Proper-complex Gaussian Distributions: Part II - Complex case

○ Complex-valued Gaussian random vector

• Def

Suppose that $\underline{x} \in \mathbb{R}^N$ and $\underline{y} \in \mathbb{R}^N$ are real-valued random vectors. Then,

$$\underline{z} \triangleq \underline{x} + j\underline{y} \qquad \in \mathbb{C}^N$$

is called **a complex-valued random vector** with N-dimension if $\underline{x}$ and $\underline{y}$ are jointly distributed.

"if" in definition $\equiv$ iff.

• Remark

(i) A complex-valued random vector $\underline{z} \in \mathbb{C}^N$ is completely characterized by the joint probability distribution function of $\mathrm{Re}\{\underline{z}\}$ and $\mathrm{Im}\{\underline{z}\}$.

(ii) A complex- 〃 〃 〃 〃
〃 〃 〃 〃 the joint characteristic function of $\mathrm{Re}\{\underline{z}\}$ and $\mathrm{Im}\{\underline{z}\}$

(iii) A complex-valued random vector with N-dimension is equivalent to a real-valued random vector with 2N-dimension.

- Def

A random vector $\underline{z} \triangleq \underline{x} + j\underline{y} \in \mathbb{C}^N$ is **a complex-valued Gaussian random vector** if the real-valued random vectors $\underline{x} \in \mathbb{R}^N$ and $\underline{y} \in \mathbb{R}^N$ are jointly Gaussian

- Theorem

A complex-valued Gaussian random vector $\underline{z} \in \mathbb{C}^N$ is completely characterized by the **mean** vector $\underline{\mu}_z \triangleq E[\underline{z}]$, the **covariance** matrix $Cov\{\underline{z}\} \triangleq E\{(\underline{z}-\underline{\mu}_z)(\underline{z}-\underline{\mu}_z)^H\}$, and the **pseudo-covariance** matrix $\widetilde{Cov}\{\underline{z}\} \triangleq E\{(\underline{z}-\underline{\mu}_z)(\underline{z}-\underline{\mu}_z)^T\}$.

/proof/

We know that $\underline{z}$ is completely characterized by the characteristic function $\varphi_{\underline{w}}(\underline{u})$ of

$$\underline{w} \triangleq \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} \triangleq \begin{bmatrix} Re\{\underline{z}\} \\ Im\{\underline{z}\} \end{bmatrix} \in \mathbb{R}^{2N}$$

and that $\varphi_{\underline{w}}(\underline{u})$ is determined by $E\{\underline{w}\}$ & $Cov\{\underline{w}\}$.

Note that

$$Cov\{\underline{z}\} = E\left[\{(\underline{x}-\underline{\mu}_x)+j(\underline{y}-\underline{\mu}_y)\}\{(\underline{x}-\underline{\mu}_x)^T - j(\underline{y}-\underline{\mu}_y)^T\}\right]$$
$$= (Cov\{\underline{x}\} + Cov\{\underline{y}\})$$
$$+ j(-Cov\{\underline{x},\underline{y}\} + Cov\{\underline{y},\underline{x}\})$$

and that

$$\widetilde{Cov}\{\underline{Z}\} = E[\{(\underline{X}-\underline{\mu_x})+j(\underline{Y}-\underline{\mu_y})\}\{(\underline{X}-\underline{\mu_x})+j(\underline{Y}-\underline{\mu_y})\}^T]$$
$$= (Cov\{\underline{X}\} - Cov\{\underline{Y}\}) + j(Cov\{\underline{X},\underline{Y}\} + Cov\{\underline{Y},\underline{X}\})$$

Since $Cov\{\underline{X}\} = \frac{1}{2}\{Re(Cov\{\underline{Z}\}) + Re(\widetilde{Cov}\{\underline{Z}\})\}$

$$Cov\{\underline{Y}\} = \frac{1}{2}\{Re(Cov\{\underline{Z}\}) - Re(\widetilde{Cov}\{\underline{Z}\})\}$$

$$Cov\{\underline{X},\underline{Y}\} = \frac{1}{2}\{-Im(Cov\{\underline{Z}\}) + Im\,\widetilde{Cov}\{\underline{Z}\}\}$$

and

$$Cov\{\underline{Y},\underline{X}\} = \frac{1}{2}\{Im(Cov\{\underline{Z}\}) + Im(\widetilde{Cov}\{\underline{Z}\})\}$$

$$Cov\{\underline{W}\} = \begin{bmatrix} Cov\{\underline{X}\} & Cov\{\underline{X},\underline{Y}\} \\ Cov\{\underline{Y},\underline{X}\} & Cov\{\underline{Y}\} \end{bmatrix}$$

is determined by $Cov\{\underline{Z}\}$ and $\widetilde{Cov}\{\underline{Z}\}$

Also,

$E\{\underline{W}\} = \begin{bmatrix} E\{\underline{X}\} \\ E\{\underline{Y}\} \end{bmatrix} = \begin{bmatrix} Re\{\underline{\mu_z}\} \\ Im\{\underline{\mu_z}\} \end{bmatrix}$ is determined by $E\{\underline{Z}\} = \underline{\mu_z}$.

Q.E.D.

