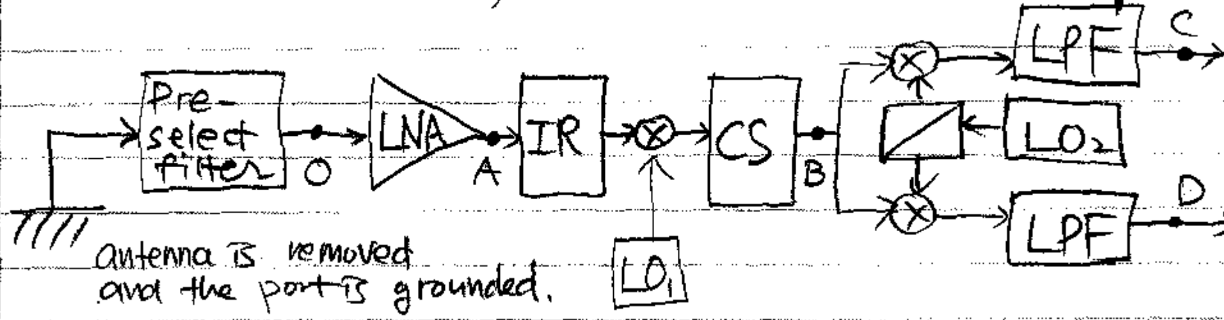
Complex baseband representation of real-valued WSS bandpass random processes

○ Why real-valued WSS bandpass random processes?

- Consider a heterodyne receiver with **no input**



At point O, there is no signal.
At point A, there is noise generated by the LNA
At point B, there exists the noise component that is within the desired frequency band.
At point C, there is the in-phase component of the noise at B.
At point D, there is 〃 quadrature 〃 〃 〃 〃 B.

- **Noise** generated by the **LNA** (From Dr. M.P. Fitz's lecture note.)

The noise at point A is known to be well modeled as a

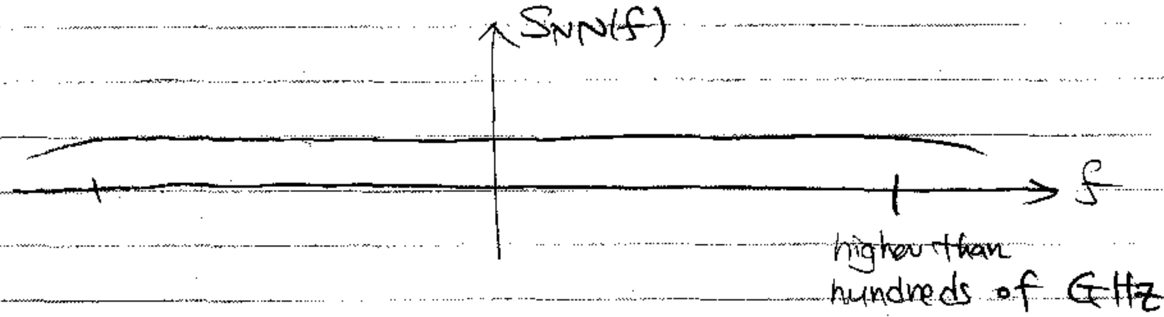
real-valued
stationary
Gaussian
} random process with the PSD

given by

$$S_{NN}(f) = \frac{2Rh|f|}{e^{\frac{h|f|}{KT}} - 1}, \quad \begin{cases} S_{NN}(f) \to 0 \text{ as } f \to \infty \\ f \ll \infty \\ S_{NN}(f) \simeq 2RKT \end{cases}$$

where $h = 6.62 \times 10^{-34}$ is Planck's constant,
$K = 1.3799 \times 10^{-23}$ is Boltzman's 〃
R is the resistance
T is the temperature in Kelvin

If we plot a typical PSD, then



Note that ~~$e^x \simeq 1+x$ for small~~ x. (Why?)

So, within the frequency band of interest, the PSD is flat. That's why often we model the noise as a **white noise**. By convention, $S_{NN}(f) = \frac{N_0}{2}$, N_0 is called the one-sided PSD, $\frac{N_0}{2}$ is called the two-sided PSD.

"N not"

- Noise at point B.

Due to the IR and CS filtering, the noise present at point B is the component of $N(t)$ that is within the desired frequency band. Let it be $X(t)$.

Then, $X(t)$ is a real-valued, stationary, Gaussian, **bandpass** random process

- Noise at point C & D.

