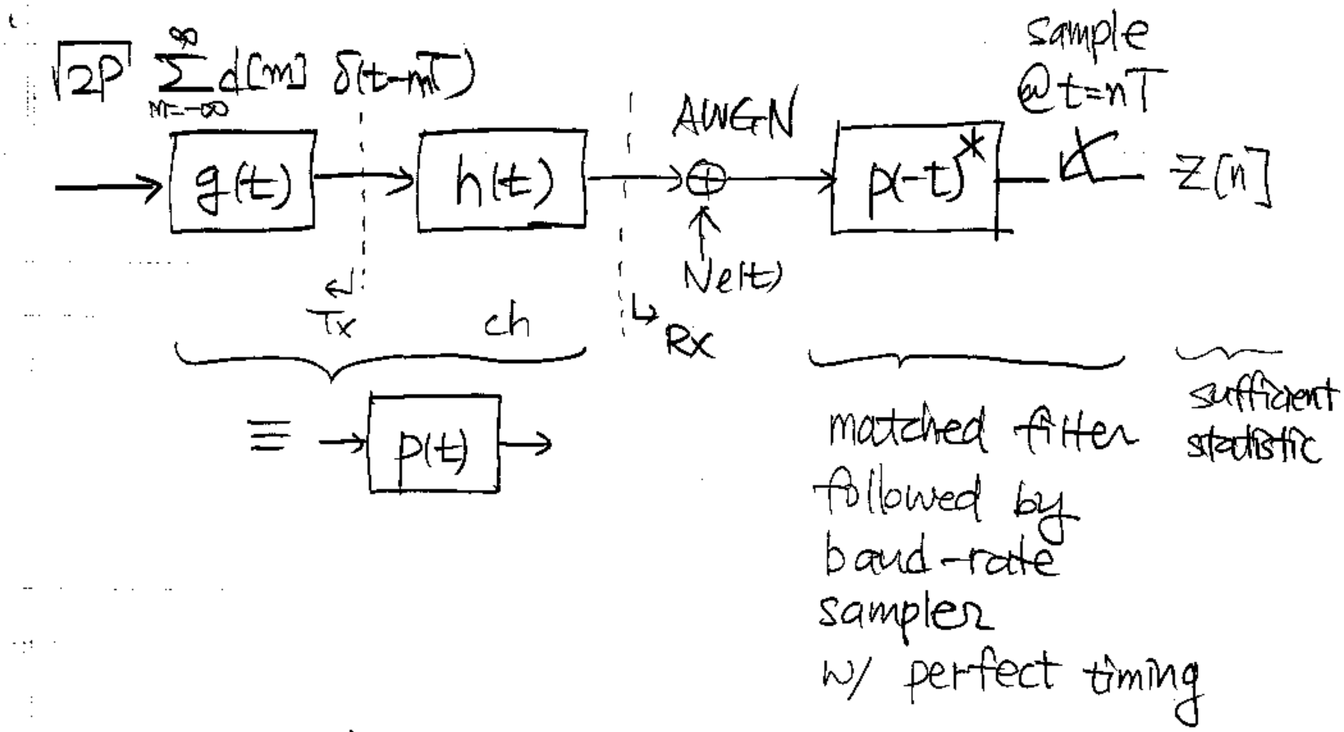
Whitened Matched Filter (WMF) Front-End

o Matched Filter Front-End



- $z[n] = \sqrt{2P}\sum_{m=-\infty}^{\infty} d[m]\,\tilde{p}((n-m)T) + N_e[n]$

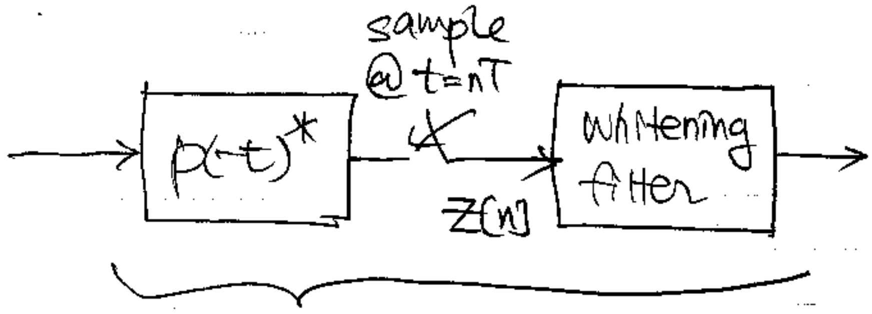
  where $\tilde{p}(t) \triangleq p(t) * p(-t)^* = \int_{-\infty}^{\infty} p(\tau+t)\,p(\tau)^* d\tau$

  and $E\{N_e[m]^* N_e[m+n]\} = 2N_0\,\tilde{p}(nT)$

- In general, we have ISI and colored noise sequence

o Whitened Matched Filter (WMF)

- The objective is to whiten the noise sequence $N_e[n]$ using a DT system that processes $z[n]$, i.e.,



- PSD of the noise component in $z[n]$

Define $g[n] \triangleq \tilde{p}(nT)$

(i) $g[n] = g[-n]^*$, i.e., $g[n]$ is conjugate symmetrical.

∵ $\tilde{p}(t)$ is conjugate symmetrical function of t.