- Notation

$$\underline{Z} \sim N^c(\underline{\mu}; C, \tilde{C}) \quad \text{or} \quad \underline{Z} \sim N_c(\underline{\mu}; C, \tilde{C})$$

$$\underline{Z} \sim \tilde{N}(\underline{\mu}; C, \tilde{C})$$

mean cov pseudo-cov.

- Def

$\underline{z} \sim \tilde{N}(\underline{\mu}; C, \tilde{C})$ is called **proper** if the pseudo-covariance vanishes, i.e., $\tilde{C} = 0$

- $\underline{z} \sim CN(\underline{\mu}, C)$

- Theorem w/o proof

If $\underline{z} \sim \tilde{N}(\underline{\mu}; C, 0)$ has $C > 0$, then

$f_{\underline{X},\underline{Y}}(\underline{x}, \underline{y})$ where $\underline{z} \triangleq \underline{X} + j\underline{Y}$

$$= \frac{1}{\pi^N \det(C)} \exp\left(-(\underline{z}-\underline{\mu})^H C^{-1} (\underline{z}-\underline{\mu})\right)$$

$$\triangleq f_{\underline{z}}(\underline{z})$$

- Example

real positive

$z \sim \tilde{N}(\mu; c, 0) \in \mathbb{C}^1$ and $c > 0$.

Then

$Cov\{X\} = \frac{c}{2}$, $Cov\{XY\} = 0$

$Cov\{Y\} = \frac{c}{2}$,

$$\Rightarrow f_{XY}(x, y) = \frac{1}{\sqrt{2\pi}\sqrt{\frac{c}{2}}} \exp\left(-\frac{(x - Re\{\mu\})^2}{2 \cdot \frac{c}{2}}\right)$$

$$\times \frac{1}{\sqrt{2\pi}\sqrt{\frac{c}{2}}} \exp\left(-\frac{(y - Im\{\mu\})^2}{2 \cdot \frac{c}{2}}\right)$$

$$= \frac{1}{\pi \cdot c} \exp\left(-(z-\mu)^* c^{-1} (z-\mu)\right)$$

$$\triangleq f_z(z)$$

- Def.

$$z \sim \tilde{N}(\underline{\mu}; C, \tilde{C})$$

is called **proper** if the pseudo-covariance vanishes, i.e., $\tilde{C} = 0$

- Theorem

If $z \sim N^c(\underline{\mu}; C, 0)$ with $C (= C^H) > 0$, $\mu \in \mathbb{C}^N$

then

$$= f_{x,y}(\underline{x}, \underline{y})$$

$$=$$

$$= \frac{1}{\pi^N \det C} \exp\left(-(\underline{z}-\underline{\mu})^H C^{-1} (\underline{z}-\underline{\mu})\right)$$

/proof/

(i) $N=1$

Suppose that $x \sim \tilde{N}(\mu; c, 0)$ and

$x \triangleq x_c + j x_s \quad \mu \triangleq \mu_c + j \mu_s$

Then

$$E\{(x-\mu)^2\} = E([\mathrm{Re}(x-\mu)]^2) - E([\mathrm{Im}(x-\mu)]^2) + j2E[\mathrm{Re}(x-\mu)\mathrm{Im}(x-\mu)]$$

$$= \mathrm{Cov}\{x_c\} - \mathrm{Cov}\{x_s\} + j2\mathrm{Cov}\{x_c, x_s\}$$

$$= 0$$

$$\Rightarrow \mathrm{Cov}\{x_c\} = \mathrm{Cov}\{x_s\}, \quad \mathrm{Cov}\{x_c, x_s\} = 0 \quad \cdots *$$

$$E\{|x-\mu|^2\} = \mathrm{Cov}\{x_c\} + \mathrm{Cov}\{x_s\} = c \quad \cdots **$$

$$\therefore \mathrm{Cov}\{x_c\} = \mathrm{Cov}\{x_s\} = \frac{c}{2}, \quad \mathrm{Cov}\{x_1, x_2\} = 0$$

Hence,

$$f_{X_c X_s}(x_c, x_s) = \frac{1}{\sqrt{2\pi\,\sigma_{X_c}^2}} \exp\left(-\frac{(x_c-\mu_c)^2}{2\sigma_{X_c}^2}\right)$$

$$\times \frac{1}{\sqrt{2\pi\,\sigma_{X_s}^2}} \exp\left(-\frac{(x_s-\mu_s)^2}{2\sigma_{X_s}^2}\right)$$

$$= \frac{1}{\pi C} \exp\left(-\frac{|x-\mu|^2}{C}\right)$$

$$= f_X(x), \qquad x \in \mathbb{C}.$$

If $\mu = 0$, then proper 1-D complex
$\equiv$ circularly symmetric 2-D
real.