If we define $X_c(t;\omega)$ and $X_s(t;\omega)$ as the inphase and quadrature components of $X(t,\omega)$, a sample path then

$$X(t) = X_c(t)\cos 2\pi f_c t - X_s(t)\sin 2\pi f_c t$$

Thus, at point C, we have $X_c(t)$,
and at " D, " " $X_s(t)$

- Noise is one of the most important system impairment in communications. Especially, the **thermal** noise is unavoidable as far as an LNA is used.

Since $X(t)$ is a (real-valued, stationary, Gaussian, bandpass) random process,

We first study what is the relation between $X(t)$, $X_c(t)$, and $X_s(t)$ when $X(t)$ is modeled as a (real-valued, wide-sense stationary, bandpass) random process.

We then include the Gaussian assumption. Note that If Gaussianity is assumed, then the wide-sense stationarity implies stationarity.

○ The main results **(in time domain)**

- Let $X(t)$ be a real-valued, WSS, bandpass random process given by

$$X(t) = X_c(t)\cos 2\pi f_c t - X_s(t)\sin 2\pi f_c t.$$

Then, (i) $E[X_c(t)] = E[X_s(t)] = 0, \ \forall t$

(ii) $E[X_c(t)X_c(t+\tau)] = E[X_s(t)X_s(t+\tau)] = R_{X_cX_c}(\tau) \quad \forall t, \tau$

and (iii) $E[X_c(t)X_s(t+\tau)] = R_{X_cX_s}(\tau)$
$= -R_{X_cX_s}(-\tau), \ \forall t, \tau.$

- All three results are interesting.

(i) A real-valued WSS bandpass process has **mean zero**

(ii) $X_c(t)$ and $X_s(t)$ are **WSS** and have the **same autocorrelation** function

(iii) $X_c(t)$ and $X_s(t)$ are **jointly WSS** and have the **cross-correlation** function that is **odd**.

- Proof of (i)

Let $E[X_c(t)] \triangleq \mu_c(t)$ and $E[X_s(t)] \triangleq \mu_s(t)$.
Then $\mu_c(t)$ & $\mu_s(t)$ have bandwidth less than f_c.
Thus,

$$E[X(t)] = E[X_c(t)\cos 2\pi f_c t - X_s(t)\sin 2\pi f_c t]$$

$$= \mathrm{Re}[\mu_\ell(t)\, e^{j2\pi f_c t}] \quad \text{----}(*)$$

where $\mu_\ell(t) \triangleq \mu_c(t) + j\mu_s(t)$.

By the wide-sense stationarity, $(*)$ must be a constant μ for all t. Since $(*)$ does not have any DC term μ must be zero $\forall t$

Thus, $\mu_\ell(t) = \mu_c(t) + j\mu_s(t)$
$= 0, \ \forall t$,
which implies $\mu_c(t) = \mu_s(t)$
$= \mu_\ell(t) = \mu(t) = 0, \forall t.$

Think it in the freq. domain.

- Proof of (ii) & (iii)

As we used the wide-sense stationarity to prove (i), we use the fact again to prove (ii) & (iii)

$$
\begin{aligned}
R_{XX}(\tau) &= E\left[x(t)\,x(t+\tau)\right] \\
&= E\left[\frac{x_\ell(t)e^{j2\pi f_c t} + x_\ell(t)^* e^{-j2\pi f_c t}}{2} \times \frac{x_\ell(t+\tau)e^{j2\pi f_c(t+\tau)} + x_\ell(t+\tau)^* e^{-j2\pi f_c(t+\tau)}}{2}\right] \\
&= \frac{1}{2}E\Big[\big(x_\ell(t)\,x_\ell(t+\tau)\,e^{j2\pi f_c(2t+\tau)} + x_\ell(t)^* x_\ell(t+\tau)^* e^{-j2\pi f_c(2t+\tau)}\big)/2 \\
&\qquad + \big(x_\ell(t)\,x_\ell(t+\tau)^* e^{-j2\pi f_c\tau} + x_\ell(t)^* x_\ell(t+\tau)\,e^{j2\pi f_c\tau}\big)/2\Big] \\
&= \frac{1}{2}\mathrm{Re}\left\{E\left[x_\ell(t)\,x_\ell(t+\tau)\right]e^{j2\pi f_c(2t+\tau)}\right\} \\
&\quad + \frac{1}{2}\mathrm{Re}\left\{E\left[x_\ell(t)^* x_\ell(t+\tau)\right]e^{j2\pi f_c\tau}\right\} \quad \forall t,\tau.
\end{aligned}
$$

