○ Nyquist's zero ISI condition

• The received signal is modeled as

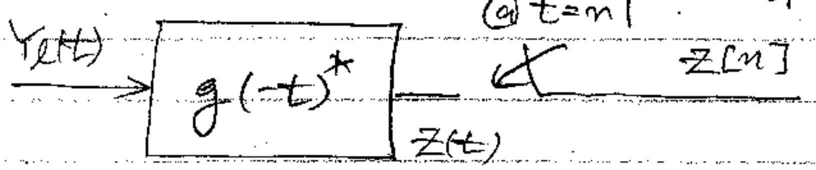
$$Y_\ell(t) = \sqrt{2P}\sum_{k=-\infty}^{\infty} d[k]\, g(t-kT) + N(t)$$

where $d[k]$ is the k-th symbol,
$g(t-kT)$ " " " " waveform,
$1/T$ " " symbol transmission rate
& $d[k]$'s are i.i.d. r.v.'s.

• We studied that if

$$\int_{-\infty}^{\infty} g(t-mT)\, g(t-m'T)^* dt \propto \delta_{m,m'}$$

then the simple receiver front-end not only



generates the sufficient statistic but also makes the symbol-by-symbol MAP detection processes only $Z[n]$ for $d[n]$ (when $\{d[n]\}_n$ is an indep. seq.)

• Definition. Nyquist condition for zero ISI
No ISI iff

$$\tilde{g}(nT) \propto \begin{cases} 1, & n=0 \\ 0, & \text{elsewhere} \end{cases}$$

where $\tilde{g}(t) = g(t) * g(-t)^*$ is called the time autocorrelation of $g(t)$.

- Theorem

Nyquist's zero ISI condition is equivalent to

$$\sum_{l=-\infty}^{\infty} \left| G(f+\frac{l}{T}) \right|^2 = \text{constant}, \ \forall f$$

/proof/

Let $|G(f)|^2 \overset{\mathcal{F}}{\longleftrightarrow} \tilde{g}(t)$.

Then, the zero ISI condition in time-domain is given by

$$\tilde{g}(nT) = \begin{cases} \text{const} (>0), & \text{for } n=0 \\ 0 & , \text{for } n\neq 0 \end{cases}$$

or, equivalently,

$$\tilde{g}(t) \sum_{l=-\infty}^{\infty} \delta(t-lT) = \text{const} \cdot \delta(t)$$

By applying Fourier transform,

$$|G(f)|^2 * \frac{1}{T} \sum_{l=-\infty}^{\infty} \delta(f-\frac{l}{T}) = \text{const}, \ \forall f$$

$$\Longleftrightarrow \frac{1}{T} \sum_{l=-\infty}^{\infty} \left| G(f-\frac{l}{T}) \right|^2 = \text{const}, \ \forall f.$$

- Corollary

A necessary condition for zero ISI is

$$2 \times \text{BW of } g(t) \geq \frac{1}{T}$$

≡ Theorem (Nyquist's Theorem for minimum bandwidth requirement)

There exists $g(t)$ inducing no ISI iff the bandwidth of $g(t)$ is greater than or equal to $\frac{1}{2T}$.
(Bandpass bandwidth W. Then, $W \geq \frac{1}{T}$)

- Example (AWGN)

(i) $W = 1.5$ MHz from 800 MHz to 801.5 MHz
BPSK (1b/symbol)
2 Mbps (2Mb/s) $\rightarrow$ 2 M symbols/sec

$\rightarrow$ zero ISI impossible

(ii) $W = 1.5$ MHz
QPSK
2 Mbps $\rightarrow$ 1 M symbols/sec

$\rightarrow$ zero ISI possible.

(Faster-than-Nyquist transmission.)

- Remember
Zero ISI is desirable. But not mendatory for reliable communication!

