Proper-complex Gaussian Distributions : Part I - Real-valued case

○ (Real-valued) Gaussian random vectors

- Def. A random vector $\underline{x}$ with

$$E\{\underline{x}\} = \underline{\mu} \quad \text{and} \quad Cov\{\underline{x}\} = C$$

is called **Gaussian** if its characteristic function is given by

$$\phi_{\underline{x}}(\underline{\omega}) \triangleq E\{e^{j\underline{\omega}^T\underline{x}}\}$$
$$= e^{j\underline{\omega}^T\underline{\mu} - \frac{1}{2}\underline{\omega}^T \mathbf{C} \underline{\omega}}$$

- Remarks

(i) A Gaussian random vector is completely characterized by its **mean** vector $\underline{\mu}$ and the **covariance** matrix C.

$\therefore$ $x \sim N(\underline{\mu}, C)$ Multivariate Gaussian

(ii) If $C > 0$, then the probability density function is given by

$$f_{\underline{x}}(\underline{x}) = \frac{1}{\sqrt{2\pi}^N \sqrt{\det C}} \exp\left(-\frac{1}{2}(\underline{x}-\underline{\mu})^T \mathbf{C}^{-1} (\underline{x}-\underline{\mu})\right)$$

where N is the dimensionality of $\underline{x}$

(ii)' $N=2$, $C = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$, $\rho \neq \pm 1$

$$f_{x_1,x_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\Big[$$

$$-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\}\Bigg]$$

(i)' For $N=2$, $\underline{X} \sim N(\mu_1,\mu_2;\sigma_1,\sigma_2,\rho)$ or

$\underline{X} \sim N(\mu_1,\mu_2;\sigma_1^2,\sigma_2^2,\rho)$

bivariate Gaussian

(i)'' For $N=1$, $\sigma>0$,

$$f_X(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2}\right)$$

(i)'' $X \sim N(\mu,\sigma^2)$ univariate Gaussian

(iii) $X\sim N(\mu,\sigma^2) \Rightarrow E\{(X-\mu)^n\}=\begin{cases}0, & \text{for odd } n\\ \frac{n!}{(\frac{n}{2})!}\left(\frac{\sigma}{\sqrt{2}}\right)^n & \text{for even } n\end{cases}$

ex/ $E\{X^2\}=1\cdot\sigma^2$
$E\{X^4\}=1\cdot3\cdot\sigma^4$
$E\{X^6\}=1\cdot3\cdot5\cdot\sigma^6$
$\vdots$

(iv) If C^{-1} does not exist, the pdf cannot be written in terms of elementary fts. multi-dimensional Dirac delta fts must be used.

Sometimes, we encounter an impulse fence.

Wozencraft & Jacob "Principles of Communication Engineering"

variance of a uniformly distributed random variable is $b^2/12$, where b is the width of the density function. Specifying the variance essentially constrains the effective width of the density function. Figure 2.40 illustrates this effect for the Gaussian density function.

A precise statement of the constraint is due to Chebyshev. Let y be a

Bell(shaped) Curve

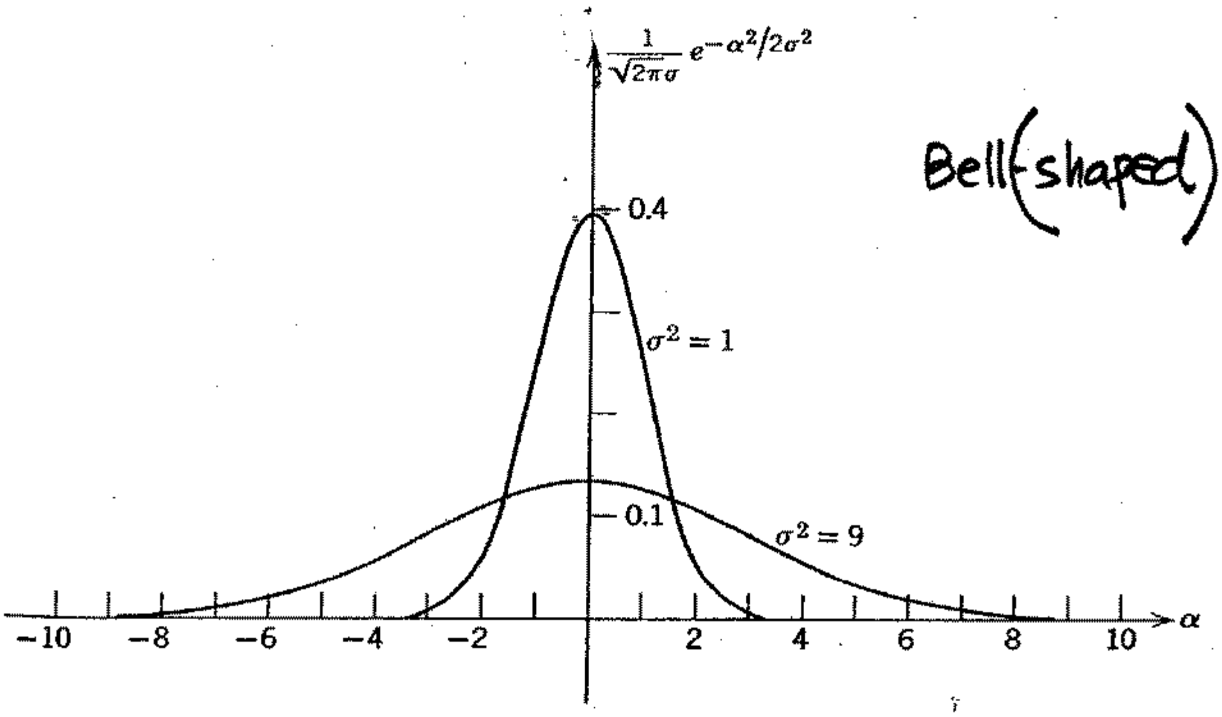


Figure 2.40 The Gaussian probability density function for two values of variance.

zero-mean random variable with finite variance σ_y^2. *Chebyshev's inequality* states that for any positive number ϵ

$$P[|y| \geq \epsilon] \leq \frac{{\sigma_y}^2}{\epsilon^2}; \qquad \bar{y} = 0. \tag{2.146}$$

Equation 2.146 can be proved as follows. By definition,

$$\overline{y^2} = \int_{-\infty}^{\infty} \alpha^2 \, p_y(\alpha) \, d\alpha.$$

Since the integrand is positive,

$$\overline{y^2} \geq \int_{|\alpha| \geq \epsilon} \alpha^2 \, p_y(\alpha) \, d\alpha.$$

This bound can be weakened further by replacing α^2 with its smallest value, ϵ^2, which yields s shown in Fig. 2.26, this densit function may be visualized as tw "fences" of impulses at $\alpha_1 = 1$ and $\alpha_2 = 2$.

For a simple example of the use of a joint density function to calculate a probability, consider the event A defined by

$$A = \{\omega : x_1^2(\omega) + x_2^2(\omega) < c^2\}$$

and the two-dimensional Gaussian density function of Eq. 2.58, with the

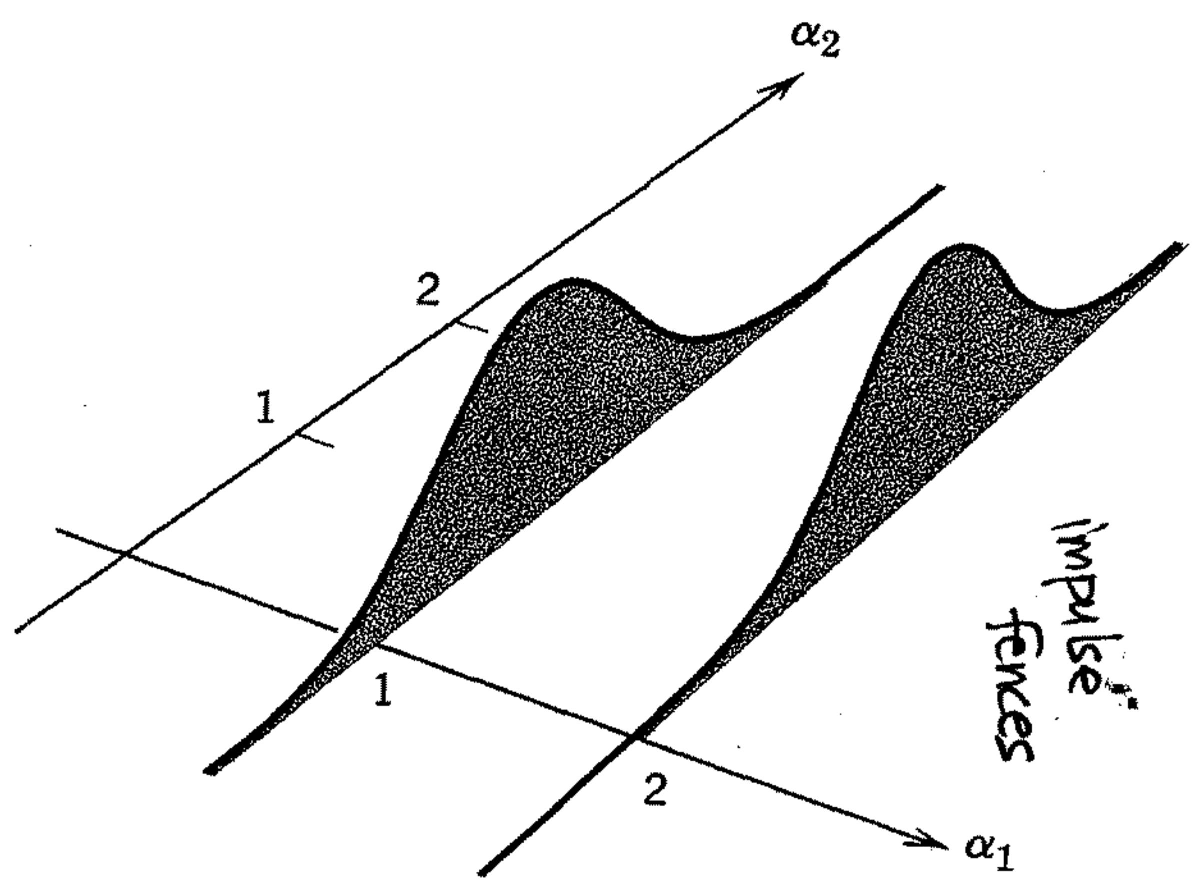


Figure 2.26 Two "fences" of impulses. The value of the one-dimensional impulse at $\alpha_1 = 1$ (or $\alpha_1 = 2$) depends on α_2.