By the defi. of WSS,

$\therefore$ $E\left[x_\ell(t)\,x_\ell(t+\tau)\right] = 0 \quad \forall t,\tau$ & $\cdots(*)$

$E\left[x_\ell(t)^* x_\ell(t+\tau)\right] \triangleq R_{x_\ell x_\ell}(\tau)$ $\cdots(**)$

pseudo-autocorrelation function of the complex envelope $x_\ell(t)$.

Note that $(*)$ implies

$$
\begin{aligned}
&E\left[(x_c(t)+j\,x_s(t))(x_c(t+\tau)+j\,x_s(t+\tau))\right] \\
&= R_{x_cx_c}(t,t+\tau) - R_{x_sx_s}(t,t+\tau) \\
&\quad + j\,R_{x_sx_c}(t,t+\tau) + j\,R_{x_cx_s}(t,t+\tau) \\
&= 0, \quad \forall t,\tau.
\end{aligned}
$$

Thus, $\quad R_{x_cx_c}(t,t+\tau) = R_{x_sx_s}(t,t+\tau)$ and
$\quad R_{x_sx_c}(t,t+\tau) = -R_{x_cx_s}(t,t+\tau)$ $\cdots(*\!*\!*)$

Note also that $(**)$ implies

$$
\begin{aligned}
&E\left[(x_c(t)-j\,x_s(t))(x_c(t+\tau)+j\,x_s(t+\tau))\right] \\
&= R_{x_cx_c}(t,t+\tau) + R_{x_sx_s}(t,t+\tau) \\
&\quad - j\,R_{x_sx_c}(t,t+\tau) + j\,R_{x_cx_s}(t,t+\tau) \\
&= R_{x_\ell x_\ell}(\tau) \quad \forall t,\tau.
\end{aligned}
$$

Thus, combined w/ (***), we obtain

$$\begin{cases} R_{x_\ell x_\ell}(\tau) = 2R_{x_c x_c}(\tau) + j\,2R_{x_c x_s}(\tau) \\ R_{x_c x_c}(\tau) = R_{x_s x_s}(\tau) \\ R_{x_c x_s}(\tau) = -R_{x_s x_c}(\tau) = -R_{x_c x_s}(-\tau) \\ R_{xx}(\tau) = R_{x_c x_c}(\tau)\cos 2\pi f_c\tau - R_{x_c x_s}(\tau)\sin 2\pi f_c \tau \end{cases}$$

- Remarks

- We naturally define in this course the autocorrelation function of a complex-valued random process $X_\ell(t)$ as

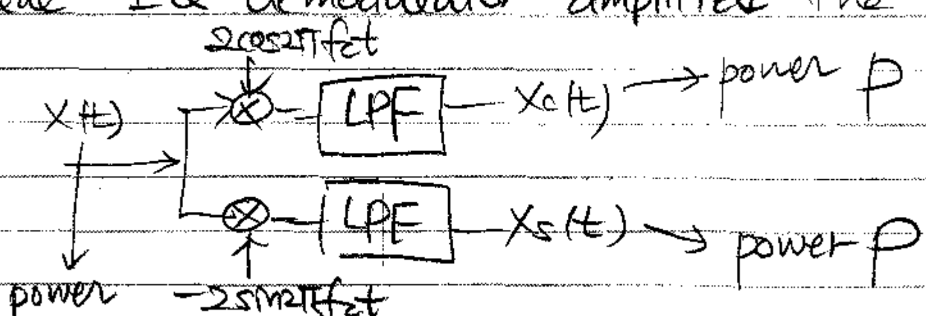
$$E\left[X_\ell(t)^{*} X_\ell(t+\tau)\right] \triangleq R_{x_\ell x_\ell}(t, t+\tau)$$

- The autocorrelation function of $X(t)$ can be written as

$$R_{xx}(\tau) = Re\left\{\left(R_{x_c x_c}(\tau) + j R_{x_c x_s}(\tau)\right) e^{j2\pi f_c \tau}\right\}$$

$$= Re\left\{\frac{R_{x_\ell x_\ell}(\tau)}{2}\, e^{j2\pi f_c\tau}\right\}$$

- The term $\frac{1}{2}$ comes from the fact that the ideal I-Q demodulator amplifies the power by 2.