(ii) $N \geq 2$

Suppose that $\underline{X} \triangleq \underline{X}_c + j\underline{X}_s$, $\underline{\mu} \triangleq \underline{\mu}_c + j\underline{\mu}_s$

$$\underline{Y} = \begin{bmatrix} \underline{X}_c \\ \underline{X}_s \end{bmatrix}$$

Then,

$$\mathrm{Cov}\{\underline{Y}\} = \begin{bmatrix} \mathrm{Cov}\{\underline{X}_c\} & \mathrm{Cov}\{\underline{X}_c, \underline{X}_s\} \\ \mathrm{Cov}\{\underline{X}_s, \underline{X}_c\} & \mathrm{Cov}\{\underline{X}_s\} \end{bmatrix}$$

$$\triangleq \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

From $\tilde{C} = 0 = \left(\mathrm{Cov}\{\underline{X}_c\} - \mathrm{Cov}\{\underline{X}_s\}\right)$
$+ j\,\mathrm{Cov}\{(\underline{X}_c, \underline{X}_s) + \mathrm{Cov}(\underline{X}_s, \underline{X}_c)\}$,

we have $A = C$, $B = -B^T$, and

$$C = 2(A - jB)$$

② $n=N$ $\quad t_1, t_2 \cdots, t_N$

- Let $\underline{\tilde{X}} = \begin{bmatrix} X_c(t_1) \\ X_c(t_2) \\ \vdots \\ X_c(t_N) \\ X_s(t_1) \\ \vdots \\ X_s(t_N) \end{bmatrix} = \begin{bmatrix} \underline{X_c} \\ \underline{X_s} \end{bmatrix}$, then the mean and covariance function are given by

$$E[\tilde{X}] = 0$$

$$Cov[\tilde{X}] = E\begin{bmatrix} \underline{X_c}\,\underline{X_c}^T & \underline{X_c}\,\underline{X_s}^T \\ \underline{X_s}\,\underline{X_c}^T & \underline{X_s}^T\,\underline{X_s}^T \end{bmatrix}$$, respectively

where

$$\{E[X_c X_c^T]\}_{ij} = R_{X_cX_c}(t_j - t_i) \triangleq A \quad \text{symmetric}$$

$$\{E[\underline{X_s}\,\underline{X_s}^T]\}_{ij} = R_{X_sX_s}(t_j - t_i) = R_{X_cX_c}(t_j - t_i) = A$$

$$\{E[\underline{X_c}\,\underline{X_s}^T]\}_{ij} = R_{X_cX_s}(t_j - t_i) \triangleq B \quad \text{skew symmetric}$$

$$\{E[X_s X_c^T]\}_{ij} = R_{X_sX_c}(t_j - t_i) = -R_{X_cX_s}(t_j - t_i) = -B = B^T$$

$$\therefore Cov[\tilde{X}] = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

- ~~Let~~ $\underline{X} = \underline{X_c} + jX_s$

~~If~~ Gaussian$_x$ and A is invertible then the density function is given

Let R a partitioned matrix

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Then, the inverse is (if exists)

$$R^{-1} = \begin{bmatrix} (A-BD^{-1}C)^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix}$$

Verify it! $RR^{-1} = I$

O.K!

In particular, $R = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ w/ $\left(\begin{array}{l} B^T = -B \\ A^T = A \end{array} \right.$

then

$$R^{-1} = \begin{bmatrix} (A+BA^{-1}B)^{-1} & -A^{-1}B(A+BA^{-1}B)^{-1} \\ +A^{-1}B(A+BA^{-1}B)^{-1} & (A+BA^{-1}B)^{-1} \end{bmatrix}$$

• Inverse of a complex matrix $\Lambda = 2(A - jB)$

w/ $A^T = A$, $B^T = -B$

Hermitian

$x^H A x$ = real!

$x^H A^H x$

$= x^H A x$

A^*

$$\Lambda^{-1} = \frac{1}{2}(A+BA^{-1}B)^{-1}(I + jBA^{-1})$$

Note

$$(\Lambda^{-1})^H = \frac{1}{2}(I + jA^{-1}B)(A+BA^{-1}B)^{-1} = \Lambda^{-1}$$

Verify it!

$$\Lambda\Lambda^{-1} = (A-jB)(I+jA^{-1}B)(A+BA^{-1}B)^{-1}$$

$$= (A+BA^{-1}B)(A+BA^{-1}B)^{-1} = I$$

- Verify that

$$\frac{1}{2}\begin{pmatrix} \underline{x}_c \\ \underline{x}_s \end{pmatrix}^T \begin{pmatrix} A & B \\ -B^T & A \end{pmatrix}^{-1} \begin{pmatrix} \underline{x}_c \\ \underline{x}_s \end{pmatrix} = \underline{x}^H \, E(\underline{X}\underline{X}^H)^{-1} \, \underline{x}$$

- 2nd we want to show

$$\cancel{(2\pi)}^N \sqrt{\det\begin{pmatrix} A & B \\ B^T & A \end{pmatrix}} = \cancel{\pi^N \det} \; (\cancel{2}(A-jB))$$

$$\Leftrightarrow 2^{2N} \det\begin{pmatrix} A & B \\ B^T & A \end{pmatrix} = \left(\det(2(A-jB))\right)^2 = \left(\det \Lambda\right)^2$$

RHS

$$\Lambda^T = 2(A-jB^T) = 2(A+jB) = 2\,(I+jBA^{-1})A$$

Hence,

$$\Lambda^{-1} = \frac{1}{4}\,(A+BA^{-1}B)^{-1}\,\Lambda^T A^{-1}$$

Then,

$$\det(\Lambda^{-1}) = \frac{1}{2^N}\left(\det\,(A+BA^{-1}B)\right)^{-1} \det(\Lambda)\left(\det(A)\right)^{-1}$$

$$\therefore \cancel{\det \Lambda = 4^N \det(A+BA^{-1}B)\,(\det\Lambda)^{-1}\det(A)}$$

$$\Rightarrow \left(\det \Lambda\right)^2 = 4^N \det(A+BA^{-1}B)\det(A)$$

LHS. (... see Lancaster.)