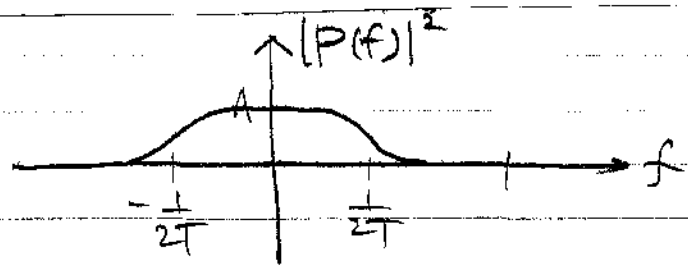
(ii) $Q(e^{j2\pi f})^* = Q(e^{j2\pi f})$, i.e., $\underbrace{Q(e^{j2\pi f})}_{\text{PSD of the noise component}/\sqrt{2P}}$ is real

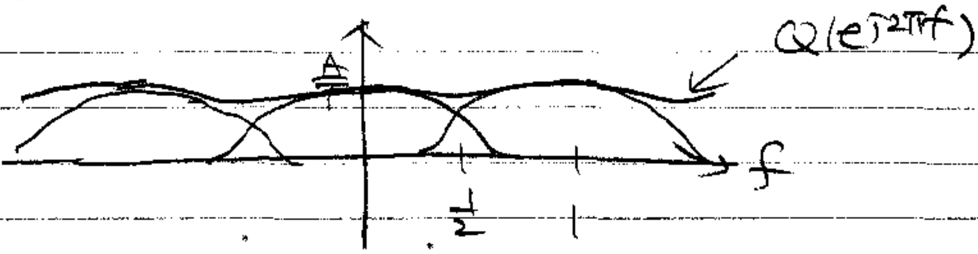
∵ $g[n]$ is conjugate symmetrical
$\Leftrightarrow Q(e^{j2\pi f})$ is real.

(iii) $Q(e^{j2\pi fT}) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \left| P(f - \frac{m}{T}) \right|^2$

$\Leftrightarrow Q(e^{j2\pi f}) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \left| P(\frac{f-m}{T}) \right|^2$

∵ $g[n]$ is the sampled version of $\tilde{p}(t)$ at rate $1/T$
$\Leftrightarrow \mathcal{F}\{g[n]\}$ = rate × folded spectrum of $\tilde{p}(t)$





- Thus, what we want is a DT system whose "magnitude square of the DTFT" is proportional to $\frac{1}{Q(e^{j2\pi f})}$