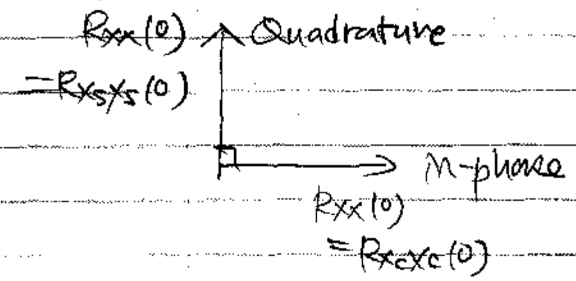
$$P \because R_{xx}(0) = \frac{1}{2} Re\{R_{x_\ell x_\ell}(0)\} = \frac{1}{2}\left(2R_{x_c x_c}(0) + j\,2R_{x_c x_s}(0)\right) = R_{x_c x_c}(0)$$

- The cross power is zero, i.e. $E[X_c(t)X_s(t)] = 0, \forall t$

$\because R_{X_cX_s}(\tau) = -R_{X_cX_s}(-\tau) \Rightarrow R_{X_cX_s}(0) = 0$.

To the contrary,

$$R_{X_cX_c}(0) = R_{X_sX_s}(0) = R_{XX}(0)$$



If $X(t)$ is Gaussian, then $X_c(t)$ & $X_s(t)$ are also Gaussian w/

$$E\{(X_c(t))^2\} = E\{(X_s(t))^2\} = E\{X(t)^2\} \triangleq \sigma^2$$

$$E\{X_c(t)X_s(t)\} = 0$$

$$\therefore f_{X_c(t)X_s(t)}(x,y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$$

circularly symmetric 2-D Gaussian random vector. $X_c(t)$ & $X_s(t)$ for each t are zero-mean & i.i.d Gaussian.

- $R_{X_eX_e}(\tau) = E[X_e(t)^* X_e(t+\tau)] = \left(E[X_e(t+\tau)^* X_e(t)]\right)^*$

$= R_{X_eX_e}(-\tau)^*$ **The auto-correlation function is conjugate symmetrical** (actually, for any complex-valued WSS random process.)

- $\tilde{R}_{X_eX_e}(\tau) \triangleq E[X_e(t)X_e(t+\tau)] = 0$

The pseudo-auto-correlation function is always zero. (not for arbitrary complex-valued WSS random process. A sufficient condition for this to be true is

"the complex-valued WSS process is the complex envelope of a real-valued WSS bandpass process.")

○ Proper-complex random ○○○

- Def. A complex-valued random process $X_e(t)$ is called proper-complex if its pseudo-autocorrelation function is zero, i.e., $E\{X_e(t)X_e(t+\tau)\}=0$, $\forall t,\tau$.
- Def. A complex-valued random vector $\underline{X}$ is called proper-complex if its pseudo-covariance matrix is zero, i.e.,

$$E\{(\underline{X}-\underline{\bar{X}})(\underline{X}-\underline{\bar{X}})^T\}=\underline{0}$$

- We will study later more about proper-complex random vectors & processes, especially for Gaussian cases.

○ The main result **(in frequency domain)**

- We interpret the main results derived in the time domain now in the frequency domain.

main results

$$R_{X_cX_c}(\tau)=R_{X_sX_s}(\tau)=R_{X_cX_c}(-\tau) \quad \cdots (*)$$

$$R_{X_cX_s}(\tau)=-R_{X_cX_s}(-\tau)=-R_{X_sX_c}(\tau) \quad \cdots (**)$$

$$R_{XX}(\tau)=R_{X_cX_c}(\tau)\cos 2\pi f_c\tau - R_{X_cX_s}(\tau)\sin 2\pi f_c\tau \quad \cdots (***)$$

- The PSD of $X_c(t)$ & $X_s(t)$

From (*), $S_{X_cX_c}(f) = S_{X_sX_s}(f) = S_{X_cX_c}(-f)$ real & even ⋯⋯(*)'