- Def

  $\underline{z} \sim \tilde{N}(\underline{\mu}; C, \tilde{C})$ is called a **white Gaussian random vector w/ mean μ** if $\tilde{C} = 0$ and there exists σ^2 such that

$$C = 2\sigma^2 I$$

Note that
$$\mathrm{Cov}(\mathrm{Re}\{z_i\}\mathrm{Re}\{z_j\}) = \sigma^2 \delta_{ij}$$
$$\mathrm{Cov}(\mathrm{Re}\{z_i\}\mathrm{Im}\{z_j\}) = 0$$
$$\mathrm{Cov}(\mathrm{Im}\{z_i\}\mathrm{Im}\{z_j\}) = \sigma^2 \delta_{ij}$$

every real, imaginary parts are i.i.d.

- **Properties** of proper Gaussian random vectors

  (i) $E[\underline{z}]$ and $\mathrm{Cov}\{\underline{z}\}$ completely characterize the distribution. (Already proved.)

  (ii) An affine transform of $\underline{z}$, e.g.,
  $$\underline{w} = A\underline{z} + \underline{b} \qquad A \in \mathbb{C}^{M\times N}, \quad \underline{b} \in \mathbb{C}^{M\times 1}$$
  is **also proper** Gaussian with mean $E[\underline{w}] = A\underline{\mu}_z + \underline{b}$ and Covariance $\mathrm{Cov}[\underline{w}] = A C_{zz} A^H$.

  (iii) Any **linear combination** of z_i's ($w = \underline{a}^H \underline{z}$) becomes a circularly symmetric (w.r.t. $\underline{a}^H \underline{\mu}_z$) complex Gaussian random variable

  i.e.,
  $$E\{\mathrm{Re}[w]\} = \mathrm{Re}[\underline{a}^H \underline{\mu}_z]$$
  $$E\{\mathrm{Im}[w]\} = \mathrm{Im}[\underline{a}^H \underline{\mu}_z]$$
  $$\mathrm{Cov}[\mathrm{Re}\{w\}\mathrm{Im}\{w\}] = 0$$
  $$\mathrm{Cov}[\mathrm{Re}\{w\}] = \mathrm{Cov}[\mathrm{Im}\{w\}].$$

white

∘ Examples (proper Gaussian)

(i) When $z \sim \tilde{N}(0; 2\sigma^2, 0)$ the joint density of X and Y $(z = x + jY)$ is given by

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$$

(ii) In (i), $\Pr(\mathrm{Re}\{z\} \geq a) = Q\left(\frac{a}{\sigma}\right)$

(iii) In (i), $\Pr(|z| \geq b) = Q\left(0, \frac{b}{\sigma}\right)$

(iv) When $\underline{z} \sim \tilde{N}(\underline{\mu}; 2\sigma^2 I, 0)$, $\underline{z} \in \mathbb{C}^N$

$$\begin{aligned}
&\Pr(\|\underline{z} - \underline{\mu}\| \geq \|\underline{z} - \underline{c}\|) \\
&= \Pr(\|\underline{z} - \underline{\mu}\|^2 \geq \|\underline{z} - \underline{c}\|^2) \\
&= \Pr(\|\underline{z}\|^2 - 2\mathrm{Re}(\underline{\mu}^H \underline{z}) + \|\underline{\mu}\|^2 \\
&\qquad \geq \|\underline{z}\|^2 - 2\mathrm{Re}(\underline{c}^H \underline{z}) + \|\underline{c}\|^2) \\
&= \Pr\left(\mathrm{Re}\{(\underline{c} - \underline{\mu})^H \underline{z}\} \geq \frac{\|\underline{c}\|^2 - \|\underline{\mu}\|^2}{2}\right) \\
&= \Pr\left(\mathrm{Re}\{w\} \geq \frac{\|\underline{c}\|^2 - \|\underline{\mu}\|^2}{2}\right)
\end{aligned}$$

Note that the complex Gaussian random variable $w \triangleq (\underline{c} - \underline{\mu})^H \underline{z}$ have

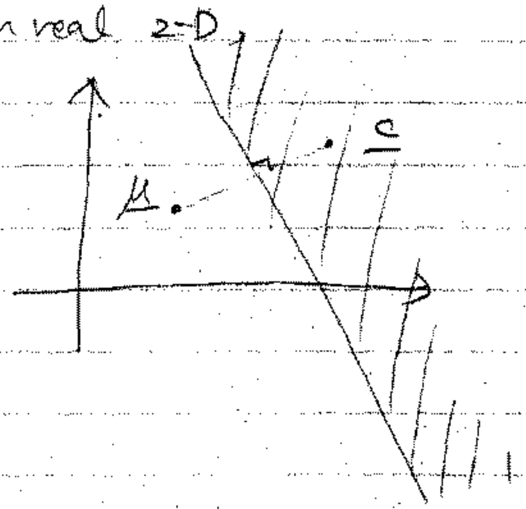
$$E[w] = (\underline{c} - \underline{\mu})^H \underline{\mu}$$