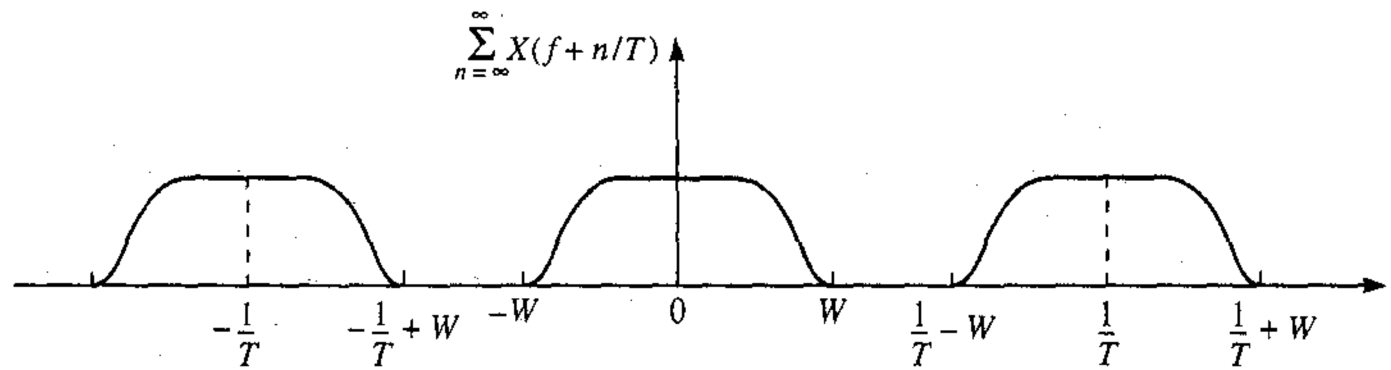


FIGURE 9-2-4 Plot of $B(f)$ for the case $T < 1/2W$.

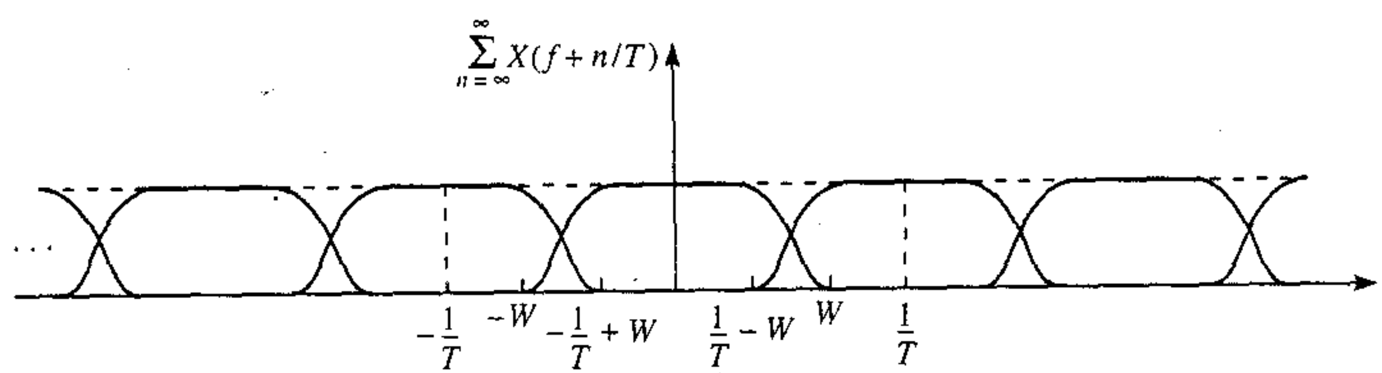


FIGURE 9-2-6 Plot of $B(f)$ for the case $T > 1/2W$.

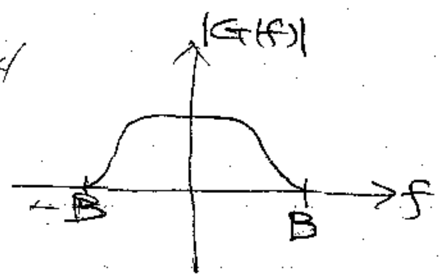
○ Excess bandwidth

- Suppose we are given a linearly modulated signal

$$s(t) = \mathrm{Re}\left[\left\{\sum_{n=-\infty}^{\infty} I_n g(t-nT)\right\} e^{j2\pi f_c t + \theta}\right]$$

where I_n is the n-th symbol, $1/T$ is the symbol transmission rate, and $B/2$ is the bandwidth of $g(t)$ ($\Rightarrow$ B is the bandwidth of the bandpass signal $s(t)$)

Ex)



- Then, **the (symbol) excess bandwidth** β is defined by the relation

$$BT = \frac{1+\beta}{2}$$

Often, called $s(t)$ has $\beta \times 100$ % (symbol) excess bandwidth

- Note

(iii) In theory, β can take any value on the interval $\beta \in (-1, \infty)$

(iv) In reality, β usually ranges $0 < \beta \leq 1$. for narrow band systems. ≤ enough to shape a pulse