- The cross PSD b/w $X_c(t)$ & $X_s(t)$

From (**),

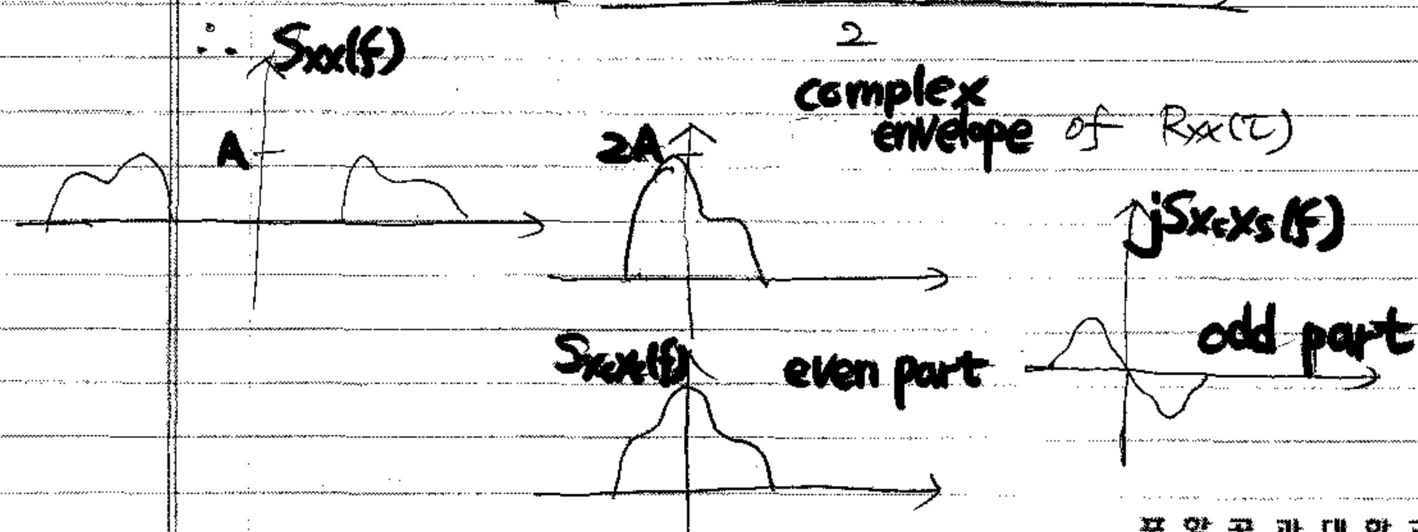
$$S_{X_cX_s}(f) = -S_{X_cX_s}(-f) = -S_{X_sX_c}(f)$$ pure imaginary & odd ⋯(**)'

- Relation among $S_{XX}(f)$, $S_{X_cX_c}(f)$, and $S_{X_cX_s}(f)$

From (***), (*)', and (**)', we obtain

$$S_{XX}(f) = \frac{S_{X_cX_c}(f-f_c) + S_{X_cX_c}(f+f_c)}{2} - \frac{S_{X_cX_s}(f-f_c) - S_{X_cX_s}(f+f_c)}{2j}$$

$$= \frac{S_{X_cX_c}(f-f_c) + jS_{X_cX_s}(f-f_c)}{2} + \frac{S_{X_cX_c}(-(f+f_c)) + jS_{X_cX_s}(-(f+f_c))}{2}$$



- We can view $R_{XX}(\tau)$ as a real-valued bandpass signal. Then, $R_{X_cX_c}(\tau)$ is the in-phase component and $R_{X_cX_s}(\tau)$ is the quadrature component.

Thus,

$$\mathcal{F}\{R_{X_cX_c}(\tau)\} = \text{even part of } \mathcal{F}\{R_{XX,\ell}(\tau)\}$$

&

$$j\,\mathcal{F}\{R_{X_cX_s}(\tau)\} = \text{odd part of } \mathcal{F}\{R_{XX,\ell}(\tau)\}$$

- Caution.

$$R_{XX}(\tau) \triangleq \mathrm{Re}\left\{R_{XX,\ell}(\tau)\, e^{j2\pi f_c\tau}\right\}$$

$$= \mathrm{Re}\left\{\frac{R_{X_\ell X_\ell}(\tau)}{2}\, e^{j2\pi f_c\tau}\right\}$$

the complex envelope of the autocorrelation $R_{XX}(\tau)$

the autocorrelation of the complex envelope $X_\ell(t)$.

$$R_{XX,\ell}(\tau) = \frac{R_{X_\ell X_\ell}(\tau)}{2}$$