$$\mathrm{Cov}[w] = (\underline{c} - \underline{\mu})^H (2\sigma^2 I)(\underline{c} - \underline{\mu})$$

$$= Q\left(\frac{\frac{\|\underline{c}\|^2 - \|\underline{\mu}\|^2}{2} - (\mathrm{Re}(\underline{c}^H \underline{\mu}) - \|\underline{\mu}\|^2)}{\sigma \|\underline{c} - \underline{\mu}\|}\right) \qquad \frac{\|\underline{c}\|^2 - 2\mathrm{Re}(\underline{c}^H \underline{\mu}) + \|\underline{\mu}\|^2}{2}$$

$$= Q\left(\frac{\frac{\|\underline{c} - \underline{\mu}\|^2}{2}}{\sigma \|\underline{c} - \underline{\mu}\|}\right) = Q\left(\frac{\|\underline{c} - \underline{\mu}\|}{2\sigma}\right)$$

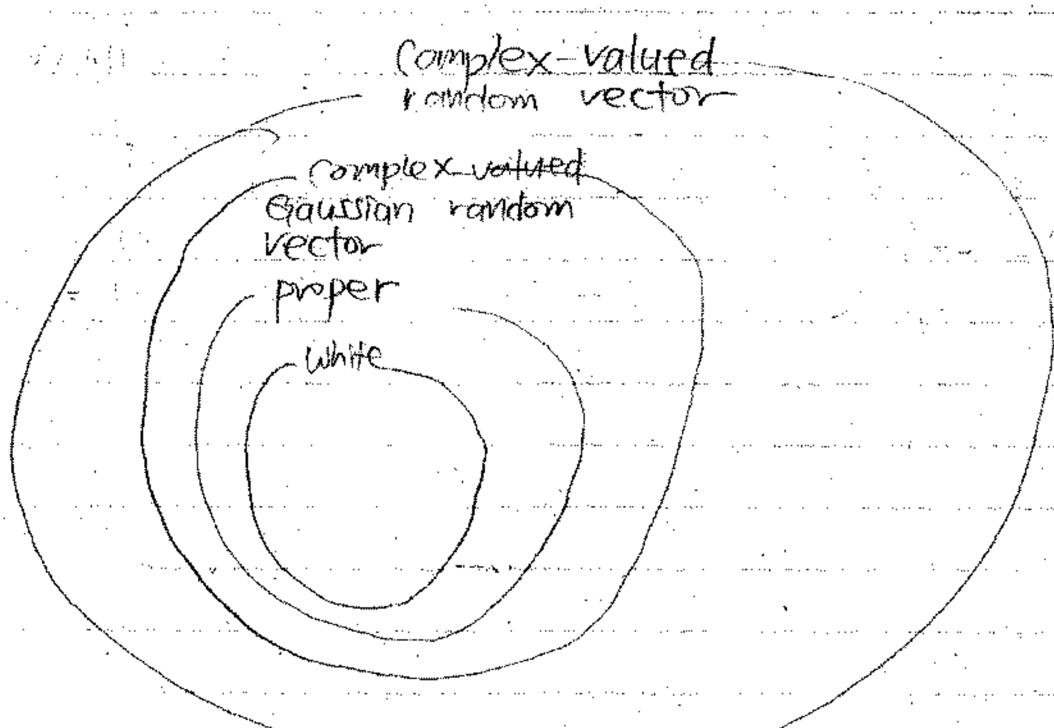
cf) In real 2-D,



(v) In (iv), $Pr(\|\underline{z}\| \geq c)$

$$= Q_N\left(\frac{\|\underline{\mu}\|}{\sigma}, \frac{c}{\sigma}\right)$$

•



Baek Soo's problem 9/23/02

반드시 0이어야

Q When $\underline{z} \sim \tilde{N}(\underline{\mu}; C, 0) \in \mathbb{C}^N$

find

$$\Pr(\|\underline{z}-\underline{\mu}\| \geq \|\underline{z}-\underline{p}\|)$$

sol

$$\Pr(\|\underline{z}-\underline{\mu}\|^2 \geq \|\underline{z}-\underline{p}\|^2)$$

$$= \Pr\Big(\mathrm{Re}\{\underbrace{(\underline{p}-\underline{\mu})^H \underline{z}}_{\triangleq w}\} \geq \frac{\|\underline{p}\|^2 - \|\underline{\mu}\|^2}{2}$$

$$E\{w\} = (\underline{p}-\underline{\mu})^H \underline{\mu}$$

$$\mathrm{Cov}\{w\} = (\underline{p}-\underline{\mu})^H C (\underline{p}-\underline{\mu})$$