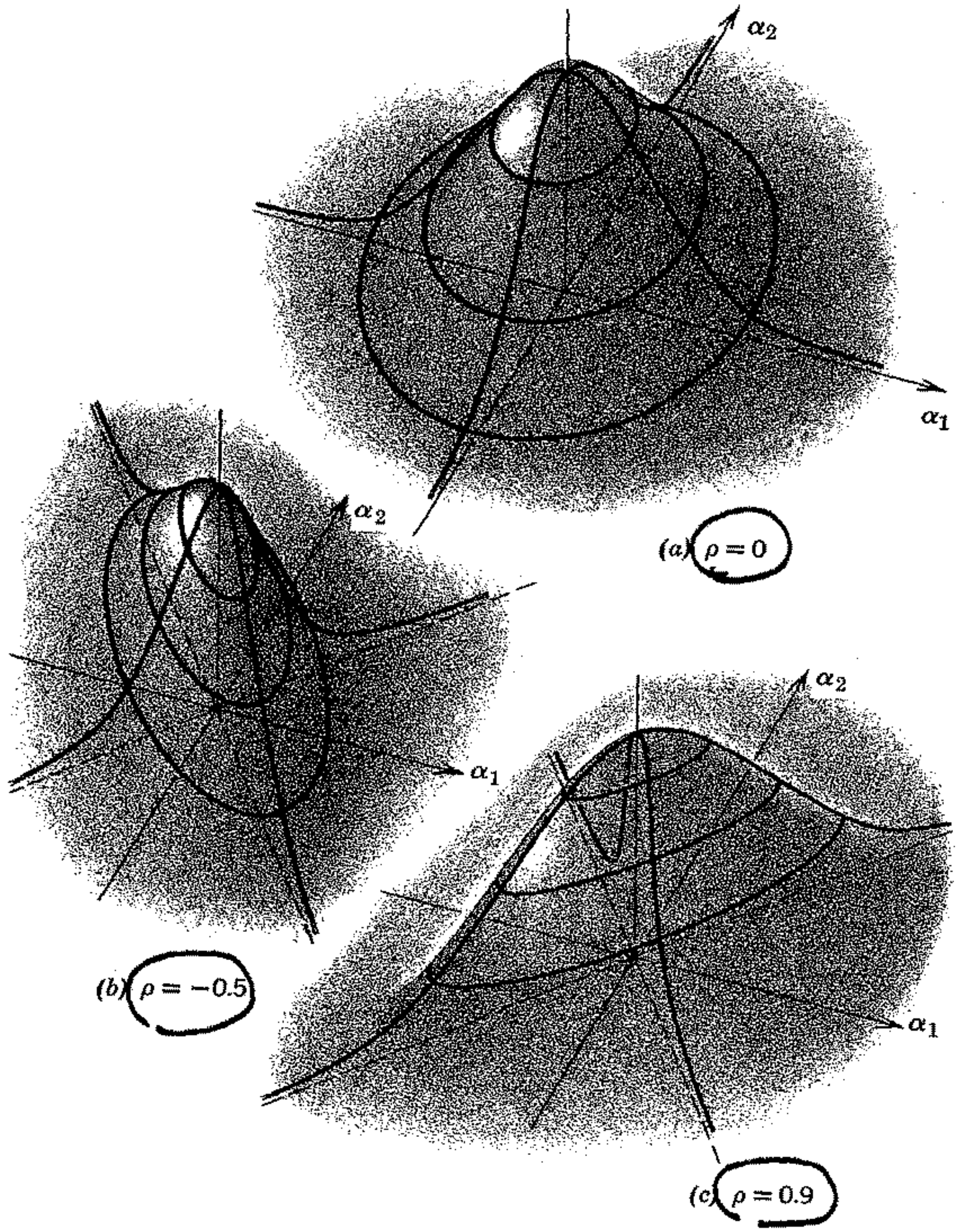


Figure 2.24 Examples of the two-dimensional Gaussian density function.

(v) when $\underline{X} \triangleq \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$, any collection of random vectors $\underline{X}_1, \underline{X}_2, \cdots, \underline{X}_{N'}$,

such as $\underline{X} = \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \\ \vdots \\ \underline{X}_{N'} \end{bmatrix}$ are called **jointly Gaussian.**

• Theorem.

Any transform $\underline{Y}$ of $\underline{X}$ given by

$$\underline{Y} = A\underline{x} + \underline{b}$$

$\underline{x} \in \mathbb{R}^N$, $\underline{b}, \underline{Y} \in \mathbb{R}^M$, $A \in \mathbb{R}^{M \times N}$

gives a Gaussian random vector $\underline{Y}$.

(proof)

$$\Phi_{\underline{Y}}(\underline{\omega}) = E\{e^{j\underline{\omega}^T\underline{Y}}\} = E\{e^{j\underline{\omega}^T(A\underline{x}+\underline{b})}\}$$

$$= e^{j\underline{\omega}^T\underline{b}}\, E\{e^{j(A^T\underline{\omega})^T\underline{x}}\}$$

$$= e^{j\underline{\omega}^T\underline{b}}\, e^{j(A^T\underline{\omega})^T\underline{\mu} - \frac{1}{2}(A^T\underline{\omega})^T C (A^T\underline{\omega})}$$

$$= \exp\left(j\underline{\omega}^T(A\underline{\mu}+\underline{b}) - \frac{1}{2}\underline{\omega}^T(ACA^T)\underline{\omega}\right)$$

which is a Gaussian characteristic function with

mean $A\underline{\mu}+\underline{b}$ and covariance ACA^T

• Remark

$$C = U\Lambda U^T = \sum_{n=1}^{N} \lambda_n \underline{u}_n \underline{u}_n^T$$

where $U = [\underline{u}_1\, \underline{u}_2 \cdots \underline{u}_N]$ satisfied $U^{-1} = U^T$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$

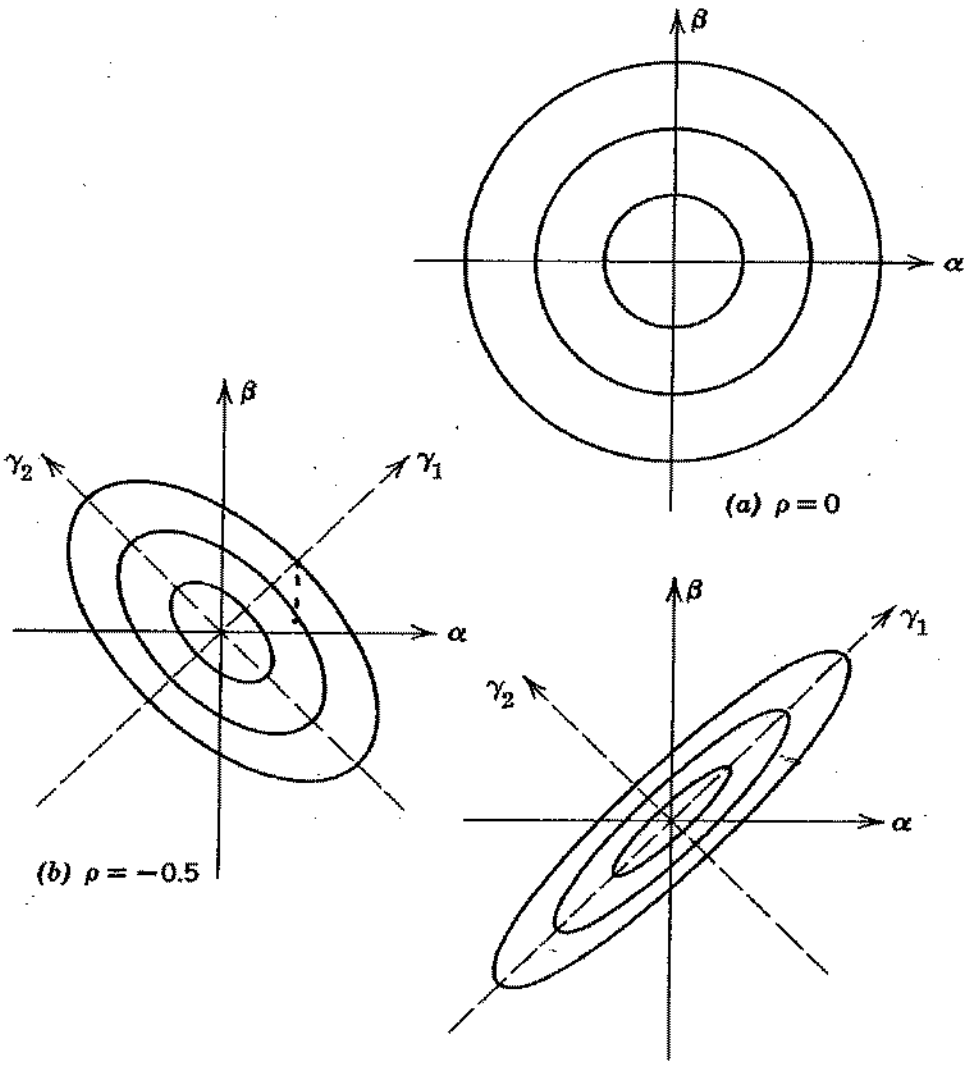


Figure 3.17 Contour plots of constant probability density for the two-dimensional Gaussian density function of Eq. 3.31. The density functions themselves are illustrated in Fig. 2.24 for $\sigma^2 = 1$.

