Sampling

○ Sampling VS. Interpolation

CT → DT DT → CT

$x(t) \to x[n]$ $x[n] \to x(t)$

CT : continuous time
DT : discrete time

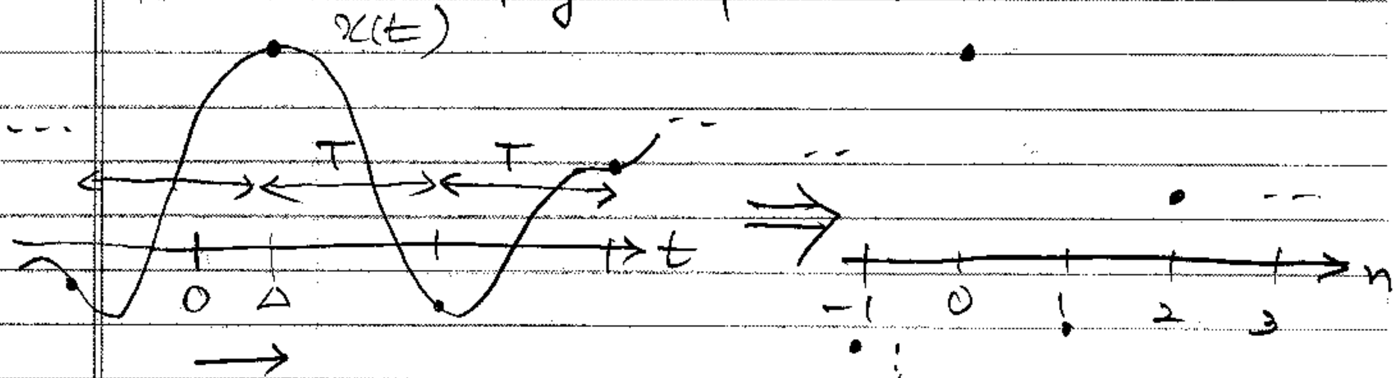
○ Two ways of understanding

Sampling ⨯ time domain
Interpolation ⨯ frequency domain

○ Uniform sampling understood in time domain

sampling < uniform sampling / non-uniform sampling

/ex/ uniform sampling w/ period T & offset Δ



offset

sampling period

sampling rate

$x[0] = x(\Delta + 0 \cdot T)$
$x[1] = x(\Delta + 1 \cdot T)$
$x[2] = x(\Delta + 2 \cdot T)$

○ Frequency-domain approach

- Fourier transform
  - continuous-time Fourier transform (CTFT)
  - discrete-time Fourier transform (DTFT)

- CTFT $$X_c(f) \triangleq \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

  CTIFT (inverse) $$x(t) \triangleq \int_{-\infty}^{\infty} X_c(f)\, e^{j2\pi f t}\, df$$

  The key assumption is $\int_{-\infty}^{\infty} e^{j2\pi f t}\, dt = \delta(f)$

- DTFT $$X_d(f) \triangleq \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi f n}$$

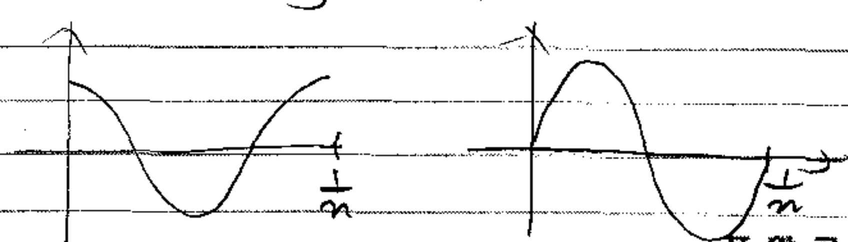
  DTIFT $$x[n] \triangleq \int_{-\frac{1}{2}}^{\frac{1}{2}} X_d(f)\, e^{j2\pi f n}\, df$$

  The key assumption is $\sum_{n=-\infty}^{\infty} e^{j2\pi f n} = \sum_{m=-\infty}^{\infty} \delta(f-m)$

- Property of DTFT

  $X_d(f)$ is periodic w/ period 1

  $\because$ $x[n]\underbrace{e^{-j2\pi f n}}_{\cos 2\pi f n \,-\, j \sin 2\pi f n}$ has period $\frac{1}{n}$ as a ft of f.