- Given $Q(e^{j2\pi f})$, there are infinitely many DT systems with DTFT $S(e^{j2\pi f})$ s.t.

$$|S(e^{j2\pi f})|^2 = \frac{C}{Q(e^{j2\pi f})} \quad \forall f, \exists c.$$

- If $Q(e^{j2\pi f})$ is rational, then the story becomes totally different. We can be assisted by the theory of z-transform to systematically find the best $S(e^{j2\pi f})$

○ Rational Transfer Function approximation to $Q(z)$, the z-transform of $q[n]$

- Property
  From $q[n] = q[-n]^*$, the z-transform

$$Q(z) = \sum_{n=-\infty}^{\infty} q[n]\, z^{-n} \quad \text{satisfies}$$

called the **reflected transfer function of Q(z)**

$$\tilde{Q}(z) = Q\left(\frac{1}{z^*}\right)^*$$

/ex/ $q[n] = -2j\,\delta[n+1] + 3\delta[n] + 2j\,\delta[n-1]$

$= q[-n]^*$

$Q(z) = -2j z + 3 + 2j z^{-1}$

$Q\left(\frac{1}{z^*}\right)^* = \left(-2j\frac{1}{z^*} + 3 + 2j\left(\frac{1}{z^*}\right)^{-1}\right)^*$

$= 2j z^{-1} + 3 - 2j z$

$= Q(z)$

- Rational Transfer function **approximation** (Lee 2.5.2)

Suppose that the z-transform $Q(z)$ is well approximated as

$$Q(z) = z^r \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = A \cdot z^r \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})}$$

$$= A z^m \frac{\prod_{k=1}^{M} (z - c_k)}{\prod_{k=1}^{N} (z - d_k)}$$

where $A = \frac{b_0}{a_0}$ and $m = N - M + r$.

- Consequence

From the property & the rational approximation, we have the following consequences.

$$Q(z) = A z^{m} \frac{\prod_{k=1}^{M} (z - c_k)}{\prod_{k=1}^{N} (z - d_k)} = Q\left(\frac{1}{z^*}\right)^*$$

$$= \left(A \left(\frac{1}{z^*}\right)^{m} \frac{\prod_{k=1}^{M} \left(\frac{1}{z^*} - c_k\right)}{\prod_{k=1}^{N} \left(\frac{1}{z^*} - d_k\right)} \right)^*$$

$$= A^* z^{-m} \frac{\prod_{k=1}^{M} \frac{1}{z}(1 - z c_k^*)}{\prod_{k=1}^{N} \frac{1}{z}(1 - z d_k^*)}$$

$\equiv$ (c_k is a **zero** / d_k is a **pole** of $Q(z)$) $\Leftrightarrow$ ($\frac{1}{c_k^*}$ is also a **zero** / $\frac{1}{d_k^*}$ is also a **pole** of $Q(z)$)