Further insight into the behavior of p_{n_1,n_2} as a function of ρ can be gained from the contour plots of constant probability density shown in Fig. 3.17. The contours are most easily visualized in terms of coordinates γ_1, γ_2 rotated 45° from α, β. If we let

$$\alpha = \gamma_1 \cos\frac{\pi}{4} - \gamma_2 \sin\frac{\pi}{4}, \tag{3.35a}$$

$$\beta = \gamma_1 \sin\frac{\pi}{4} + \gamma_2 \cos\frac{\pi}{4}, \tag{3.35b}$$

$\Rightarrow$ Any N-variate Gaussian distribution can be obtained by **rotating**, **reflecting**, **dilating**, **translating** the Gaussian distribution $N(\underline{0}, I_N)$

- Remark

Suppose that $\underline{Y}$ and $\underline{Z}$ are jointly Gaussian r. vectors. Then, the conditional probability distribution of $\underbrace{\underline{Y} \text{ given } \underline{Z}=\underline{z}}_{\underline{Y}|\underline{Z}=\underline{z}}$ is also **Gaussian** with mean

$$E\{\underline{Y}|\underline{Z}=\underline{z}\} = E\{\underline{Y}\} + Cov(\underline{Y},\underline{Z})Cov(\underline{Z})^{-1}(\underline{z}-E\{\underline{Z}\})$$

and covariance

$$Cov\{\underline{Y}|\underline{Z}=\underline{z}\} = Cov(\underline{Y}) - Cov(\underline{Y},\underline{Z})\,Cov(\underline{Z})^{-1}Cov(\underline{Z},\underline{Y})$$

This result is very important! You can find the usefulness in LMMSE, ALMMSE estimation

Implications

(i) $E\{\underline{Y}|\underline{Z}=\underline{z}\}$ = unconditional mean $E\{\underline{Y}\}$ of $\underline{Y}$ + a linear correction term as a function of $\underline{z}-E\{\underline{Z}\}$

$\Rightarrow$ If $\underline{z}=E\{\underline{Z}\}$, then $E\{\underline{Y}|\underline{Z}=\underline{z}\} = E\{\underline{Y}|\underline{Z}=E\{\underline{Z}\}\}$
$= E\{\underline{Y}\}$

This does not hold in general $\underline{Y}, \underline{Z}$ jointly distributed.

(ii) Conditional covariance is less than or equal to un-conditional "

$\boxed{\text{1-D}}$

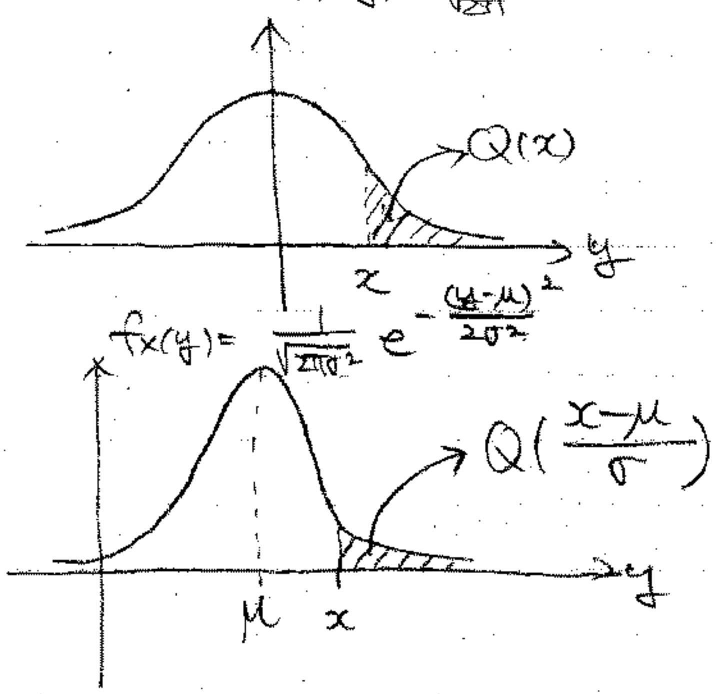
○ Gaussian distributions for communication engineering

- $X \sim N(0,1)$
  - Definition

$$\mathbf{Q(x)} \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}}\, dy$$

$$f_X(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$f_X(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$



- The integral is not an elementary integral.
  Hence, $\underbrace{F(x)}$ $= 1 - Q(x)$ and $\underbrace{\mathrm{erf}(x)}$ are
  distribution function of X ↓
  MATLAB built-in
  widely tabulated, where

$$\mathbf{erf(x)} \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy \quad \textbf{(error ft)}$$

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) \quad \left(\text{complementary error ft}\right)$$

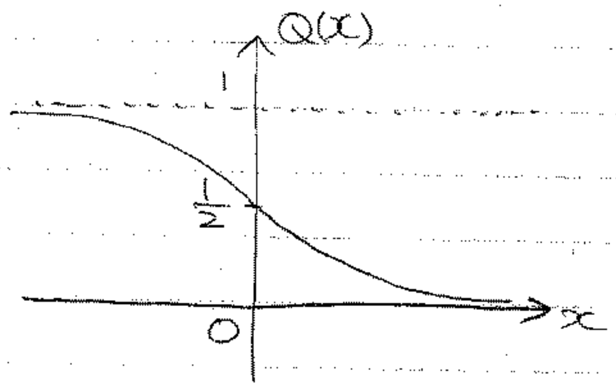
Note

$$Q(x) = \frac{1}{2}\left\{1 - \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right\}$$

$$= \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

- Properties as a function

(0) (0th order derivative)



$$0 < Q(x) < 1, \quad \forall x \in \mathbb{R}$$

$$Q(x) + Q(-x) = 1, \quad \forall x \in \mathbb{R} \left(\Rightarrow Q(0) = \frac{1}{2}\right)$$

(1) (1st order derivative)

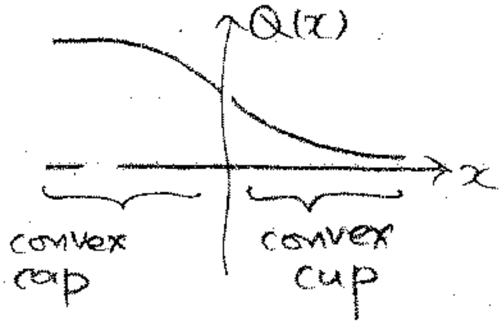
$$Q'(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} < 0 \qquad \forall x \in \mathbb{R}$$

(monotonically decreasing)

(ii) (2nd order derivative)

$$Q''(x) = \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$Q''(x) > 0$ for $x > 0$ (convex cup)
$Q''(x) < 0$ for $x < 0$ (convex cap)



(← Figure. Wozencraft P83. Caution: Log scale in y-axis)

Since

$$0 < \int_{\alpha}^{\infty} \frac{1}{\beta^2} e^{-\beta^2/2}\, d\beta < \frac{1}{\alpha^3} \int_{\alpha}^{\infty} \beta e^{-\beta^2/2}\, d\beta = \frac{1}{\alpha^3} e^{-\alpha^2/2},$$

we have the bounds

$$\frac{1}{\sqrt{2\pi}\alpha} e^{-\alpha^2/2}\left(1 - \frac{1}{\alpha^2}\right) < Q(\alpha) < \frac{1}{\sqrt{2\pi}\alpha} e^{-\alpha^2/2}; \qquad \alpha > 0. \qquad (2.121)$$

These two bounds are plotted together with $Q(\alpha)$ in Fig. 2.36.