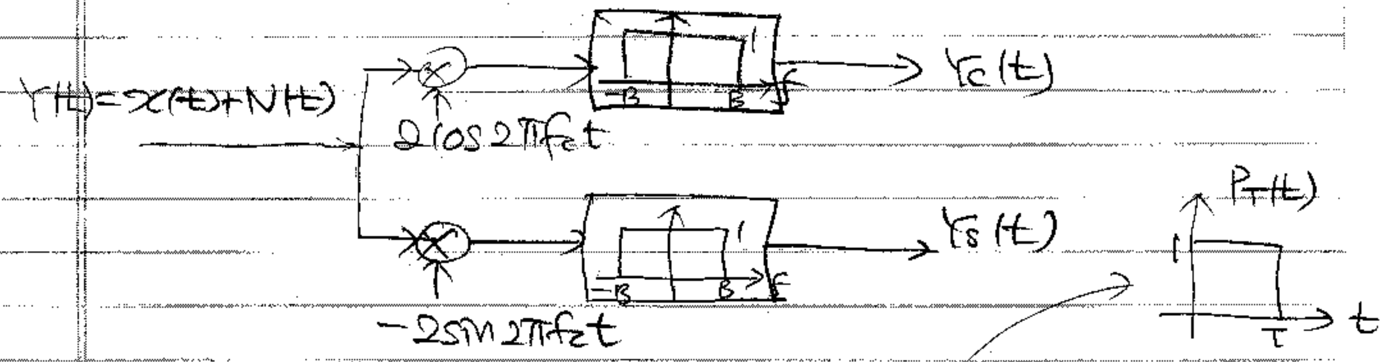
$$= \Pr\left(\mathrm{Re}\{w\} \geq \frac{\|p\|^2 - \|\underline{\mu}\|^2}{2} \right)$$

$$= Q\left(\frac{\frac{\|\underline{p}\|^2 - \|\underline{\mu}\|^2}{2} - 2\mathrm{Re}\{(\underline{p}-\underline{\mu})^H \underline{\mu}\}}{\sqrt{\frac{1}{2}(\underline{p}-\underline{\mu})^H C (\underline{p}-\underline{\mu})}} \right) \qquad \|p\|^2 - 2\mathrm{Re}(p^H \mu) + \|\mu\|^2$$

$$= Q\left(\frac{\|\underline{p}-\underline{\mu}\|^2}{\sqrt{2(\underline{p}-\underline{\mu})^H C (\underline{p}-\underline{\mu})}} \right)$$

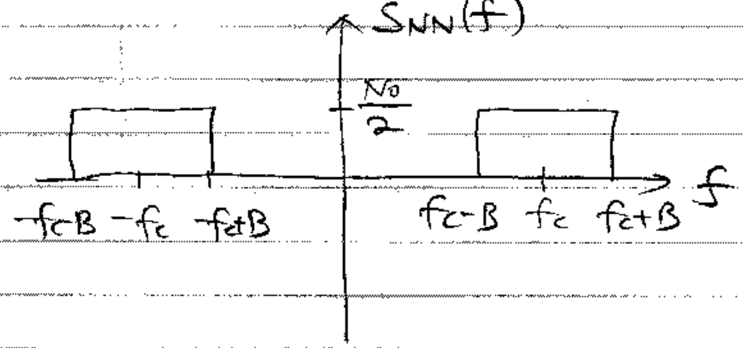
○ Example 1

- Consider the following quadrature demodulation.



where
$$x(t) = Re\left\{\sqrt{2P}\sum_{m=-\infty}^{\infty} b[m]\, p_T(t-mT)\, e^{j2\pi f_c t}\right\}$$

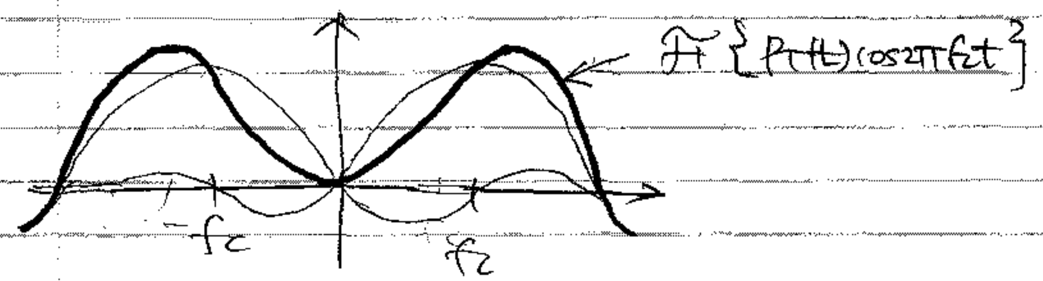
and $N(t)$ is a **bandlimited AWGN w/ two-sided PSD** $N_0/2$, center frequency f_c, and the BW $2B$