(i) Since $\beta = 0$ is the case with the minimum bandwidth for zero ISI, if $g(t)$ sampled at nT generates no ISI ($n \neq 0$), then β can be thought as the extra (normalized) bandwidth

spent in order to shape the pulse.
Even if $\tilde{g}(t)$ results in nonzero ISI, β can serve as the measure of bandwidth expansion w.r.t. sinc pulse modulation

ex/ $1/T = 10$ MHz, 0% excess bandwidth $\Rightarrow B = \frac{\beta + 1}{2T} = \frac{10 \times 10^6}{2} = 5$ MHz
DSB BPSK

$1/T = 10$ MHz, 50% excess bandwidth DSB BPSK
$\Rightarrow B = \frac{1+\beta}{2T} = 5 \text{ MHz} \times 1.5 = 7.5$ MHz

$1/T = 10$ MHz, 100% " " " "
$\Rightarrow B = 5 \text{ MHz} \times 2 = 10$ MHz

(ii) A frequency-domain raised cosine pulse with the rolloff factor β has the symbol excess bandwidth β.

(v) In spread-spectrum systems, $\beta \gtrsim 100$ is not unusual.

the sufficient statistic is the same as long as ~~...~~ no ISI and perfect sampling.
∴ small β saves BW.

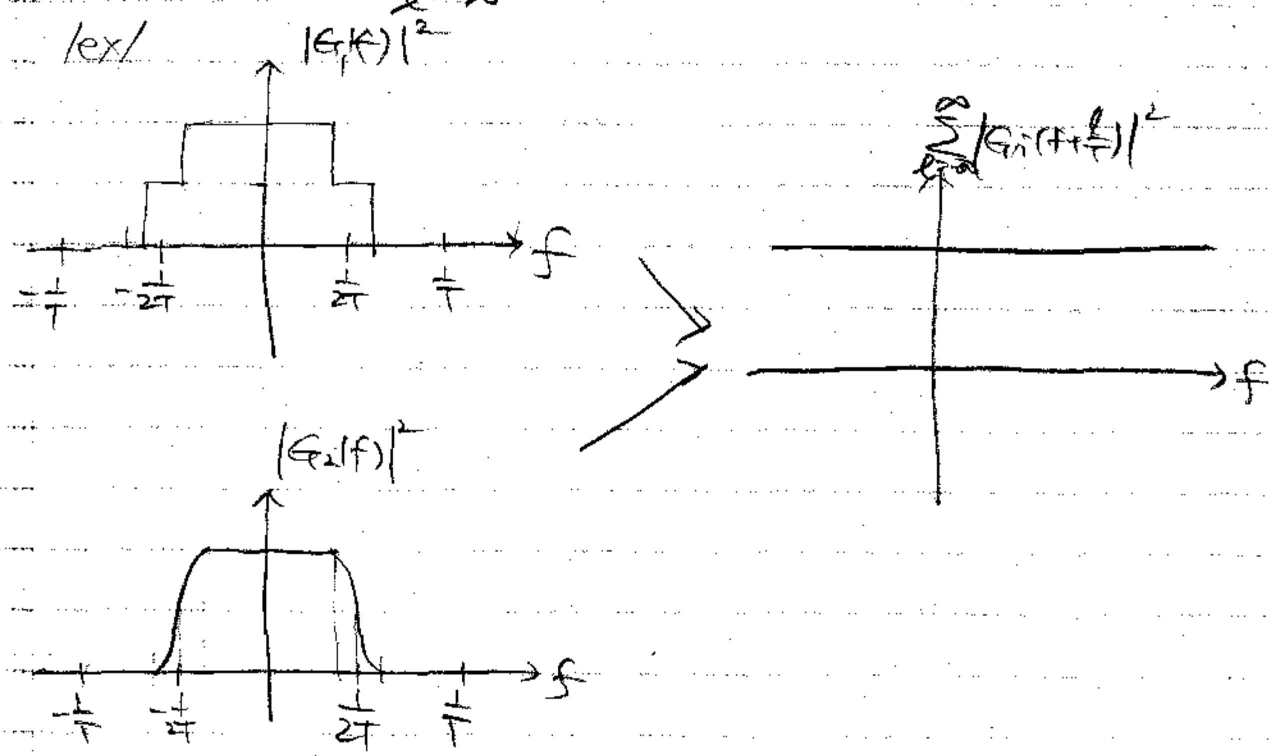
However small β pulse is difficult to generate.