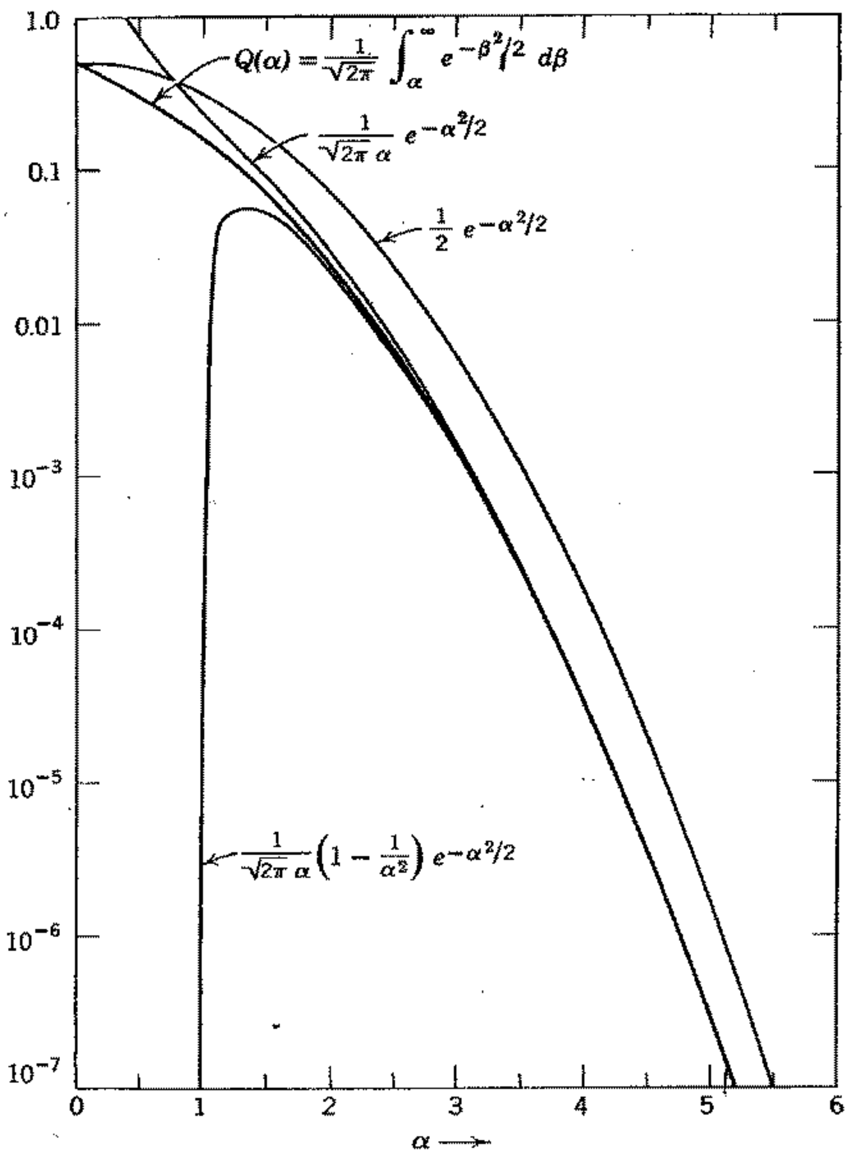


Figure 2.36 The function $Q(\alpha)$ and three bounds.

– Upper and lower **bounds on Q(x)**

(i) $Q(x) \leq e^{-\frac{x^2}{2}}$ $\quad x \geq 0$ $\quad$ (a Chernoff bound)

(ii) $Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}}$ $\quad \binom{x \geq 0}{\ }$ $\quad$ (2-D integration)

In (ii), the equality holds iff $x = 0$

(iii) $\frac{1}{\sqrt{2\pi}}\left(1-\frac{1}{x^2}\right)\frac{e^{-\frac{x^2}{2}}}{x} < Q(x) < \frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{x^2}{2}}}{x}$

for $x > 0$ $\quad$ (Integration by parts)

– Comparison of two upper bounds

$\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{x^2}{2}}}{x}$ is tighter than $\frac{1}{2}e^{-\frac{x^2}{2}}$ for

$\sqrt{\frac{2}{\pi}} < x$

$\frac{1}{2}e^{-\frac{x^2}{2}}$ is tighter than $\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{x^2}{2}}}{x}$ for

$0 \leq x < \sqrt{\frac{2}{\pi}}$ $(\simeq 0.7979)$ $\quad$ (← See Wozencraft Figure 2.36)

However, actually, both are inaccurate.

– Approximation for $Q(x)$

(i) erfc(x) can be represented by a Taylor series in terms of the **Hermite polynomial**

of degree (K-1) as

$$\frac{d^k}{dx^k}\,\mathrm{erfc}(x) = (-1)^k \frac{2}{\sqrt{\pi}} H_{k-1}(x)\, e^{-x^2}$$

(ii) Alternative expression for $Q(x)$ is given by

$$Q(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta \,. \quad (x \geq 0)$$

proper ~~[illegible]~~ integral

~~[illegible]~~ integral
improper

$$\left(= \frac{1}{\sqrt{2\pi}}\int_x^{\infty} \exp\left(-\frac{y^2}{2}\right) dy \right)$$

Numerical integration! required

(iii) n-th order approximation of $Q(x)$

$$Q(x) \simeq Q_n(x) = \frac{1}{\sqrt{2\pi}\,x}\, e\left(-\frac{x^2}{2}\right) \times \left\{ 1 + \sum_{k=1}^{n} (-1)^k \frac{1\cdot 3\cdot 5 \cdots (2k-1)}{x^{2k}} \right\}$$

for $x > 0$

- Property of $Q(\sqrt{2x})$ for $x > 0$.

(i) $Q'(\sqrt{2x}) < 0$ for $x > 0$ monotonically decreasing

(ii) $Q''(\sqrt{2x}) > 0$ for $x > 0$ convex cup ft

We can use Jensen's ineq.

(sol)

$$Q'(\sqrt{2x}) = -\frac{1}{\sqrt{2\pi}}\, e^{-x}\, \frac{1}{\sqrt{2x}}$$

$$Q''(\sqrt{2x}) = \frac{1}{\sqrt{2\pi}}\, e^{-x}\, \frac{1}{\sqrt{2x}}\left(1 + \frac{1}{2x}\right)$$

- $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

— When $\mu_X = \mu_Y = 0$

(i) Cartesian coordinate

$$f_{XY}(x, y) = \frac{1}{\sqrt{2\pi}^2 \sqrt{\det(\sigma^2 I)}} \exp\left(-\frac{1}{2} [x \ y] (\sigma^2 I)^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \right)$$

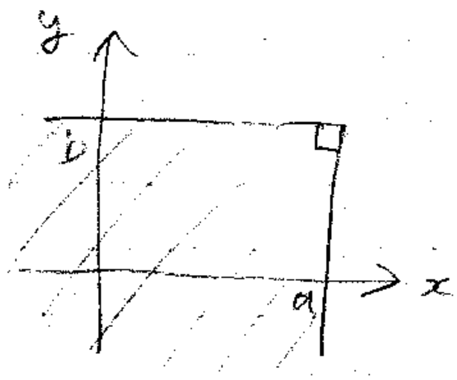
$$= \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2} \right)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)} \ \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right)}$$

$$= \quad f_X(x) \quad \times \quad f_Y(y)$$

$\Rightarrow$ X & Y are independent

/ex/ $\Pr(X \le a, Y \le b)$
$= \Pr(X \le a) \Pr(Y \le b)$
$= \left\{1 - Q\left(\frac{a}{\sigma}\right)\right\}\left\{1 - Q\left(\frac{b}{\sigma}\right)\right\}$



(ii) Polar coordinate

If we rotate by

$$\begin{bmatrix}\tilde{X}\\ \tilde{Y}\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}X\\ Y\end{bmatrix}$$

then

$$f_{\tilde{X}\tilde{Y}}(\tilde{x},\tilde{y}) = f_{XY}(\tilde{x}\cos\theta+\tilde{y}\sin\theta, -\tilde{x}\sin\theta+\tilde{y}\cos\theta)$$

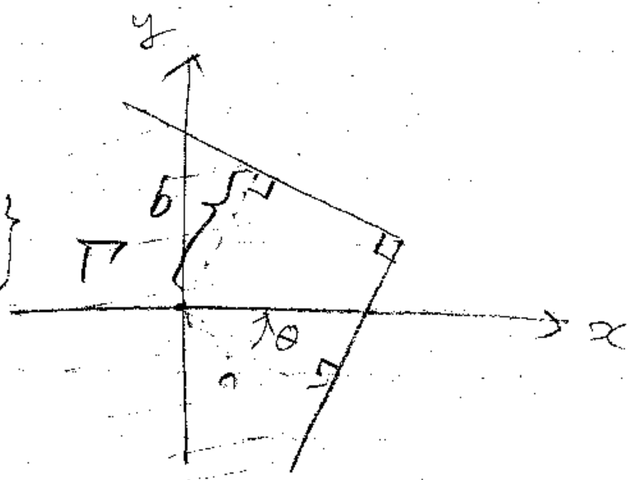
$$= \frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\{(\tilde{x}\cos\theta+\tilde{y}\sin\theta)^2 + (-\tilde{x}\sin\theta+\tilde{y}\cos\theta)^2\}\right)$$

$$= \frac{1}{2\pi\sigma^2}\exp\left(-\frac{\tilde{x}^2+\tilde{y}^2}{2\sigma^2}\right)$$

$$= f_{XY}(\tilde{x},\tilde{y}) \qquad \forall\theta\in[0,2\pi)$$

$\Rightarrow$ rotationally invariant distribution
$\equiv$ circularly symmetric distribution

/ex/ $\Pr((X,Y)\in\Gamma)$
$= \Pr(\tilde{X}\le a, \tilde{Y}\le b)$
$= \{1-Q(\frac{a}{\sigma})\}\{1-Q(\frac{b}{\sigma})\}$



In the polar coordinate,

$$f_{R,\Theta}(r,\theta) \triangleq f_{XY}(r\cos\theta, r\sin\theta)\, r$$

$$= \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right), \quad r\ge 0,\ 0\le\theta<2\pi$$

where $R = \sqrt{X^2+Y^2}$, $\Theta = \tan^{-1}\frac{Y}{X}$

$$= \underbrace{\frac{1}{2\pi}}\cdot\underbrace{\frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)}$$

$$= f_\Theta(\theta) \times f_R(r)$$

$\Rightarrow$ Θ and R are independent.

uniform random variable on $[0,2\pi)$

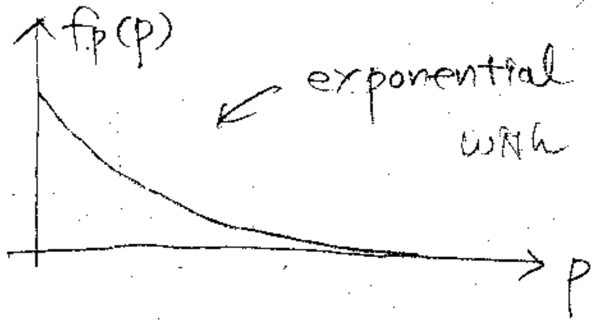
Rayleigh random variable w/ parameter σ^2

As we've seen $f_{R,\Theta}(r,\theta)$ is not a function of θ. (circularly symmetric!)