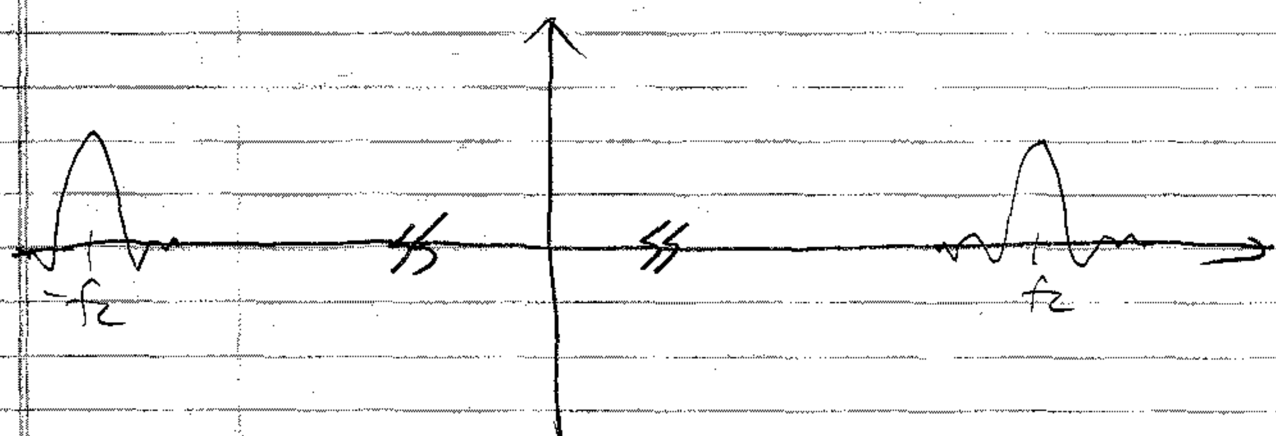
- We assume that the BW of the bandpass signal $x(t)$ is much smaller than the carrier frequency f_c. This assumption is called **the narrowband assumption**.

- Q1 What is the complex envelope of $x(t)$?

A) Since $P_T(t)$ has infinite support in frequency domain, $\sqrt{2P}\sum_{m=-\infty}^{\infty} b[m]\, p_T(t-mT)$ is not the complex envelope in the strict sense. In extreme, if $\frac{1}{T} \simeq f_c$, we have the spectrum of $p_T(t)\cos 2\pi f_c t$ given by



Fortunately, thanks to the narrowband assumption, $\frac{1}{T} \ll f_c$, which gives us the spectrum of $p_T(t)\cos 2\pi f_c t$ given by



We no longer have aliasing! Therefore, we can claim that

$$x_c(t) \approx \sqrt{2P} \sum_{m=-\infty}^{\infty} b[m]\, p_T(t-mT)$$

• Q2. Let $x_c(t)$ be the complex envelope of $x(t)$. Then, is it always true that

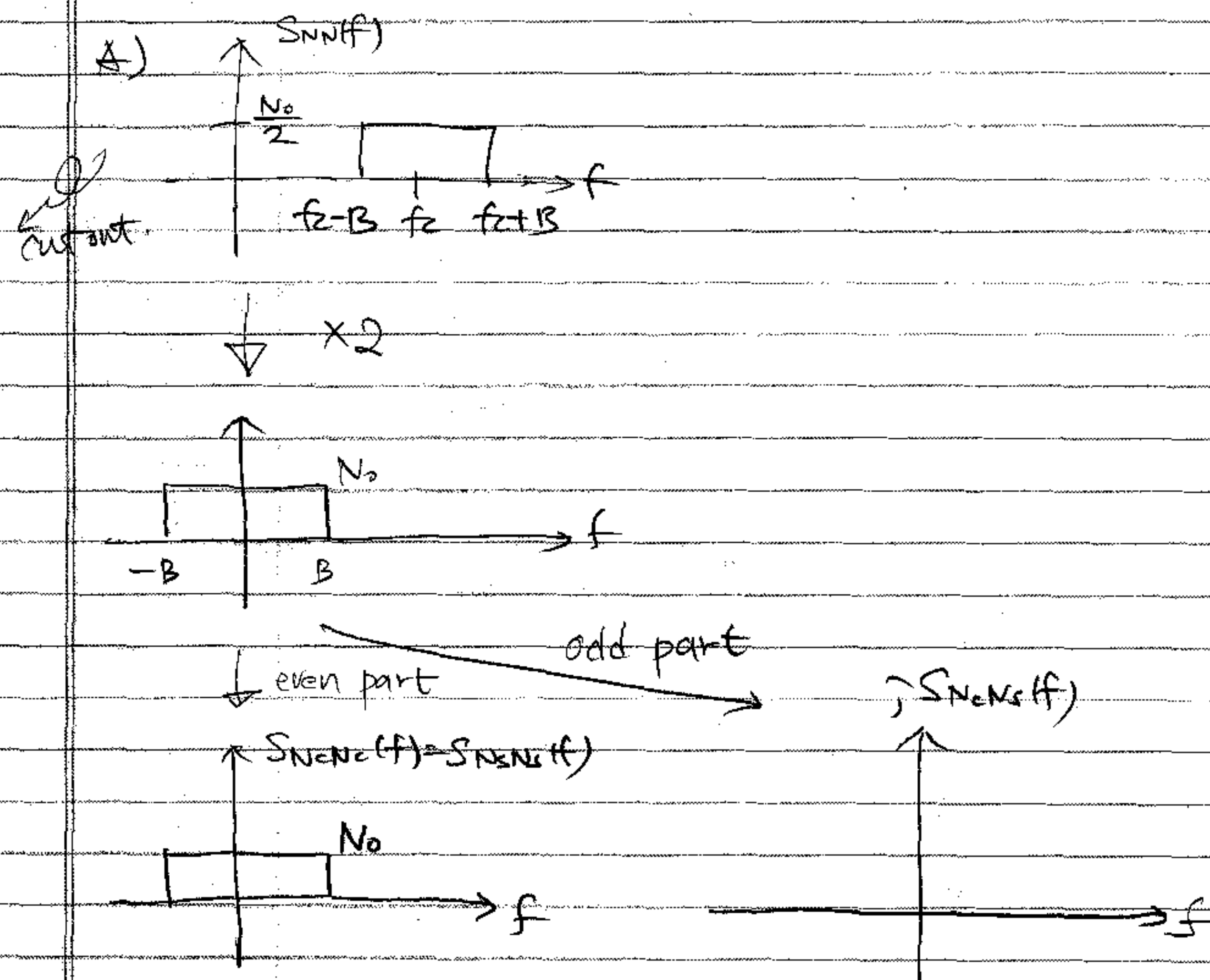
$$r_c(t) + j\, r_s(t) = x_c(t) \quad ?$$

