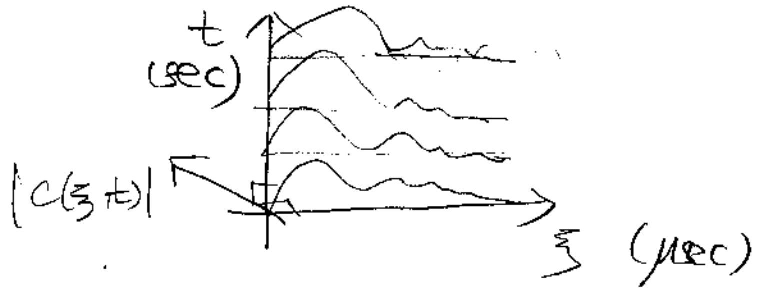
Comments on Deterministic $C(\xi;t)$

◎ $C(\xi;t) = h_t(\xi)$ = impulse response at t with relative delay ξ

◎ Usually, time scale in ξ is much smaller than time scale in t



$\Rightarrow$ We may consider Fourier transform of $C(\xi;t) = h_t(\xi)$ with respect to ξ.

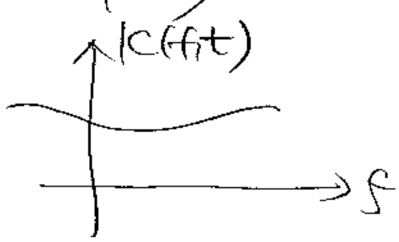
Then,

$$C(f,t) = H_t(f) \triangleq \int_{-\infty}^{\infty} C(\xi;t)\, e^{-j2\pi f\xi}\, d\xi$$

represents the frequency response of $C(\xi;t)$ at t.

∘ $C(f;t)$

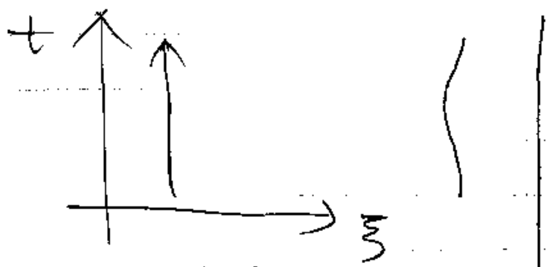
If $|C(f;t)|$ varies very slowly in f given fixed t,
then, frequency coherence is very high



Let coherence bandwidth be the maximum distance b/w two f's having almost the same value of $|C(f;t)|$.

As we've seen already, the coherence bandwidth times the maximum delay spread determines the severity of frequency selectivity.

◎ Now consider $c(\xi, t)$ as a function of t



/ex/ $c(\xi ; t) = \alpha(t)\,\delta(\xi - \Delta)$

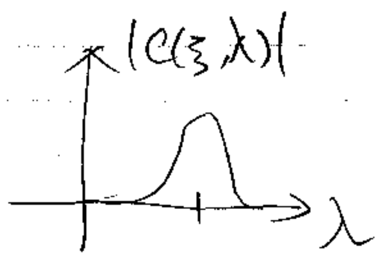
$\Rightarrow y(t) = \alpha(t)\, x(t-\Delta)$

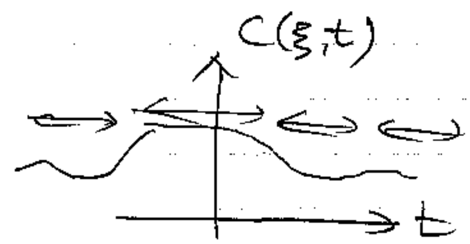
$\Rightarrow Y(f) = A(f) * \left(X(f)\, e^{-j2\pi f \Delta}\right)$

$\Rightarrow$ From $c(\xi, t)$ as a function of t, we can get the information about how fast the channel changes.