Let $P \triangleq R^2$, then

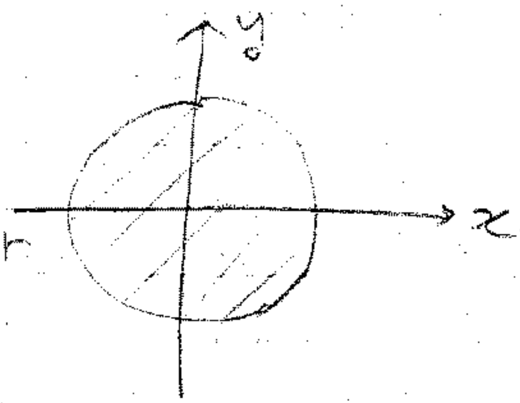
$$f_P(p) = \frac{1}{2\sqrt{p}} f_R(\sqrt{p})$$

$$= \frac{1}{2\sigma^2}\exp\left(-\frac{p}{2\sigma^2}\right), \quad p \geq 0$$



exponential random variable with parameter $\frac{1}{2\sigma^2}$

ex/
$$\Pr\left(\sqrt{X^2+Y^2} \le c\right) = \Pr(R \le c)$$
$$= \int_0^c \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) dr$$
$$= \left[-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right]_0^c$$
$$= 1 - \exp\left(-\frac{c^2}{2\sigma^2}\right).$$



Now, let's show

$$Q(x) = \int_0^{\frac{\pi}{2}} \frac{1}{\pi} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta, \quad x \ge 0$$

and

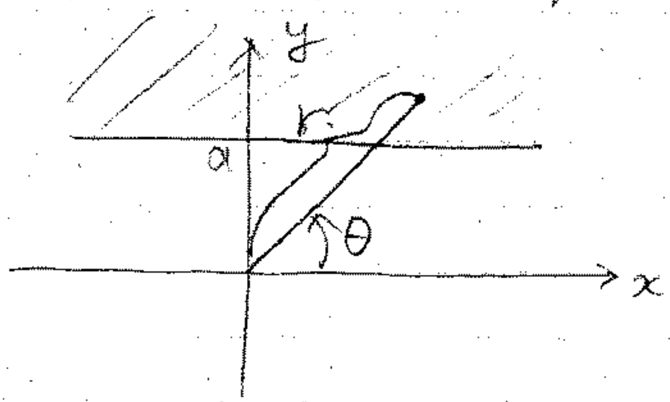
$$Q(x) \le \frac{1}{2} \exp\left(-\frac{x^2}{2}\right), \quad x \ge 0$$

Note that

$$Q(a) = \Pr(Y \ge a, -\infty < X < \infty)$$

where $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

Hence,



$$Q(a) = \Pr(R\sin\Theta \ge a, \ 0 < \Theta < \pi)$$

$$= \int_0^{\pi} \int_{\frac{a}{\sin\theta}}^{\infty} \frac{r}{2\pi} \exp\left(-\frac{r^2}{2}\right) dr\, d\theta$$

$$= \int_0^{\pi} \frac{1}{2\pi} \exp\left(-\frac{a^2}{2\sin^2\theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\pi} \exp\left(-\frac{a^2}{2\sin^2\theta}\right) d\theta$$

Since $\sin^2\theta \le 1$

$$\Rightarrow \frac{1}{\sin^2\theta} \ge 1$$

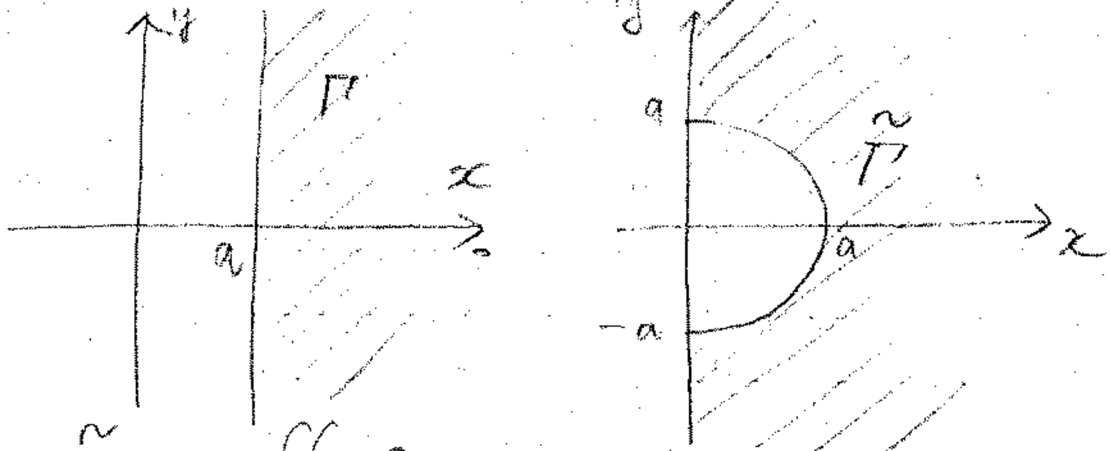
$$\Rightarrow -\frac{x^2}{2\sin^2\theta} \le -\frac{x^2}{2}$$

$$\Rightarrow \exp\left(-\frac{x^2}{2\sin^2\theta}\right) \le \exp\left(-\frac{x^2}{2}\right)$$

$$\Rightarrow Q(x) \le \int_0^{\frac{\pi}{2}} \frac{1}{\pi} \exp\left(-\frac{x^2}{2}\right), \quad x \ge 0$$

$$\Rightarrow Q(x) \le \frac{1}{2} \exp\left(-\frac{x^2}{2}\right), \quad x \ge 0$$

This can be shown as follows, too.



$$\Gamma \subset \tilde{\Gamma} \Rightarrow \iint_{\Gamma} f_{XY}(x,y)\,dx\,dy \le \iint_{\tilde{\Gamma}} f_{XY}(x,y)\,dx\,dy$$

$$\Rightarrow Q(x) \le \Pr\left(R \ge x, -\frac{\pi}{2} \le \Theta \le \frac{\pi}{2}\right)$$

$$= \frac{1}{2} \Pr(R \ge x)$$

$$= \frac{1}{2} \exp\left(-\frac{x^2}{2}\right), \quad x \ge 0$$

- When $(\mu_X, \mu_Y) \neq (0, 0)$

(i) Cartesian coordinate

$$f_{XY}(x,y) = \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(x-\mu_X)^2}{2\sigma^2}\right)}_{f_X(x)} \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(y-\mu_Y)^2}{2\sigma^2}\right)}_{f_Y(y)}$$

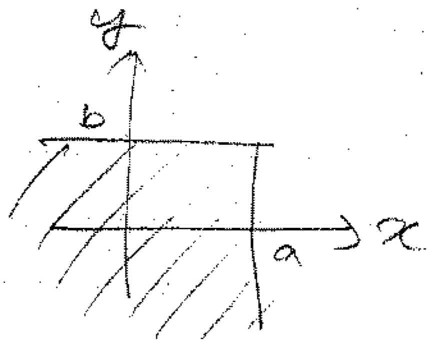
$$= f_X(x) \times f_Y(y)$$

$\Rightarrow$ X & Y are independent

/ex/ $\Pr(X \le a, Y \le b)$

$= \Pr(X \le a)\Pr(Y \le b)$

$= \left\{1 - Q\left(\frac{a-\mu_X}{\sigma}\right)\right\}\left\{1 - Q\left(\frac{b-\mu_Y}{\sigma}\right)\right\}$



(ii) Polar coordinate.