A). No. We need in this case $B \gg \frac{1}{T}$. Otherwise, some sidelobes of $P_T(t)$ are filtered out by the ideal LPF w/ bandwidth B

- Q3 Let $r_c(t)+jr_s(t) = x_\ell(t)+N_\ell(t)$
$= x_\ell(t)+(N_c(t)+jN_s(t))$

Find $S_{N_cN_c}(f)$, $S_{N_sN_s}(f)$, and $jS_{N_cN_s}(f)$.

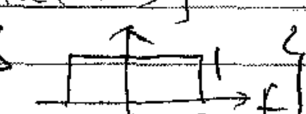
A)



- Q4. What is the autocorrelation function of $N_\ell(t)$?

A) $E\{N_\ell^*(t)N_\ell(t+\tau)\} = E\{(N_c(t)+jN_s(t))^*(N_c(t+\tau)+jN_s(t+\tau))\}$

$=2E\{N_c(t)N_c(t+\tau)\}$

$=2N_0\,\mathcal{F}^{-1}\{$  $\}$

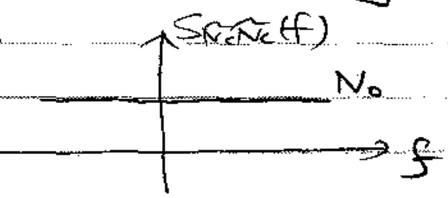
- When the narrowband assumption holds and the ideal LPF has large enough bandwidth, the **complex-baseband equivalent of the received signal** is

$$x_R(t) + \tilde{N}(t)$$

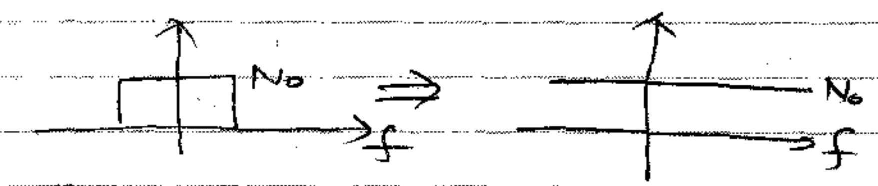
where $\tilde{N}(t)$ is a proper-complex AWGN with two-sided PSD $2N_0$.

Let $\tilde{N}(t) = \tilde{N}_c(t) + j\tilde{N}_s(t)$

Then,



The reason why



is because $\delta(\tau)$ is much easier to handle than $\mathrm{sinc}(\cdots)$.

○ Example 2

diffusive

- Consider a **specular multipath channel**

dispersive channel = frequency selective channel

$$h(t) = a\delta(t) + b\delta(t-\Delta).$$

with the channel input given by $x(t) = \mathrm{Re}\{\sqrt{2P}\sum_{m=-\infty}^{\infty} b[m]\, p_T(t-mT)\, e^{j(2\pi f_c t+\theta)}\}$.

- Q1. Find the channel output.

A) The channel output is

$$y(t) = h(t) * x(t) = a\,x(t) + b\,x(t-\Delta)$$

$$= \mathrm{Re}\left\{\sqrt{2P}\sum_{m=-\infty}^{\infty} b[m]\left(a\,p_T(t-mT) + b\,p_T(t-\Delta-mT)\right.\right.$$
$$\left.\left. e^{-j2\pi f_c\Delta}\right) e^{j(2\pi f_c t+\theta)}\right\}$$

- Q2. What is the complex envelope of $y(t)$?

A) From Q1,

$$y_\ell(t) = \sqrt{2P}\sum_{m=-\infty}^{\infty} b[m]\left(a\,p_T(t-mT) + b\,p_T(t-\Delta-mT)\,e^{-j2\pi f_c\Delta}\right)$$
$$\times\left(e^{j\theta}\right).$$

- It turns out that, effectively, $h(t)$ has the complex envelope

$$h_\ell(t) = \mathbf{2}\left(a\,\delta(t) + b\,\delta(t-\Delta)\,e^{-j2\pi f_c\Delta}\right)$$

because

$$y_\ell(t) = \frac{1}{2}x_\ell(t) * h_\ell(t)$$

The reason why we call this an **effective complex envelope** is that actually $h(t)$ does not have the complex envelope because it is not band-limited.

Of course, we can redefine $h(t)$ by bandpass filtering it with a BPF w/ center f_c and bandwidth B. However, this approach violates "simplicity" we pursue by using complex signals.

- Note that the specular multipath channel

$$h(t) = a\delta(t) + b\delta(t-\Delta)$$

introduces not only the delayed multipath in $y_\ell(t)$ but also the phase shift $e^{-j2\pi f_c \Delta}$, which is **dependent on** the center frequency f_c