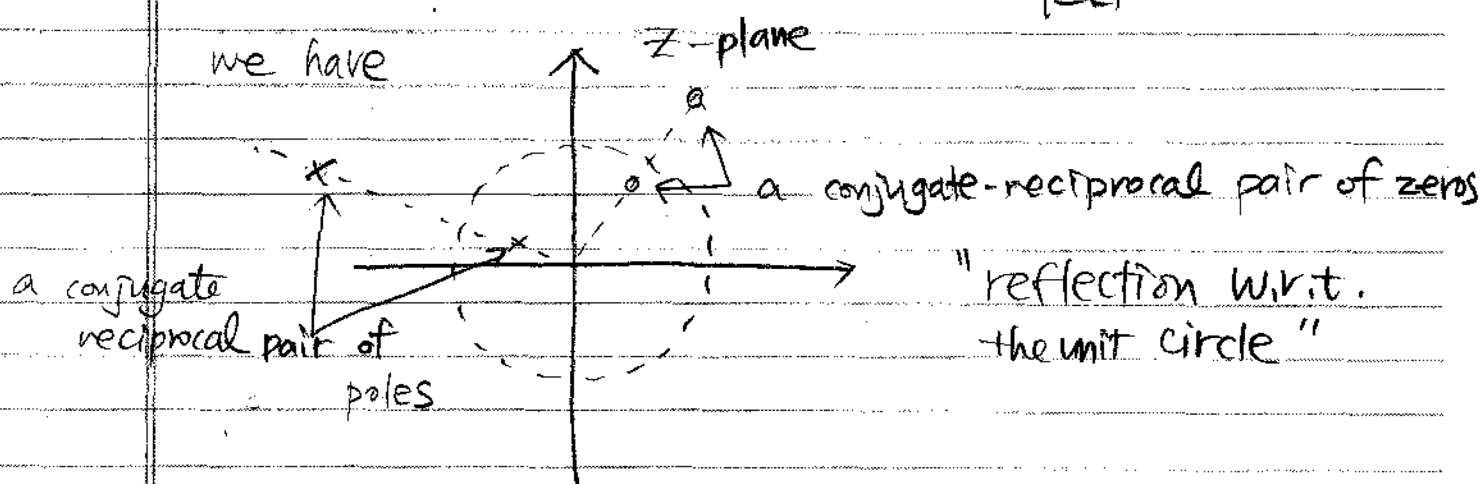
← called the **conjugate-reciprocal** of c_k.

Since

$$c_k = |c_k| e^{j\angle c_k} \Leftrightarrow \frac{1}{c_k^*} = (c_k^*)^{-1} = \left(|c_k| e^{-j\angle c_k}\right)^{-1}$$

$$= \frac{1}{|c_k|} e^{j\angle c_k}$$

we have



"reflection w.r.t. the unit circle"

- Conclusions

(i) $$Q(z) = A z^{r} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})(1 - \frac{1}{c_k^*} z^{-1})}{\prod_{k=1}^{N} (1 - d_k z^{-1})(1 - \frac{1}{d_k^*} z^{-1})}$$

If $|c_k| < 1$, then $\left|\frac{1}{c_k^*}\right| > 1$ and vice versa.
If $|d_k| < 1$, then $\left|\frac{1}{d_k^*}\right| > 1$. and vice versa.

(ii) There exists a factorization

$$Q(z) = R(z)\, R\left(\frac{1}{z^*}\right)^*$$

in such a way that
[if a zero is included in $R(z)$
then the conjugate-reciprocal zero is included in $R(\frac{1}{z^*})^*$

if a pole is " " "
" " " " pole " " " "

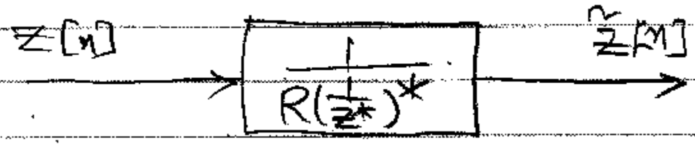
Q. How many such a factorization is possible?

A. 2^{M+N}

○ **Noise whitening** using $\frac{1}{R(\frac{1}{z^*})^*}$

- We know that the symbol waveform appears in $z[n]$ is proportional to $f[n]$, and the autocorrelation function of the AGN is proportional to $f[n]$

- Now, consider filtering $z[n]$ using $\frac{1}{R(\frac{1}{z^*})^*}$ as



Then, the symbol waveform component becomes

$$Q(Z) \times \frac{1}{R(\frac{1}{Z^*})^*} = R(Z)$$

and the noise PSD becomes

$$\alpha Q(e^{j2\pi f}) \left| \frac{1}{R(e^{j2\pi f})^*} \right|^2 = Q(e^{j2\pi f}) \frac{1}{Q(e^{j2\pi f})} = 1.$$

- Thus, if we choose **R(Z)** as a **minimum phase** rational transfer function, then we obtain

$$\tilde{Z}[n] = \sqrt{2P} \sum_{m=-\infty}^{\infty} d[m]\, r[n-m] + \tilde{N}[n]$$

where
 $r[n]$ is a **causal** sequence
 $\tilde{N}[n]$ is a **white** Gaussian noise.