○ <u>Uniform sampling understood in frequency domain</u>

· <u>Just for simplicity, we consider uniform sampling with offset 0, i.e.,</u>

· $x[n] \triangleq x(nT)$. $\forall n$

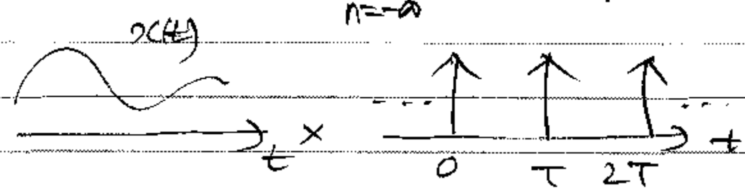
~~(Caution. Different offset results in different x[n] in general)~~

Q. Find **$X_d(f)$ in terms of $X_c(f)$.**

A.
$$\boxed{\begin{aligned} X_d(f) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(\frac{f}{T}-\frac{n}{T}\right) \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(\frac{f-n}{T}\right) \end{aligned}}$$

sol)

Consider $y(t) = x(t) \times \sum_{n=-\infty}^{\infty} \delta(t-nT)$



Note that the CTFT of the periodic signal $\sum_{n=-\infty}^{\infty} \delta(t-nT)$ is $\frac{1}{T}\sum_{n=-\infty}^{\infty} \delta\left(f-\frac{n}{T}\right)$

By definition,
$$\underbrace{FT}_{CTFT}\{y(t)\} \triangleq \int_{-\infty}^{\infty}\left(x(t)\times\sum_{n=-\infty}^{\infty}\delta(t-nT)\right)e^{-j2\pi ft}\,dt$$

$$\overset{why?}{=} \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)\,e^{-j2\pi nT}\,dt$$

why?

$$= \sum_{n=-\infty}^{\infty} x(nT)\, e^{-j2\pi f n T}$$

$$= X_d(fT) \qquad \cdots (*)$$

By modulation property,

$$\mathcal{F}\{y(t)\} = \mathcal{F}\{x(t)\} * \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t-nT)\right\}$$

$$= X_c(f) * \frac{1}{T}\sum_{n=-\infty}^{\infty} \delta\left(f-\frac{n}{T}\right)$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(f-\frac{n}{T}\right) \qquad \cdots (**)$$

From $(*)$ and $(**)$, we have

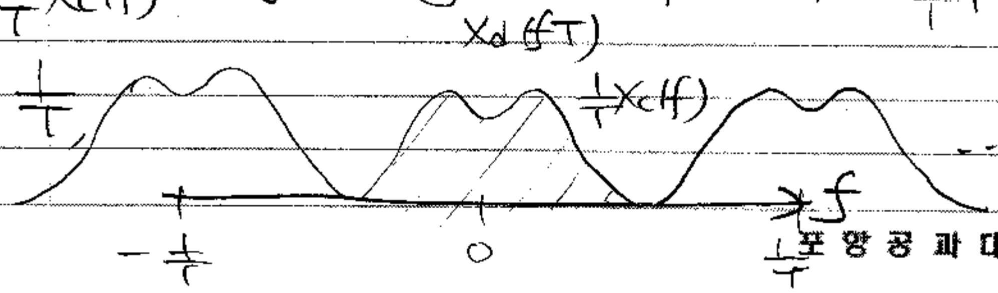
$$\boxed{X_d(fT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(f-\frac{n}{T}\right)} \qquad \cdots (***)$$

$$\Leftrightarrow \quad X_d(f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(\frac{f}{T}-\frac{n}{T}\right)$$