If we rotate with respect to $\begin{bmatrix}\mu_X \\ \mu_Y\end{bmatrix}$ by

$$\begin{bmatrix}\tilde{X} \\ \tilde{Y}\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}X-\mu_X \\ Y-\mu_Y\end{bmatrix} + \begin{bmatrix}\mu_X \\ \mu_Y\end{bmatrix}$$

then

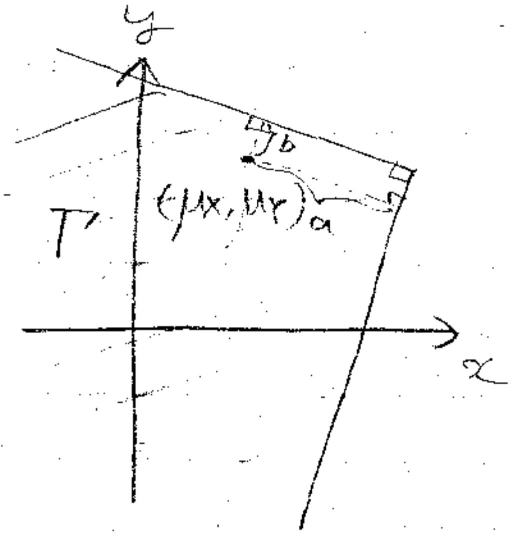
$$f_{\tilde{X}\tilde{Y}}(\tilde{x},\tilde{y}) = f_{XY}\big((\tilde{x}-\mu_X)\cos\theta + (\tilde{y}-\mu_Y)\sin\theta + \mu_X,\ -(\tilde{x}-\mu_X)\sin\theta + (\tilde{y}-\mu_Y)\cos\theta + \mu_Y\big)$$

$$\cdots = f_{XY}(\tilde{x},\tilde{y})$$

$\Rightarrow$ circularly symmetric with respect to $\begin{bmatrix}\mu_X \\ \mu_Y\end{bmatrix}$

/ex/ $\Pr((X,Y)\in\Gamma)$

$$=\left\{1-Q\left(\frac{a}{\sigma}\right)\right\}\left\{1-Q\left(\frac{b}{\sigma}\right)\right\}$$



In the polar coordinate,

$$f_{R,\Theta}(r,\theta) \triangleq f_{XY}(r\cos\theta, r\sin\theta)\,r$$

$$= \frac{r}{2\pi\sigma^2}\exp\left(-\frac{(r\cos\theta-\mu_X)^2+(r\sin\theta-\mu_Y)^2}{2\sigma^2}\right)$$

$$= \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2+\chi^2}{2\sigma^2}\right)\exp\left(\frac{r(\mu_X\cos\theta+\mu_Y s}{\sigma^2}\right.$$

where $\chi \triangleq \sqrt{\mu_X^2+\mu_Y^2}$

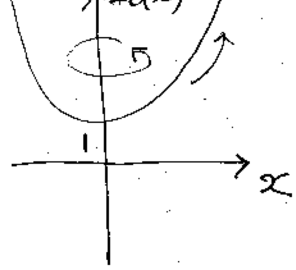
Since

$$f_R(r) = \int_0^{2\pi} f_{R\Theta}(r,\theta)\,d\theta$$

$$= \cdots = \frac{r}{\sigma^2}\exp\left(-\frac{r^2+\chi^2}{2\sigma^2}\right)$$

$$\times \underbrace{\frac{1}{2\pi}\int_0^{2\pi}\exp\left(\frac{r\chi}{\sigma^2}\cos(\theta+\alpha)\right)d\theta}_{\triangleq I_0\left(\frac{r\chi}{\sigma^2}\right)}$$

where $I_0(x) \triangleq \frac{1}{2\pi}\int_0^{2\pi}\exp(x\cos\theta)\,d\theta$



is called **the zero-th order modified Bessel function of the 1st kind.**

The distribution of R is a **Ricean** / **Rician** **distribution**

$$f_\Theta(\theta) = \frac{1}{2\pi} \exp\left(-\frac{\alpha^2}{2\sigma^2}\right) + \frac{\alpha\cos(\theta-\theta_0)}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-\alpha^2 \sin^2(\theta-\theta_0)}{2\sigma^2}\right) \times \left(1 - Q\left(\frac{A_0\cos(\theta-\theta_0)}{\sigma}\right)\right)$$

where $\alpha = \sqrt{\mu_x^2 + \mu_y^2}$

$\theta = \tan^{-1}\left(\frac{\mu_y}{\mu_x}\right)$

$$\lim_{\frac{\alpha}{\sigma}\to\infty} f_\Theta(\theta) = \delta(\theta-\theta_0)$$

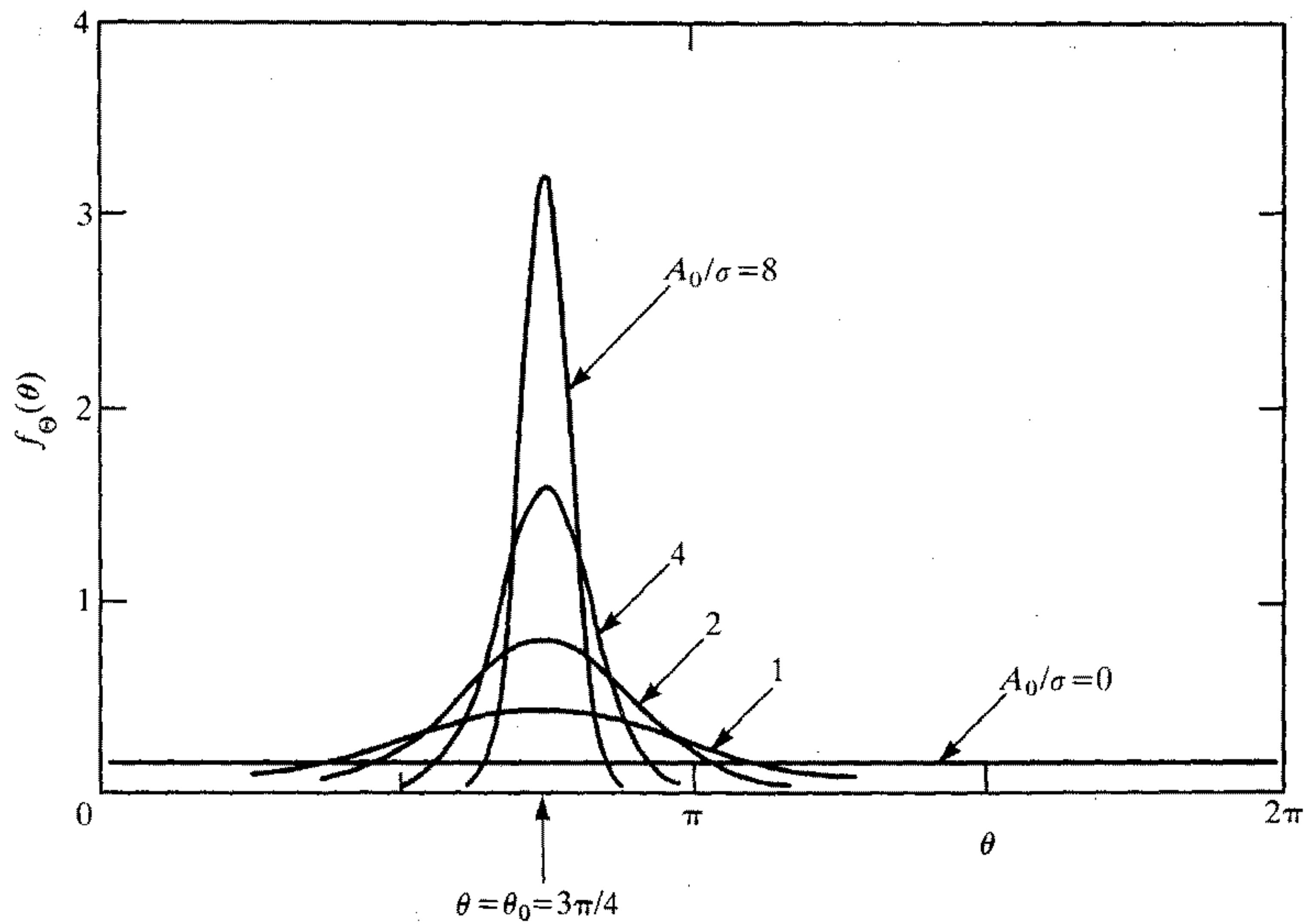


FIGURE 10.6-2
Probability density function of the phase of the sum of a sinusoidal signal and gaussian noise. Curves are plotted for a signal phase of $\theta_0 = 3\pi/4$.