- Given B & T, zero ISI g(t) is not unique

(i) Nyquist condition $\sum_{\ell=-\infty}^{\infty} |G(f+\frac{\ell}{T})|^2 = \text{const}$ does not restrict the phase response of g(t)

(ii) Different $|G_1(f)|^2$ & $|G_2(f)|^2$ may have the same $\sum_{\ell=-\infty}^{\infty} |G_i(f+\frac{\ell}{T})|^2$,

/ex/



○ Square-root raised cosine pulse

- It is easy to verify that

$$X(f) = |G_T(f)|^2 = \begin{cases} T & |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2}\left\{1+\cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right]\right\} & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0 & \text{elsewhere} \end{cases}$$

(See Figure 9-2-11, Proakis 3rd)

satisfies the Nyquist condition, where

$0 \le \beta \le 1$ is called **the roll off factor**

$\rightarrow \operatorname{sinc}(\frac{t}{T})$

called **the raised cosine spectrum w/ rolloff factor β**

- In time domain,

$$x(t) = \mathcal{F}^{-1}\{X(f)\}$$

$$= \operatorname{sinc}\left(\frac{t}{T}\right)\frac{\cos\left(\pi\beta\frac{t}{T}\right)}{1-4\beta^2\left(\frac{t}{T}\right)^2}$$

$$= \begin{cases} \dfrac{\cos\left(\pi\beta\frac{t}{T}\right)}{1-4\beta^2\left(\frac{t}{T}\right)^2} & \text{for } t=0 \\ \dfrac{\sin\frac{\pi t}{T}}{\frac{\pi t}{T}} \cdot \dfrac{\cos\left(\pi\beta\frac{t}{T}\right)}{1-4\beta^2\left(\frac{t}{T}\right)^2} & \text{for } t \ne 0 \end{cases}$$

is called **the (freq-domain) raised cosine pulse with the rolloff factor β.**

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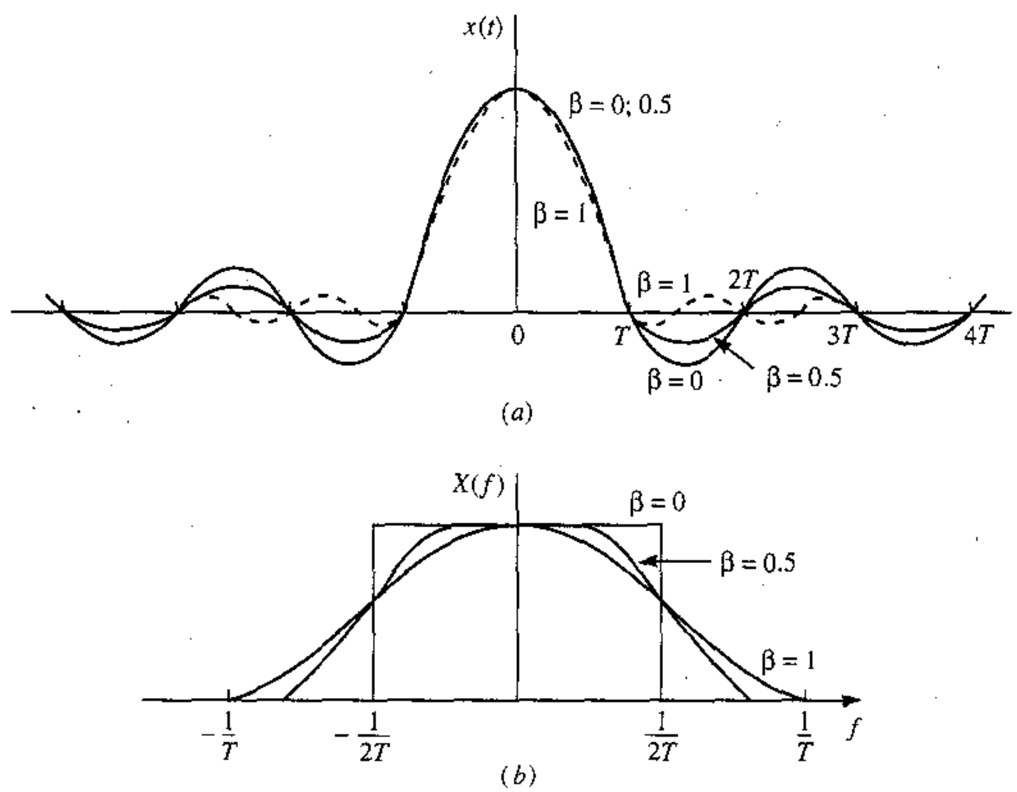


FIGURE 9-2-7 Pulses having a raised cosine spectrum.

