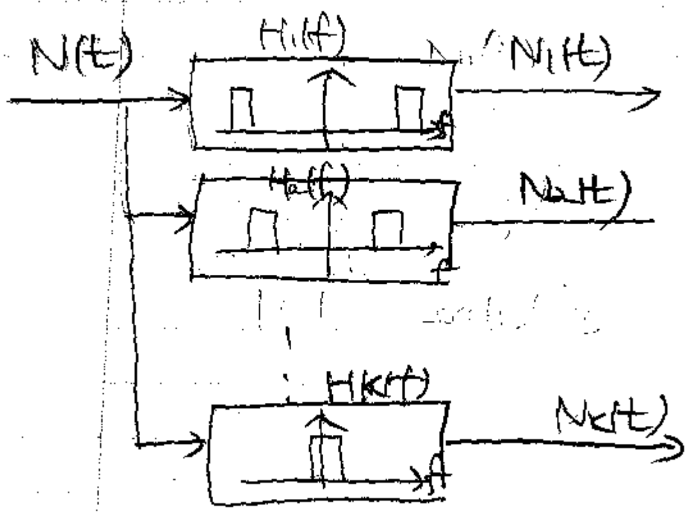


Then, any subband can be down-converted to a band-limited channel w/ AWGN



Thus, we have many band-limited AWGN channels that can transmit data in parallel.

- Moreover, the AWGNs in all subbands are independent random processes. This is because



the cross PSD $S_{N_iN_j}(f)=0, \forall f$ as

$$S_{N_iN_j}(f) = \mathcal{F}\{R_{NN}(\tau) * h_i(t) * h_j(-t)^*\}$$

$$= S_{NN}(f)\underbrace{H_i(f)H_j(f)^*}_{=0,\forall f}$$

$$=0, \forall f$$

Note that the joint Gaussianity implies uncorrelatedness equals independence.

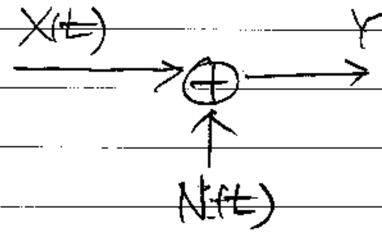
- So, we've reduced the problem to the capacity calculation problem for parallel Gaussian channels.

wide-sense stationary, zero-mean

○ Capacity of bandlimited additive colored Gaussian Noise channel

· Def. wide-sense stationary (WSS), zero-mean (ACGN)

A bandlimited additive colored Gaussian noise channel has the input-output relation as



where $X(t)$ is the channel input, which is bandlimited,
$N(t)$ is a WSS colored Gaussian noise process independent of $X(t)$ and having the same bandwidth as $X(t)$, and
$Y(t)$ is the channel output that is the sum of $X(t)$ and $N(t)$.

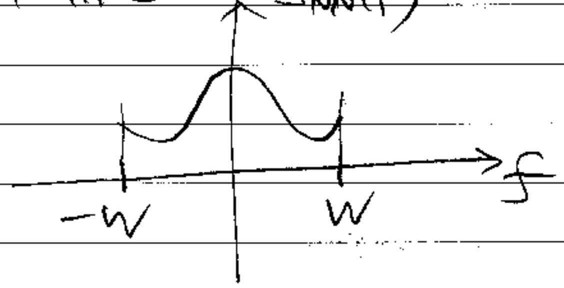
- Def.

A bandlimited WSS zero-mean CGN process is a random waveform such that, $\forall n \in \mathbb{N}$ and $\forall t_1, \cdots, t_n$, the vector $\begin{bmatrix} N(t_1) \\ \vdots \\ N(t_n) \end{bmatrix}$ is a zero-mean Gaussian random vector with the (i,j)th entry of the covariance matrix is given by $R(|t_i - t_j|)$ with $\mathcal{F}\{R(\tau)\}$ is

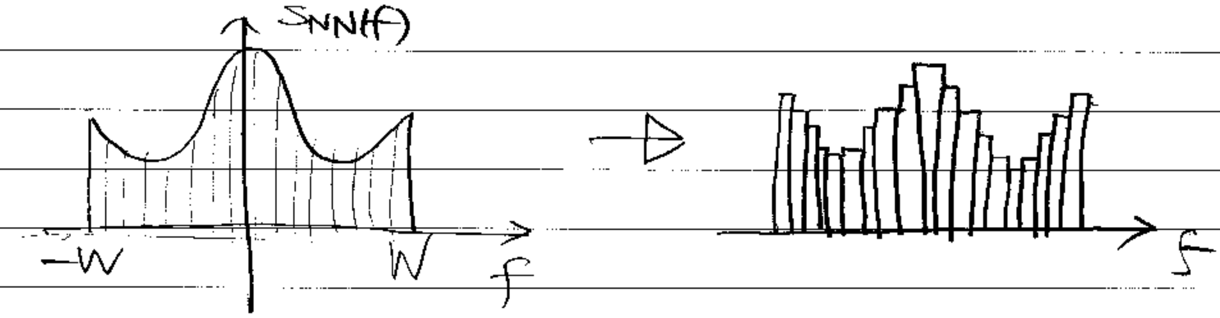
not flat on the frequency band.

- Remark.

The PSD of a bandlimited WSS zero-mean CGN may look like



- Using the idea of Riemann integration, we can divide the interval $f \in [-W, W]$ into equal-bandwidth subintervals. Then, we virtually have parallel AWGN channels.