Consider Fourier transform of $c(\xi ; t)$ with respect to t.

$$\int_{-\infty}^{\infty} c(\xi, t)\, e^{-j2\pi\lambda t}\, dt = C(\xi, \lambda)$$





- We can define coherence time from $c(\xi ; t)$ by finding the length of time interval given fixed ξ.

If $|C(\xi, \lambda)|$ has energy concentration in λ_0 then λ_0 represents the Doppler frequency.

Wireless Channels : WSSUS channel model

○ Complex baseband representation

- From the relation

$$y(t) = \int_{-\infty}^{\infty} c(\xi,t)\, x(t-\xi)\, d\xi$$

we define

$$c_\ell(\xi,t) = c(\xi,t)\, e^{-j2\pi f_c \xi}$$

where $c(\xi,t_0)$ is band pass for each t_0.

$$\left(\because\ c(\xi,t_0) = \mathrm{Re}\left\{c_\ell(\xi,t_0)\, e^{j2\pi f_c \xi}\right\}\right)$$

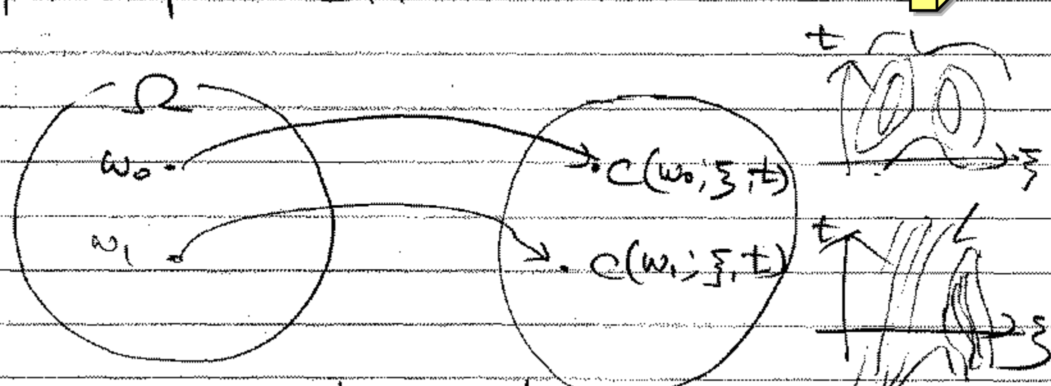
Then,

$$y_\ell(t) = \int_{-\infty}^{\infty} \frac{1}{2} c_\ell(\xi,t)\, x(t-\xi)\, d\xi$$

- In this note, we use $c(\xi,t)$ to denote $\frac{1}{2} c_\ell(\xi,t)$.

○ Proper-complex Gaussian approximation

- In some applications, we can model $c(\xi,t)$ as a proper-complex Gaussian random field.



So, the envelope $|c(\xi_0,t_0)|$ is modeled as

a Rayleigh random variable if $E\{c(\xi,t_0)\}=0$
while
a Ricean " " " " $\neq 0$.

○ Channel correlation property 1 (Time-Domain)

- Since $c(\xi,t_0)$ is modeled as a proper-complex Gaussian random variable and $c(\xi,t)$ for each (ξ,t)-pair are modeled as **joint Gaussian**, the statistical property of the random field $c(\xi,t)$ is completely characterized by

  (i) $E\{c(\xi,t)\}$ for each ξ and t, and ... (*)
  (ii) $E\{c(\xi_0,t_0)^* c(\xi_1,t_1)\}$ for every ξ_0,ξ_1,t_1,t_2

We assume **zero-mean** in what follows ... (**)

- A common assumption 1 is that

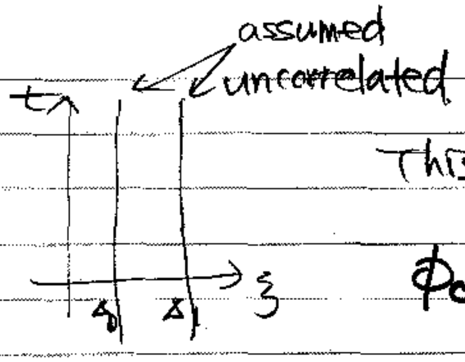
  random processes $c(\xi,t)$ as processes of t are assumed **jointly WSS** for any choice of ξ_0 and ξ_1. **(WSS model)**