• Define $X_{rc}(f)$ as the raised cosine spectrum

Then,

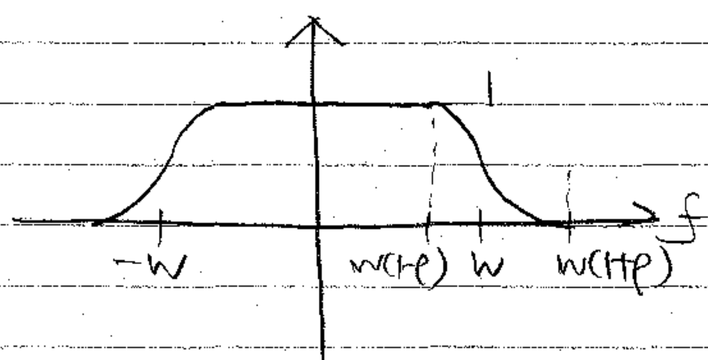
$$G_{\sqrt{RC}}(f) \triangleq \sqrt{|X_{rc}(f)|}\; e^{-j2\pi f t_0}$$

delay t_0 to make the filter realizable

is called **the square-root raised cosine spectrum w/ the rolloff factor β**

and its inverse Fourier transform is called **the (frequency-domain) square-root raised cosine pulse w/ the rolloff factor β**.

• When we (re)define the RC spectrum as



Then, the square-root RC pulse is given by

$$p(t) = \frac{\sqrt{2W}}{1-(8\beta W t)^2}\left(\frac{\sin(2\pi W(1-\beta)t)}{2\pi W t} + \frac{4\beta}{\pi}\cos(2\pi W(1+\beta)t)\right)$$

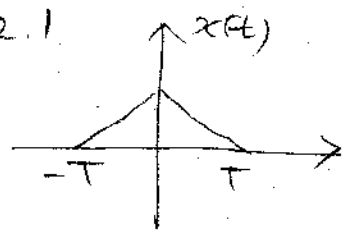
○ Eye pattern or Eye diagram.

- We want to check whether the signal $\sum_{k=-\infty}^{\infty} d[k]x(t-kT)$ induces ISI when sampled at every T.

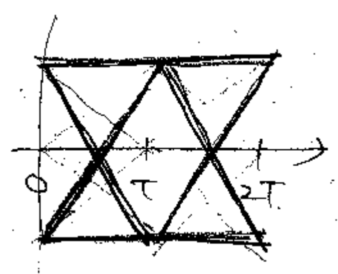
- We can use an oscilloscope to check it.
  With input $\sum_{k=-\infty}^{\infty} \underbrace{d[k]}_{\text{random sequence from } \in \{d_1, \dots, d_M\}} x(t-kT)$

  & sweep rate ~~an integer multiple of~~ $\frac{1}{nT}$, $n \in \mathbb{Z}$
  the oscilloscope displays a pattern called an eye pattern/diagram.

Example. 1



$d[k] \in \{+1, -1\}$



$2 \times 2 \times 2 = 2^3 = 8$

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pattern is centered on the screen. The horizontal time base is set to be approximately equal to the symbol duration. The inherent persistence of the cathode-ray tube displays the superimposed segments of the $v_o(t)$ signal. The eye pattern of the *pseudorandom binary sequence* (PRBS) generator is displayed if the data output of this generator is directly connected to the vertical input of the oscilloscope.

Computer-generated eye diagrams are described in later sections. The "information content" of experimentally obtained and computer-generated eye diagrams is important for the comprehension of most digital wireless transmission systems. For example, the experimentally obtained (that is, observed in a hardware laboratory experiment) eye diagram of a *non–phase-equalized, conventional* fourth-order Butterworth filter is shown in

