Complex Baseband Representation of Real Band-pass Signals: Derivation

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- Derivation
  - Since $x(t)$ is real, its Fourier transform $X(f)$ has conjugate symmetry. i.e.,
$$X(f) = X^*(-f).$$
  - Fig.
  - Thus,
$$X_+(f) \triangleq \begin{cases} 2X(f), & \text{for } f > 0, \\ 0, & \text{for } f < 0. \end{cases}$$
contains all the information in $X(f)$.

– This can be easily shown as

$$X(f) = \frac{X_+(f) + X_+^*(-f)}{2} \tag{1}$$
$$= \begin{cases} X(f), & \text{for } f > 0, \\ X^*(-f), & \text{for } f < 0. \end{cases}$$

– Definition. $x_+(t) \triangleq \mathcal{F}^{-1}\{X_+(f)\}$ is called **the pre-envelope (or analytic signal)** of $x(t)$.

– Using $f_c \in (\frac{B}{2}, B)$ and $\theta \in [0, 2\pi)$, we can define

$$X_l(f) \triangleq X_+(f + f_c)e^{-j\theta}, \tag{2}$$

i.e.,

$$X_+(f) = X_l(f - f_c)e^{j\theta}.$$

This complex baseband signal also contains all the information in $X(f)$.

– Fig.

– Definition. $x_l(t) \triangleq \mathcal{F}^{-1}\{X_l(f)\}$ is called **the complex-envelope** of $x(t)$ for f_c and θ.
(In most cases, θ is set to zero.)

– Table

– From (1) and (2), we have

$$x(t) = \frac{x_+(t) + x_+^*(t)}{2} = \mathrm{Re}\{x_+(t)\}$$

and

$$x_+(t) = x_l(t)e^{\mathrm{j}(2\pi f_c t+\theta)}.$$

Thus,

$$x(t) = \mathrm{Re}\{x_l(t)e^{\mathrm{j}(2\pi f_c t+\theta)}\}. \tag{3}$$

If we define

$$x_c(t) \triangleq \mathrm{Re}\{x_l(t)\} \text{ and } x_s(t) \triangleq \mathrm{Im}\{x_l(t)\},$$

then (3) can be written as

$$\begin{aligned} x(t) &= \mathrm{Re}\{(x_c(t) + \mathrm{j}x_s(t))e^{\mathrm{j}(2\pi f_c t+\theta)}\} \\ &= x_c(t)\cos(2\pi f_c t + \theta) - x_s(t)\sin(2\pi f_c t + \theta). \end{aligned}$$

- Real and Imaginary Parts of $x_l(t)$
  - Q. In the FD, find $x_c(t)$ and $x_s(t)$ from $x(t)$.
  - A. Since

$$x_c(t) = \frac{x_l(t) + x_l^*(t)}{2} \overset{\mathcal{F}}{\longleftrightarrow} X_c(f) = \frac{X_l(f) + X_l^*(-f)}{2}$$

and

$$x_s(t) = \frac{x_l(t) - x_l^*(t)}{2\mathrm{j}} \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{j}X_s(f) = \frac{X_l(f) - X_l^*(-f)}{2},$$

we have

$$X_l(f) = \underbrace{X_c(f)}_{\substack{\text{conjugate}\\ \text{symmetrical}\\ \text{part}}} + \underbrace{\mathrm{j}X_s(f)}_{\substack{\text{conjugate}\\ \text{anti-symmetrical}\\ \text{part}}} .$$

  - If $X(f)$ is real, then $X_c(f)$ is the even part of $X_l(f)$ and $\mathrm{j}X_s(f)$ is the odd part of $X_l(f)$.
  - Fig.