  This with the Gaussian assumption leads (**) to the following definition.

$$\phi_c(\xi_1,\xi_2;\tau) \triangleq E\{c(\xi_1,t)^* c(\xi_2,t+\tau)\} \quad \cdots (***)$$

- A common assumption 2 is that

  $c(\xi_0,t)$ and $c(\xi_1,t)$ are uncorrelated random processes. (**Uncorrelated Scattering model**)



This leads (★★) to

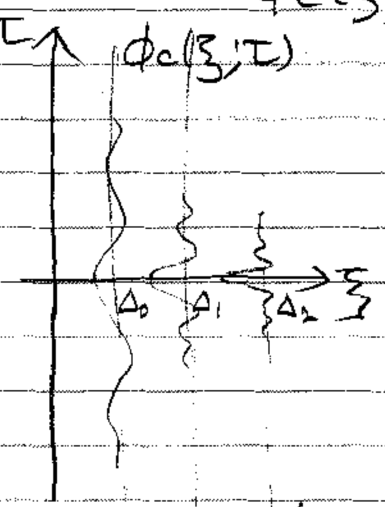
$$\phi_c(\xi_1, \xi_2; \tau) = \phi_c(\xi_1; \tau)\, \delta(\xi_1 - \xi_2)$$

In summary, **WSSUS model leads to**

$$E\{C(\xi_1, t_1)^* C(\xi_2, t_2)\} = \phi_c(\xi_1; t_1 - t_2)\, \delta(\xi_1 - \xi_2)$$

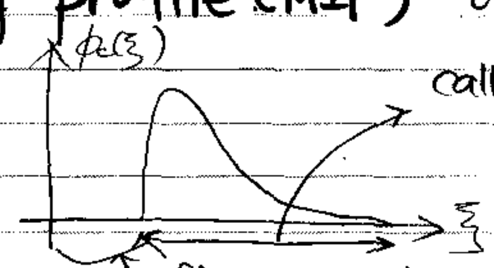
which shows the necessity to define the auto-correlation function

$$\phi_c(\xi; \tau) \triangleq E\{C(\xi, t)^* C(\xi, t+\tau)\}$$



- Note that $\phi_c(\xi; 0)$ denotes the power of each uncorrelated random process at delay ξ.
  We call $\phi_c(\xi) \triangleq \phi_c(\xi; 0)$ **the multipath intensity profile (MIP)** or Power delay profile

/ex/



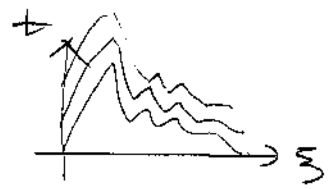
or called **multipath / delay spread of the channel.**

we often ignore this channel delay.