Feher

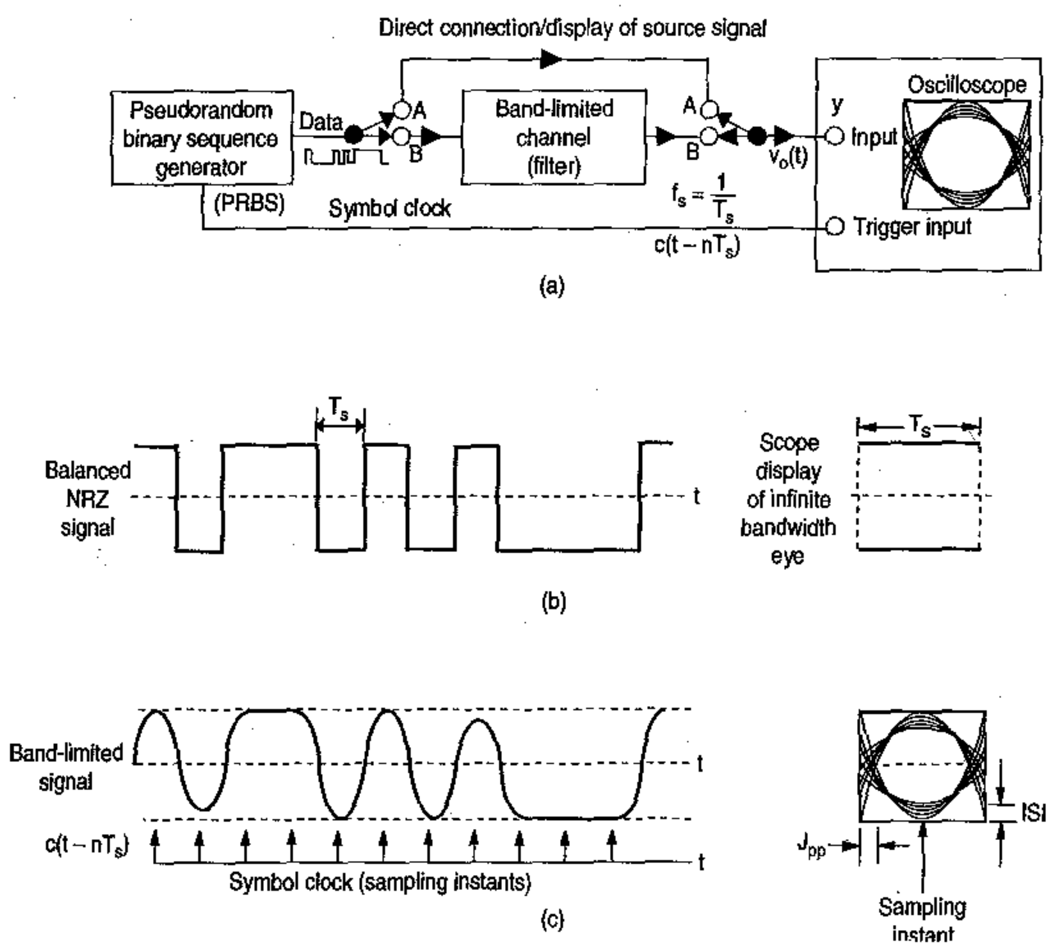

Figure 4.2.6 Eye diagram measurement setup and display. (a), setup; (b), NRZ signal (c), bandlimited eye diagram.