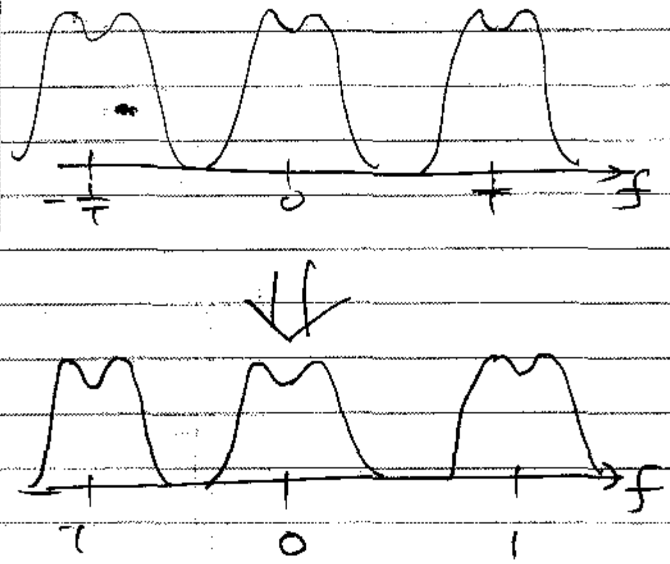
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(\frac{f-n}{T}\right)$$

- Interpretation of the result $(***)$

(i) $X_d(fT)$ consist of frequency shifted versions of $\frac{1}{T}X_c(f)$ by integer multiples of sampling rate $\frac{1}{T}$



(ii) Since we want $X_d(f)$, we just need to scale $X_d(fT)$



Alternate proof

$$X_d(f) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi f n}$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, e^{-j2\pi f n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} X_c(\tilde{f}) e^{j2\pi \tilde{f} nT} d\tilde{f} \right) e^{-j2\pi f n}$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} X_c\left(\frac{\hat{f}}{T}\right) \sum_{n=-\infty}^{\infty} e^{j2\pi(\hat{f}-f)n}\, d\hat{f}$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} X_c\left(\frac{\hat{f}}{T}\right) \sum_{m=-\infty}^{\infty} \delta(\hat{f}-f-m)\, d\hat{f}$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} X_c\left(\frac{f+m}{T}\right) \delta(\hat{f}-f-m)\, d\hat{f}$$

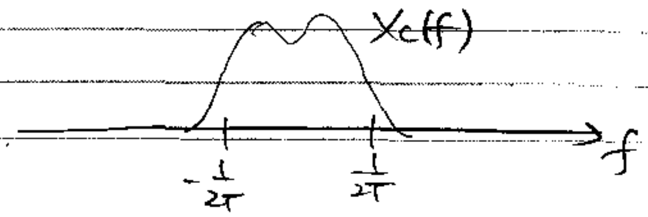
$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\left(\frac{f+m}{T}\right)$$

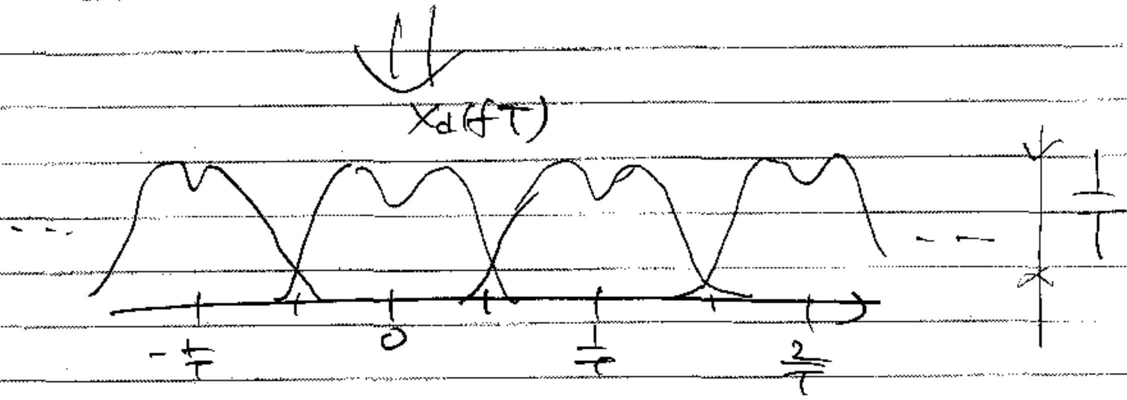
Aliasing

- If twice the bandwidth of $x(t)$ is less than or equal to the sampling rate $\frac{1}{T}$, then there is no overlapping in constructing $X_d(fT)$.