Moreover, due to the minimum-phase property of $r[n]$, $r[n]$ becomes the causal sequence that has the maximum energy containment around $n=0$, i.e.,

$$\sum_{n=0}^{N} |r[n]|^2 \geq \sum_{n=0}^{N} |\tilde{r}[n]|^2, \quad \forall N \in \{0, 1, 2, \cdots\}$$

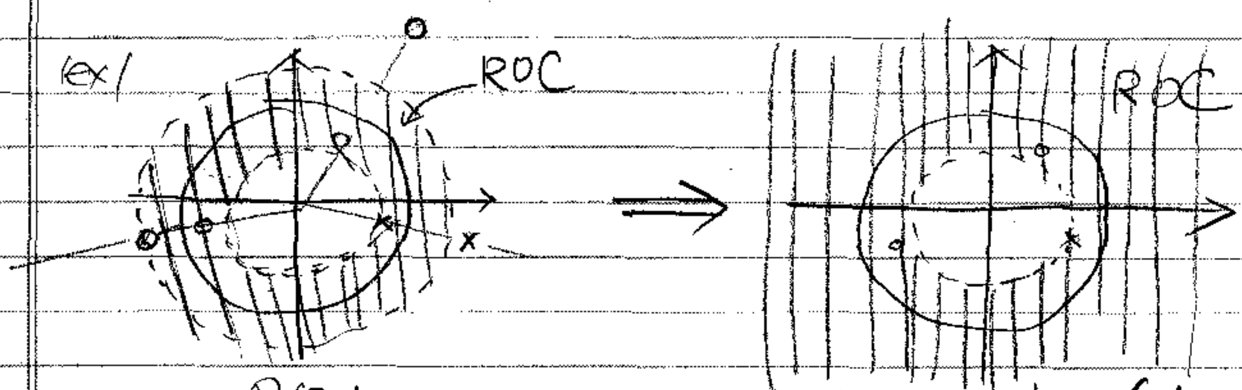
among all possible causal sequence that can be constructed from $Q(z) = R(z)R(\frac{1}{z^*})^*$

factorization.

- Reminder
  Given a z-transform that has $Q(z) = R(z)R(\frac{1}{z^*})^*$ factorization, $R(z)$ is called minimum phase-(lag) if $R(z)$ has all the poles and zeros of $Q(z)$ inside the unit circle.

  Of course, the region of convergence is chosen to include the unit circle. Thus, $r[n]$ becomes causal, stable.