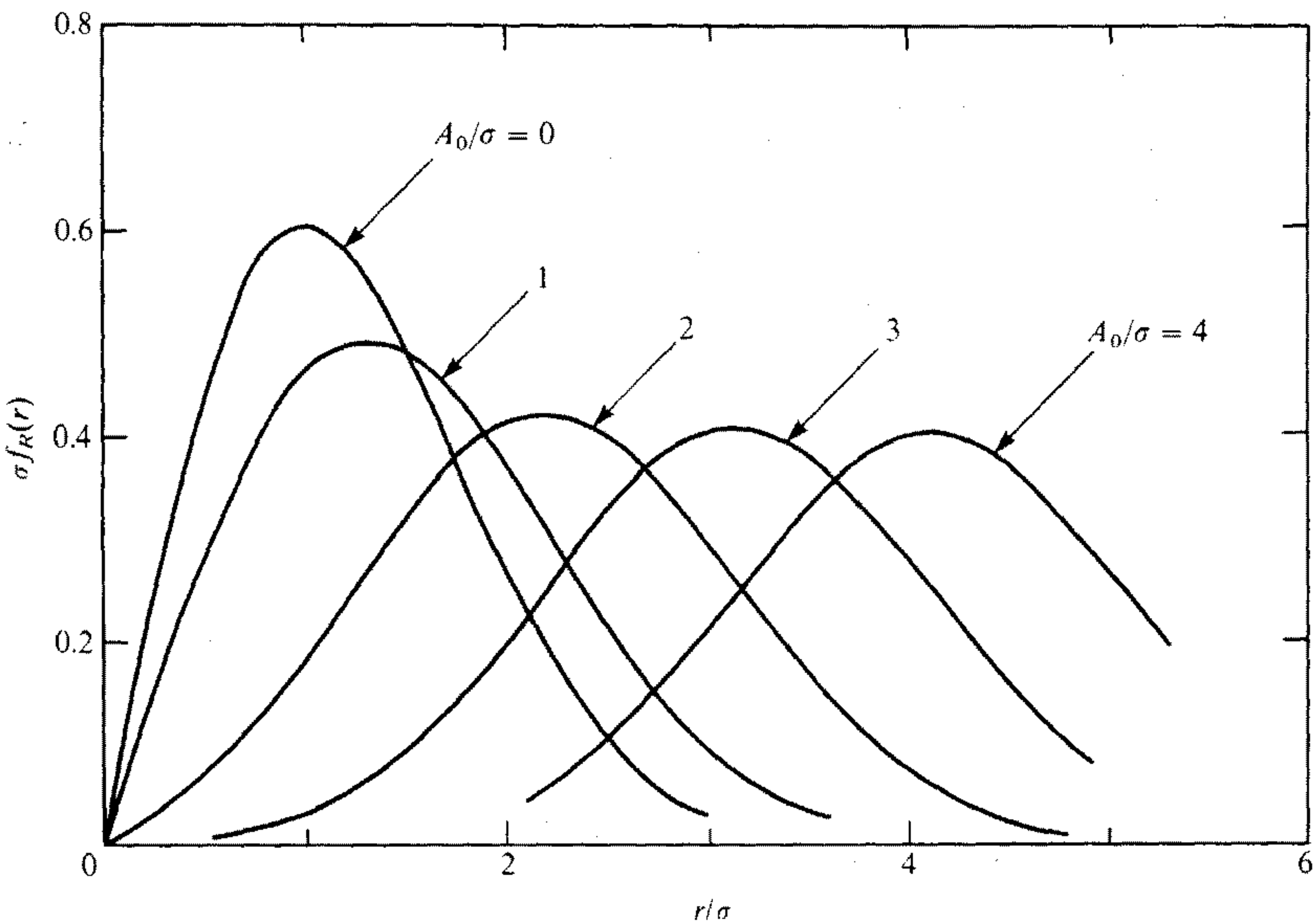


FIGURE 10.6-1
Probability densities of the envelope of a sinusoidal signal (amplitude A_0) plus noise (power σ^2) for various ratios A_0/σ.

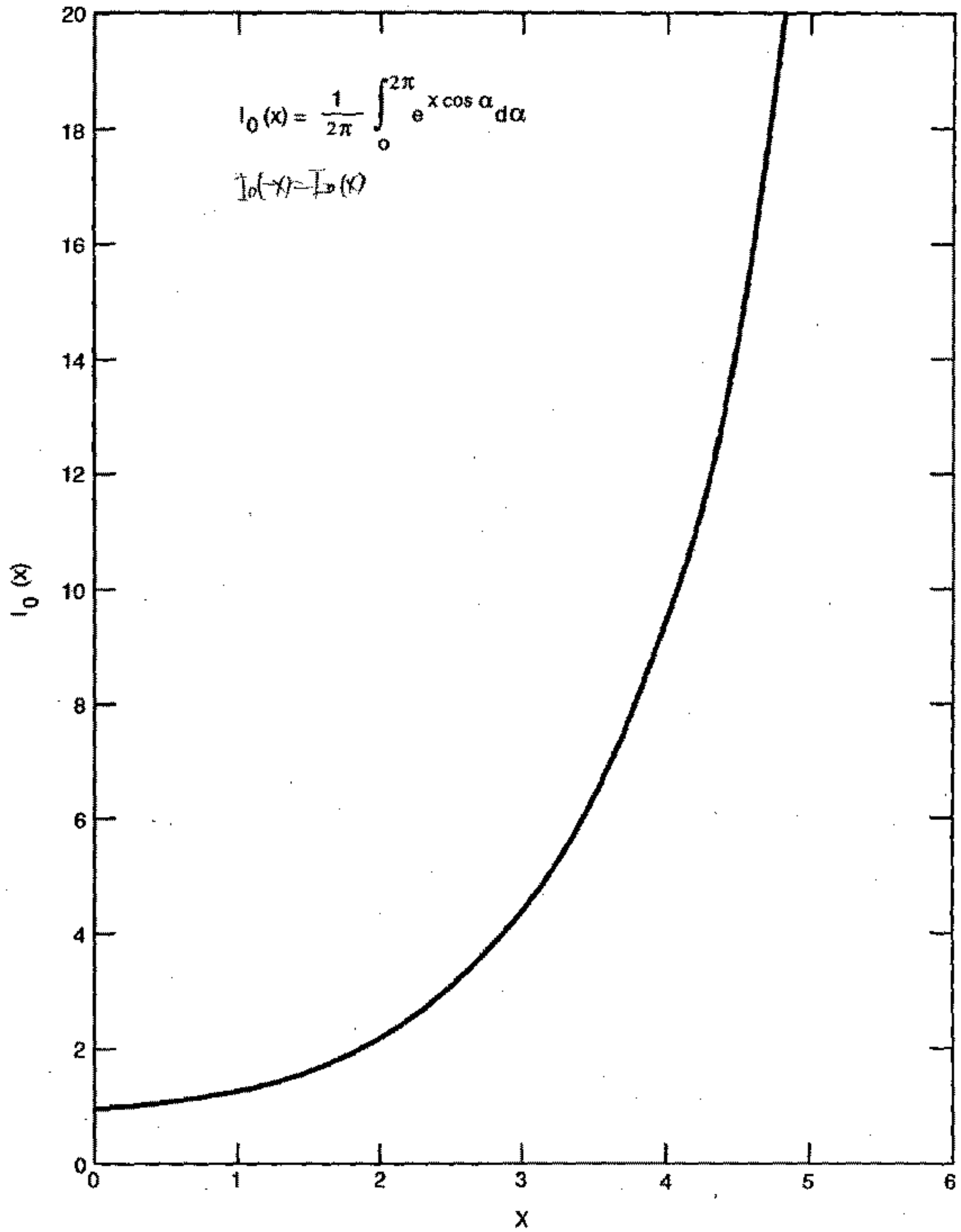


Figure 5.2 Plot of $I_0(x)$

Hence, using (5.22) in (5.20), the optimum decision rule sets $\widehat{m}(\rho(t)) = m_k$ when

$$I_0\left(\frac{2\xi_i}{N_0}\right)\exp\left(-\frac{E_i}{N_0}\right) \tag{5.23}$$

is maximum for $k = i$. Recall from (5.17) that the envelope ξ_i is given by

$$\xi_i = \sqrt{z_{ci}^2 + z_{si}^2} \tag{5.24}$$

○ 2-D case (cont.)

Def. When $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

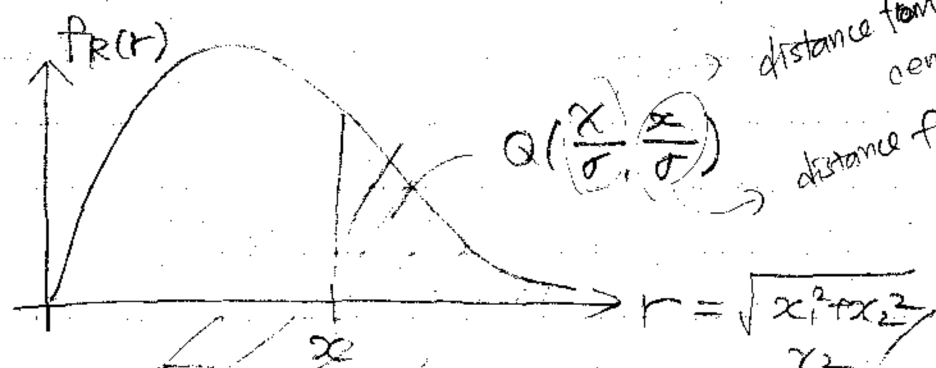
$$Q\left(\frac{\chi}{\sigma}, \frac{x}{\sigma}\right) \triangleq \Pr\left(\sqrt{X_1^2 + X_2^2} \geq x\right)$$

where $\chi \triangleq \sqrt{\mu_1^2 + \mu_2^2}$.