  ∴ Nyquist's minimum sampling rate for zero aliasing = 2 times the bandwidth of $x(t)$

(ex)





- <u>Anti-aliasing filter</u>

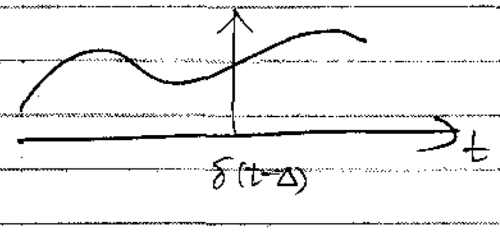
When a uniform sampling w/ rate $1/T$ is performed, after the CT signal is first low-pass filtered with bandwidth less than or equal to $\frac{1}{2T}$.

Such a LPF is called an **anti-aliasing filter**

Sampler Design

- Mathematically speaking, sampling a CT signal $x(t)$ at $t=\Delta$ can be performed as

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-\Delta)\,dt = x(\Delta)$$



However, it is impossible to implement such a system.

- Fortunately, for the sampling of <u>bandlimited signals</u>, we can be assisted from math formulas.

- Suppose that $x(t)$ is bandlimited w/ BW B. Then,

$$x\left(\frac{n}{2B}\right) = x(t)\Big|_{t=\frac{n}{2B}}$$

$$= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \Big|_{t=\frac{n}{2B}}$$

$$= \int_{-B}^{B} X(f) e^{j2\pi f t} df \Big|_{t=\frac{n}{2B}}$$

$$= \int_{-\infty}^{\infty} X(f) \; e^{j2\pi f \frac{n}{2B}} \left(u(f+B) - u(f-B)\right) df$$

$$= \int_{-\infty}^{\infty} x(t) \; \underbrace{\mathcal{F}^{-1}\left\{\left(u(f+B) - u(f-B)\right) e^{-j2\pi f \frac{n}{2B}}\right\}^*}_{} dt$$

$$\int_{-B}^{B} e^{j2\pi f\left(t-\frac{n}{2B}\right)} df$$

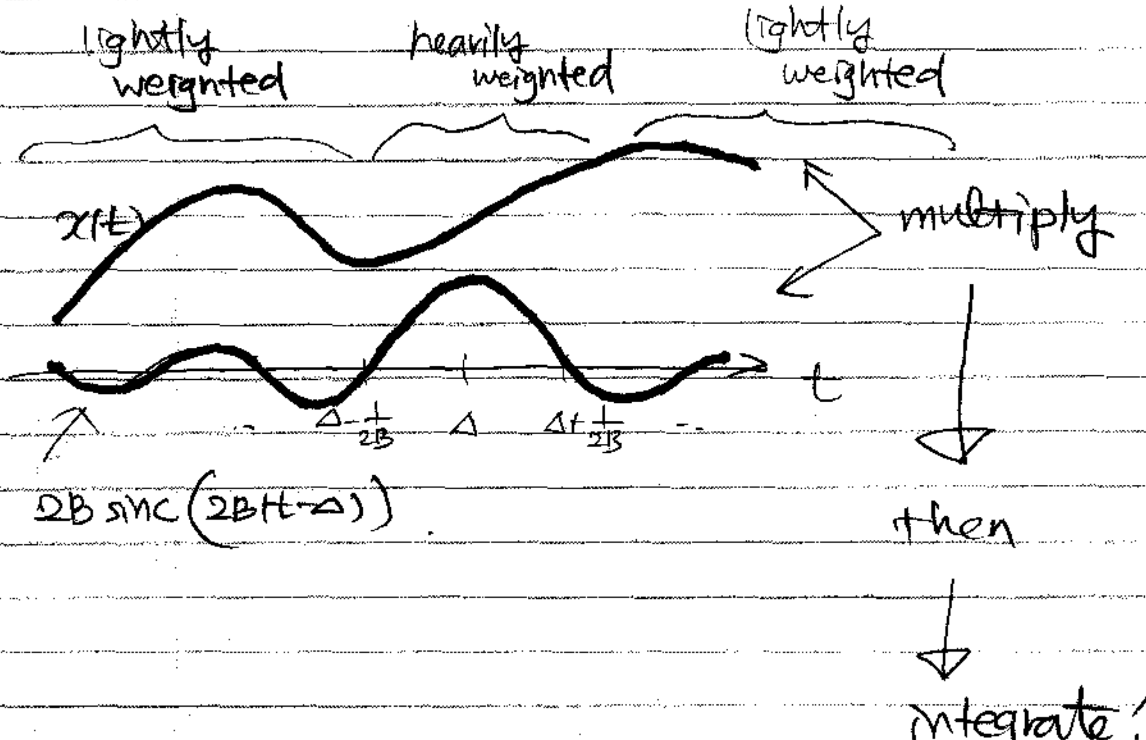
$$= \frac{e^{j2\pi f\left(t-\frac{n}{2B}\right)}}{2\pi\left(t-\frac{n}{2B}\right)}\Bigg|_{-B}^{B} = \frac{e^{j2\pi B\left(t-\frac{n}{2B}\right)} - e^{-j2\pi B\left(t-\frac{n}{2B}\right)}}{j2\pi\left(t-\frac{n}{2B}\right)}$$