ex/

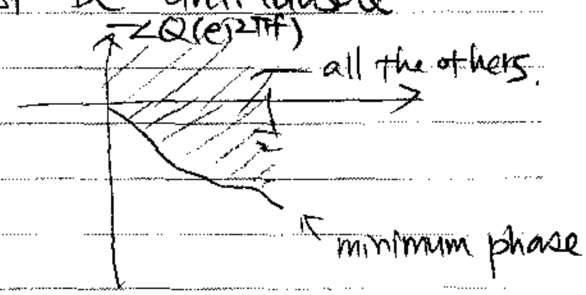


$Q(z)$ $\Longrightarrow$ minimum phase (-lag) $R(z)$

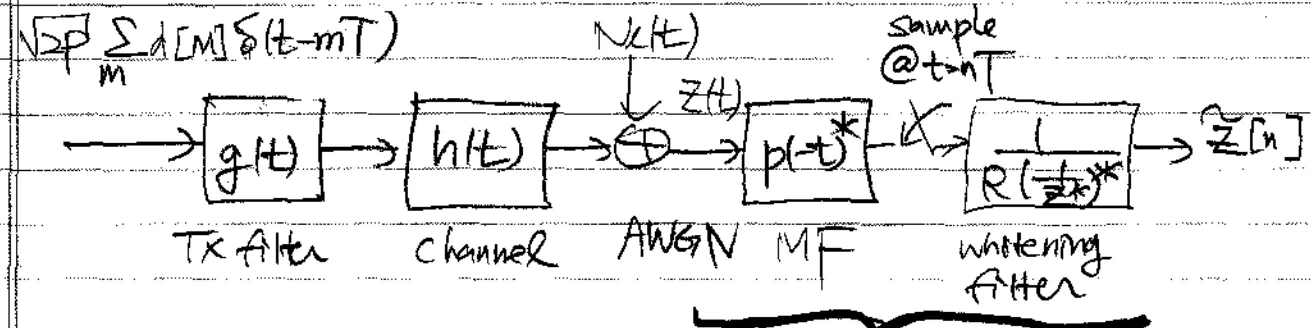
- Note that the whitening filter has all the pole & zero outside the unit circle because $R(\frac{1}{z^*})^*$ has all the poles & zeros of $Q(z)$ outside the unit circle. Of course, a pole of $R(\frac{1}{z^*})^*$ becomes a zero of $\frac{1}{R(\frac{1}{z^*})^*}$, and a zero of " " " pole " "

($\frac{1}{R(\frac{1}{z^*})^*}$ is the maximum phase system)

Thus, the whitening filter must be anti causal to be stable



○ Conclusion



called the whitened matched filter (**WMF**) front end

then we have $\{\hat{Z}[n]\}_n$ as **a sufficient statistic** equivalent to $Z(t)$

$$\hat{Z}[n] = \sqrt{2P}\sum_m d[m]\, r[n-m] + \tilde{N}[n]$$

where $r[n]$ is causal & minimum phase (-lag) $\left(-\angle Q(e^{j2\pi f})\right)$
$\tilde{N}[n]$ is white

- Limitations of the WMF front-end

(i) Matched filtering may **not** be **possible** for some applications where the channel $h(t)$ is unknown a priori. → Not a model for channel identification,

(ii) The z-transform of $f[n] \triangleq \tilde{p}(nT)$ may **not** well **approximated to a rational** response w/ reasonable # of poles & zeros

(iii) WMF front-end does not generate a s.s. if $N_L(t)$ is **not white Gaussian**.
(If colored WSS Gaussian, $\frac{P(f)^*}{\sqrt{S_{NLNL}(f)}}$ can be used w/ WMF instead of $P(f)^*$.)

- The WMF observation model is very popular model for its simplicity.

ex/ MLSD (Maximum Likelihood Sequence Detection)

Q. $\tilde{Z}[n] = \sqrt{2p} \sum_{m=0}^{M-1} d[m]\, r[n-m] + \tilde{Z}[n]$

$0 \le n \le M+N$ (← These M, N have nothing to do w/ the # of poles & zeros)

where $\begin{cases} r[n] \simeq 0 \quad \forall n > N \\ d[m] \in \{+1, -1\} \text{ i.i.d. equally likely} \end{cases}$

Find the MLSD.

A. Since noise is white, the minimum distance decision rule is the ML rule.

Let $S_1[n] = r[n] + r[n-1] + \cdots + r[n-M+1]$
$S_2[n] = -r[n] + r[n-1] + \cdots + r[n-M+1]$

⋮

$S_{2^M}[n] = -r[n] - r[n-1] + \cdots - r[n-M+1]$

Then, the ML rule becomes

$$\underline{d}' = \underset{\underline{d} \in \{-1,1\}^M}{\arg\min} \left\| \underline{\tilde{Z}} - \sqrt{2p}\, \underline{S}_i \right\|.$$

If the noise were not white, then a whitening process would have needed.