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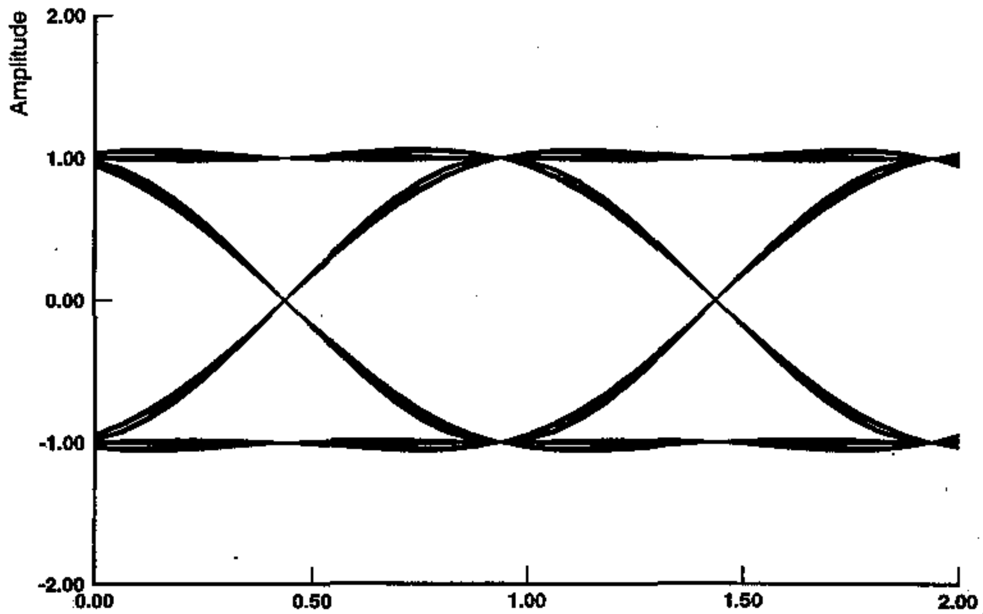
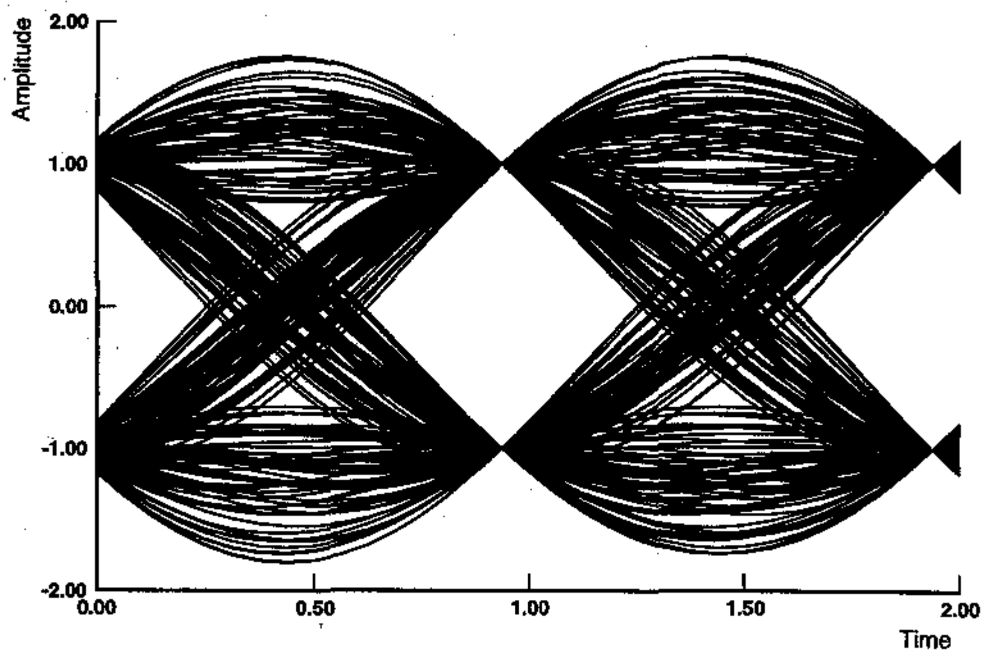


(a)

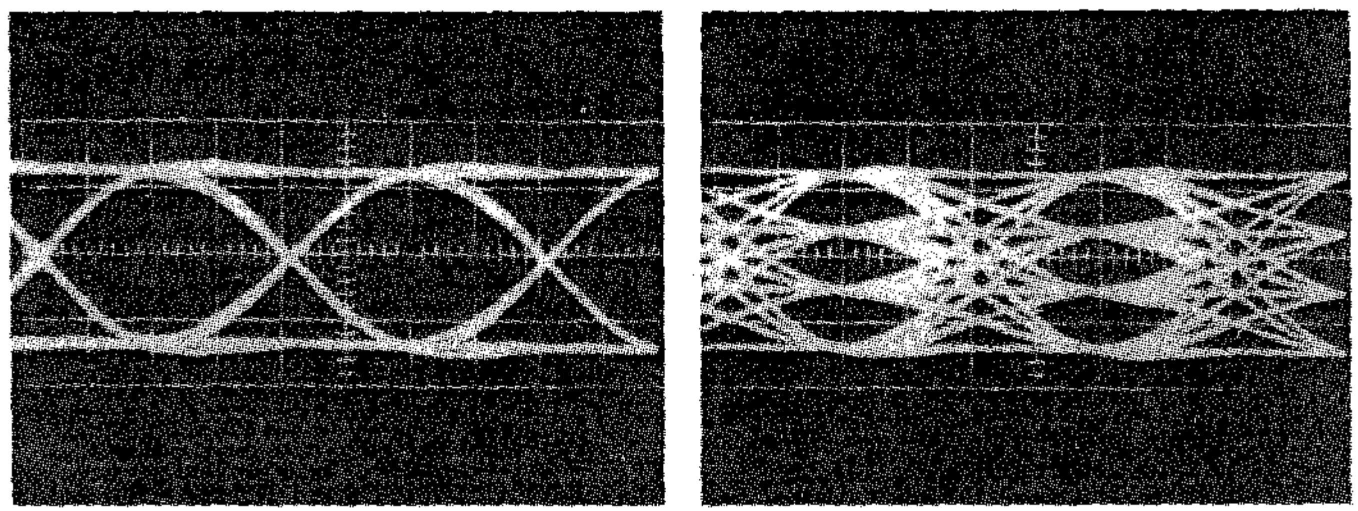


(b)