$$= \frac{\sin 2\pi B\left(t-\frac{n}{2B}\right)}{\pi\left(t-\frac{n}{2B}\right)}$$

$$= 2B \operatorname{sinc}\left(2B\left(t-\frac{n}{2B}\right)\right)$$

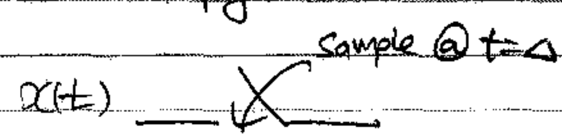
$$= 2B\int_{-\infty}^{\infty} x(t) \operatorname{sinc}\left(2B\left(t-\frac{n}{2B}\right)\right) dt$$

$$\therefore \; x(\Delta) = 2B\int_{-\infty}^{\infty} x(t) \operatorname{sinc}\left(2B(t-\Delta)\right) dt$$

- 

Thus, a sampler can be implemented by a correlator.

- The idea is simply understood by



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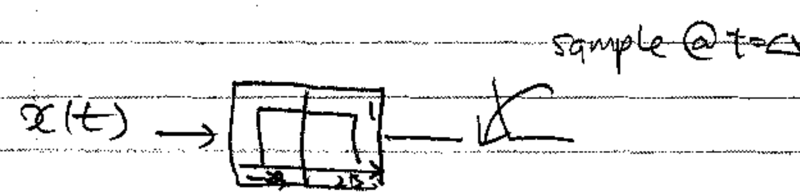

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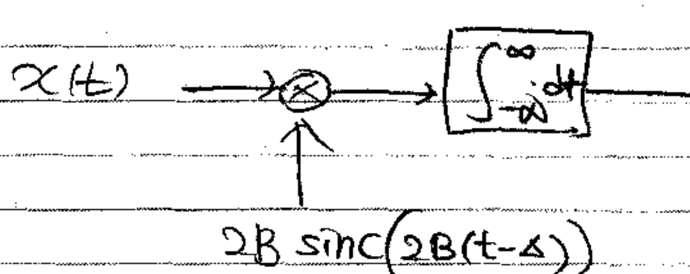

- Note that $\left\{\sqrt{2B}\,\mathrm{sinc}\left(2B\left(t-\frac{n}{2B}\right)\right)\right\}_{n=-\infty}^{\infty}$ is an orthonormal basis for the set of all finite-energy signals with bandwidth B.

$$\frac{1}{\sqrt{2B}}\,x(\Delta) = \underbrace{\int_{-\infty}^{\infty} x(t)\,\sqrt{2B}\,\mathrm{sinc}\left(2B\left(t-\frac{\Delta}{2B}\right)\right)dt}_{\text{the projection of } x(t) \text{ to the direction of } \sqrt{2B}\,\mathrm{sinc}\left(2B\left(t-\frac{\Delta}{2B}\right)\right)}$$

- We will see

$$x(t) = \underbrace{\sum_{n=-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2B}}\,x\left(\frac{n}{2B}\right)}_{\text{coefficients}}\ \underbrace{\sqrt{2B}\,\mathrm{sinc}\left(2B\left(t-\frac{n}{2B}\right)\right)}_{\text{basis functions}}}_{\text{reconstruction.}}$$