Note that each subchannel have independent noise because $E[N_i(t_1) N_j(t_2)]$

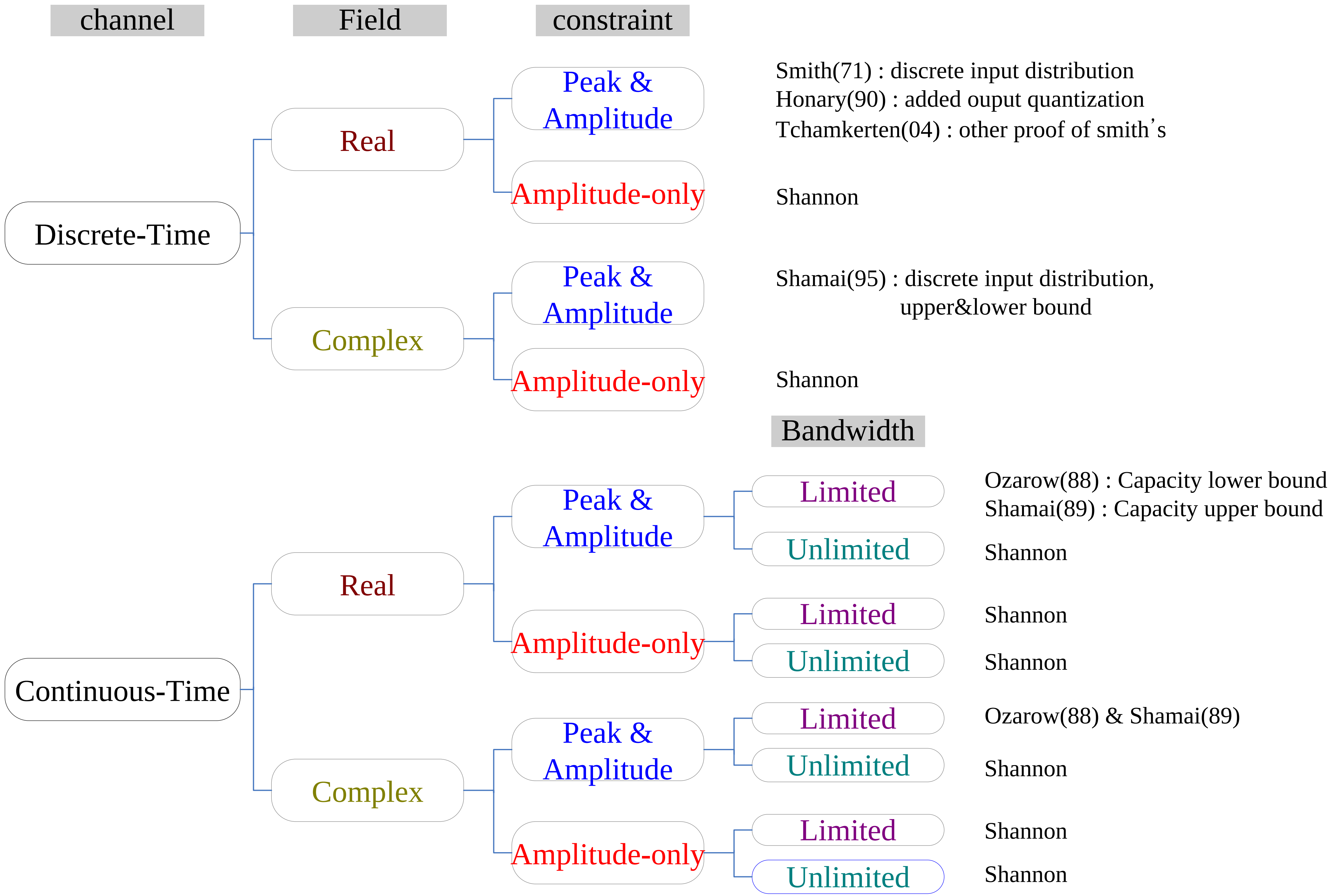
$$= R_{NN}(\tau) * \underbrace{h_i(\tau)}_{\text{bandpass filter for th ith subchannel}} * h_j(-\tau) \Big|_{\tau = t_1 - t_2} = 0, \ \forall t_1, t_2.$$

Hence,
$$C = \int_{-W}^{W} \frac{1}{2} \log\left(1 + \frac{[\nu^* - S_{NN}(f)]^+}{S_{NN}(f)}\right) df$$

where $\nu^* (\geq 0)$ is the solution to
$$\int_{-W}^{W} [\nu - S_{NN}(f)]^+ df = P$$

○ Gaussian channels with feedback.

- If we consider communicating over a memoryless extension of an AGN channel, then again feedback does not increase capacity. Proof omitted.


channel
Field
constraint
Discrete-Time
Real
Peak & Amplitude
Smith(71) : discrete input distribution
Honary(90) : added ouput quantization
Tchamkerten(04) : other proof of smith's
Amplitude-only
Shannon
Complex
Peak & Amplitude
Shamai(95) : discrete input distribution,
upper&lower bound
Amplitude-only
Shannon
Bandwidth
Continuous-Time
Real
Peak & Amplitude
Limited
Ozarow(88) : Capacity lower bound
Shamai(89) : Capacity upper bound
Unlimited
Shannon
Amplitude-only
Limited
Shannon
Unlimited
Shannon
Complex
Peak & Amplitude
Limited
Ozarow(88) & Shamai(89)
Unlimited
Shannon
Amplitude-only
Limited
Shannon
Unlimited
Shannon