Figure 4.2.11 Computer-generated eye diagrams of raised-cosine Nyquist filtered systems, including an x/sin x-shaped aperture equalizer for NRZ data (see equation 4.2.19). (a), $\alpha = 1$ roll-off factor; (b), $\alpha = 0.3$ roll-off factor. CREATE-1 program, Appendix 2, generates this type of diagrams.

Proakis "Digital Communications" 3rd Edition, McGraw Hill.

FIGURE 9-2-1 Examples of eye patterns for binary and quaternary amplitude shift keying (or PAM).



BINARY

QUATERNARY

Example 2

- Often, $\sum_{k=-\infty}^{\infty} d[k]\, x(t-kT) + N(t)$ is used as an input.

Proakis
"Digital Communications"
3rd Ed.
McGraw Hill

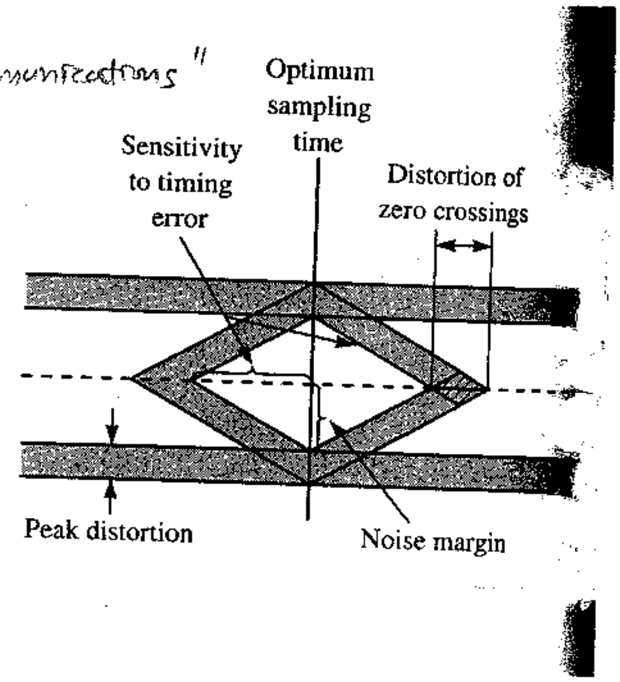


FIGURE 9-2-2 Effect of intersymbol interference on eye opening.

- Using an eye pattern, we may qualitatively assess
  - (i) how closely the Tx & the Rx filters match ($N(t)=0$)
  - (ii) how much ISI is present ($N(t)=0$)
  - (iii) how sensitive the system is to a timing error
  - (iv) how noisy the received signal is

etc.

- For 2-D signaling such as QPSK, QAM, MPSK, we display the real part and the imaginary part of the samples at $t = mT$.
  The resultant pattern is called the 2 dimensional eye pattern/diagram.

- Example.
  8-PSK
  MF output

Proakis "Digital Communications" 3rd Ed. McGraw Hill

perfect match & No noise & correct sampling instants

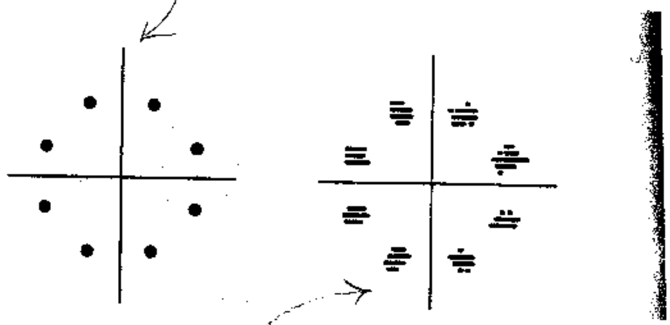


FIGURE 9-2-3 Two-dimensional digital "eye patterns."

imperfect matching or
incorrect timing or
noisy observation