↑ Chi /kai/

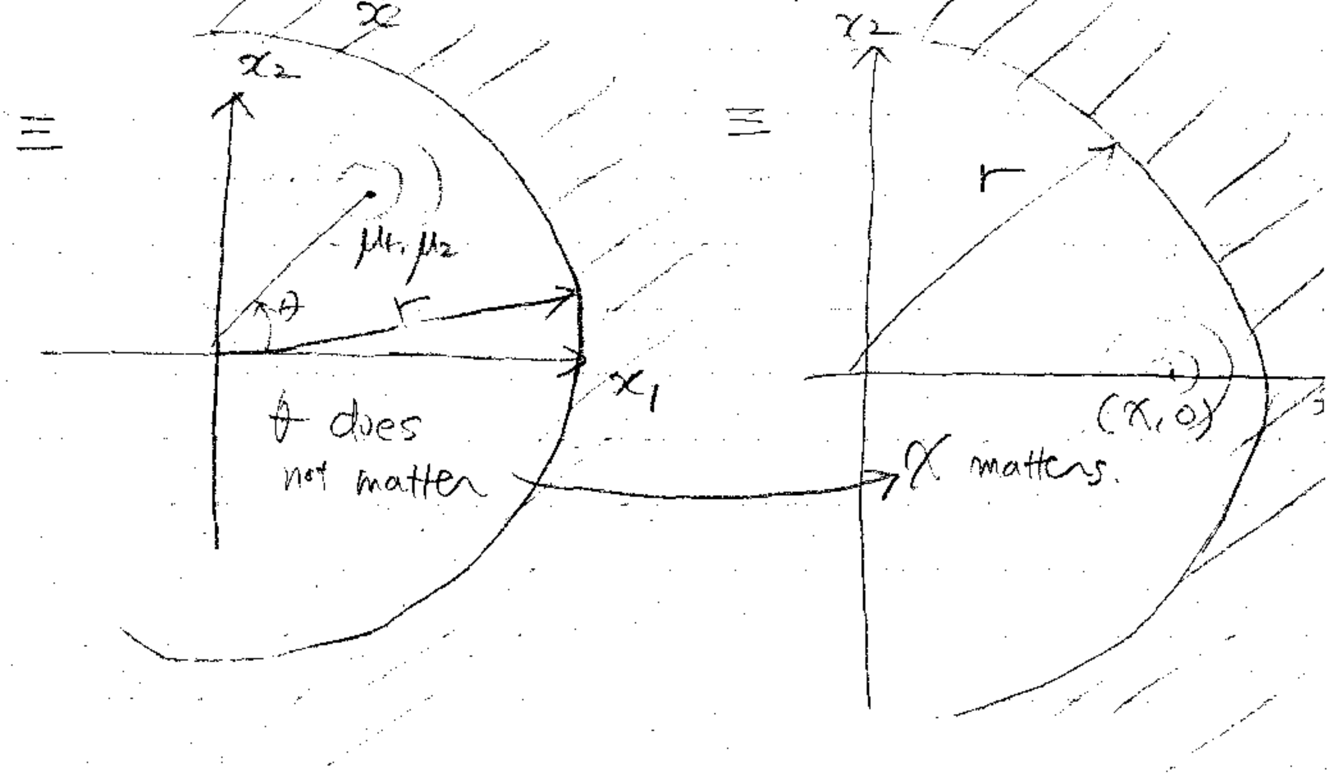
$Q(a, b)$ is called **Marcum's Q-function**

cf) $Q\left(\frac{x - \mu_1}{\sigma}\right) \triangleq \Pr(X_1 \geq x)$



→ distance from origin to the center of density

→ distance from origin to the boundary



○ 2m-D case

· The **generalized Marcum's Q-function**

If $Y = \sqrt{\sum_{i=1}^{2m} X_i^2}$, then

$$F_Y(y) = 1 - Q_m\left(\frac{x}{\sigma}, \frac{y}{\sigma}\right)$$

where

$$Q_m(a,b) = Q_1(a,b) + e^{\frac{a^2+b^2}{2}} \sum_{k=1}^{m-1} \left(\frac{b}{a}\right)^k I_k(ab)$$

w/

$$Q_1(a,b) = e^{-\frac{a^2+b^2}{2}} \sum_{k=0}^{\infty} \left(\frac{a}{b}\right)^k I_k(ab), \quad (b > a > 0)$$

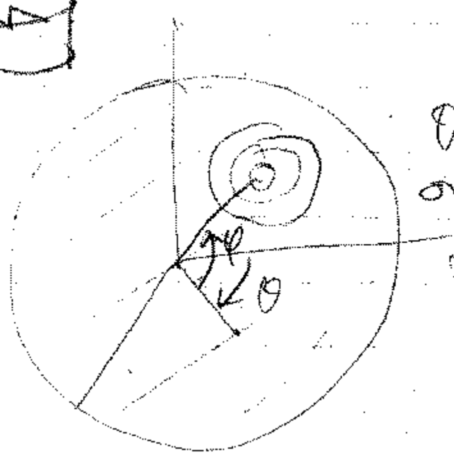
· $Y = \sum_{i=1}^{n} X_i^2$ where X_i's are independent w/ σ^2.

$\equiv$

Given $f_X(x)$, find $f_Y(y)$ w/ $Y = X^2$

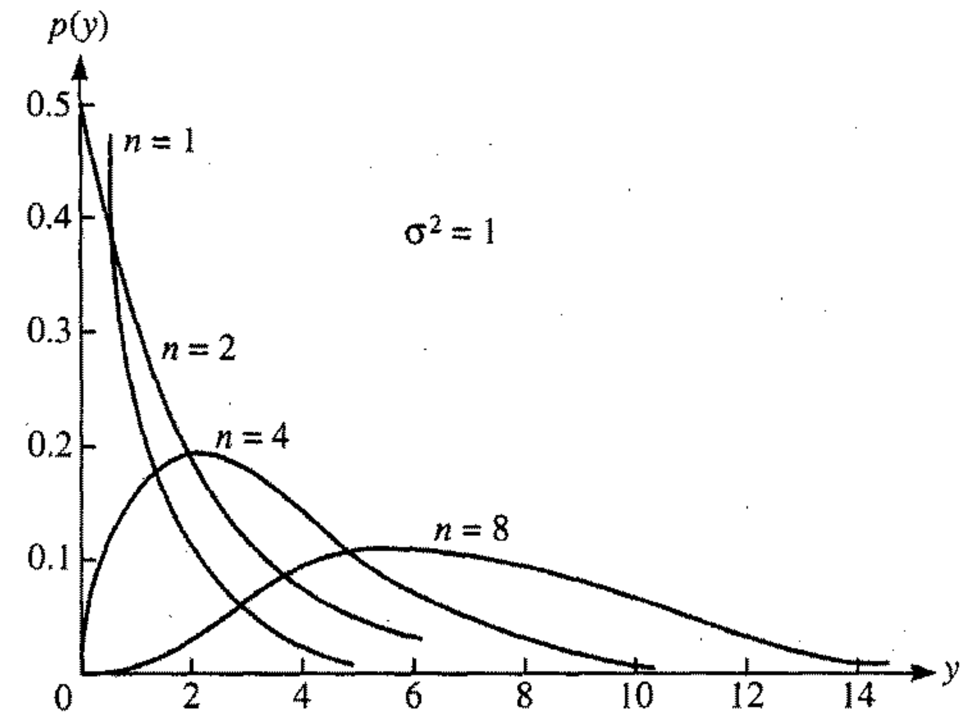
$\because$ $Y = \sqrt{\sum_{i=1}^{n} X_i^2}$ learned already! for some n.

.. named!



θ, φ does not matter
only the distance to
the center (mean)
matters.

n=2 exponential



central

The pdf of a chi-square-distributed random variable for several degrees of freedom.

gamma) pdf with n degrees of freedom. It is illustrated in Fig. 2-1-9. The case $n = 2$ yields the exponential distribution.

The first two moments of Y are

$$E(Y) = n\sigma^2$$

○ n-D case

- $Y = \sum_{i=1}^{n} X_i^2$ where X_i's are i.i.d. $N(0, \sigma^2)$

The **central chi-square distribution**

χ

$$P_Y(y) = \frac{1}{\sqrt{2\sigma^2}^{\,n}\,\Gamma(\frac{n}{2})}\, y^{\frac{n}{2}-1} e^{-\frac{y}{2\sigma^2}}, \quad y \ge 0$$

a **chi-square** (or **gamma**) **pdf with n degrees of freedom.**

In particular, $n=2$ yield an exponential density. (← Figure. Proakis 3rd. P43)

- $Y = \sum_{i=1}^{n} X_i^2$ where X_i's are independent and $X_i \sim N(\mu_i, \sigma^2)$

Let

$\chi = \sqrt{\mu_1^2 + \mu_2^2 + \cdots + \mu_n^2}$, then

$$P_Y(y) = \frac{1}{2\sigma^2}\left(\frac{y}{\chi^2}\right)^{\frac{n-2}{4}} e^{-\frac{y+\chi^2}{2\sigma^2}}\, I_{\frac{n}{2}-1}\left(\frac{\sqrt{y}\,\chi}{\sigma^2}\right)$$

where

$$I_\alpha(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{\alpha+2k}}{k!\,\Gamma(\alpha+k+1)}, \quad x \ge 0$$

if $\alpha = 0, 1, 2, \cdots$ $= \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta} \cos(n\theta)\, d\theta$

the **noncentral chi-square pdf w/ n degrees of freedom**

χ^2 the **noncentrality parameter of the distribution**

$$E\{Y\} = n\sigma^2 + \alpha^2$$
$$E\{Y^2\} = 2n\sigma^4 + 4\sigma^2\alpha^2 + (n\sigma^2+\alpha^2)^2$$
$$E\{Y^2\} - E\{Y\}^2 = 2n\sigma^4 + 4\sigma^2 s^2$$

- Note, for all forms of $Y = \sqrt{\sum X_i^2}$ or $Y = \sum X_i^2$ w/ independent X_i's with the same variance, $f_Y(y)$ is a function of $\alpha^2 = \sum_{i=1}^{n} \mu_i^2$!