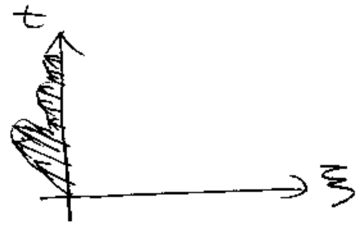
Two Extreme Cases for WSSUS channel

○ Two extreme cases

(i) $c(\xi, t) = h(\xi)$



(ii) $c(\xi, t) = \alpha(t)\,\delta(\xi)$



○ Random static channel

- $c(\xi, t) = h(\xi)$
- $E\{h(\xi)\} = 0, \ \forall \xi$

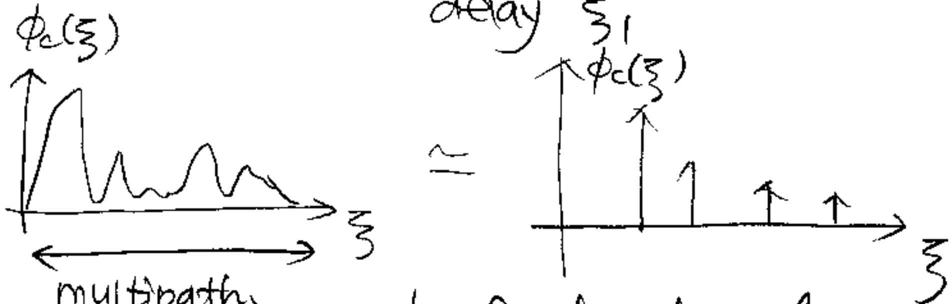
TD $E\{h(\xi_1)^* h(\xi_2)\} = \underbrace{\phi_c(\xi_1)}\,\delta(\xi_1 - \xi_2)$ ∵ uncorrelated

$\triangleq E\{|h(\xi_1)|^2\}$ multipath intensity profile (MIP)

= variance of channel response delay ξ_1

= power of the component w/ delay ξ_1

ex)



multipath delay > spread of the channel

FD • $E\{H(f)\} = E\{\int_{-\infty}^{\infty} h(\xi)\, e^{-j2\pi f\xi}\, d\xi\} = 0, \ \forall f$

$$E\{H(f_1)^* H(f_2)\} = E\left\{\left(\int_{-\infty}^{\infty} h(\xi_1) e^{-j2\pi f_1 \xi_1} d\xi_1\right)^* \left(\int_{-\infty}^{\infty} h(\xi_2) e^{-j2\pi f_2 \xi_2} d\xi_2\right)\right\}$$

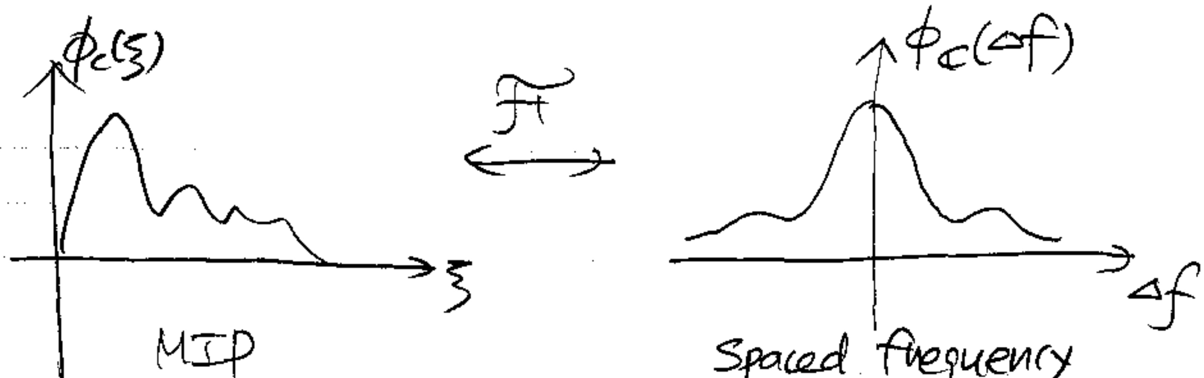
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{E\{h(\xi_1)^* h(\xi_2)\}}_{\phi_c(\xi_1)\,\delta(\xi_1-\xi_2)}\ e^{j2\pi(f_1\xi_1 - f_2\xi_2)}\, d\xi_1\, d\xi_2$$

$$= \int_{-\infty}^{\infty} \phi_c(\xi_1)\, e^{j2\pi(f_1-f_2)\xi_1}\, d\xi_1$$

$$\triangleq \phi_C(f_1-f_2)$$

Fourier transform of the MIP evaluated at f_2-f_1

"Spaced frequency coherence function"



MIP → delay spread

Spaced frequency correlation function → Cohence bandwidth

delay spread — roughly inversely proportional — Cohence bandwidth

is determined by the MIP.

If coherence BW >> signal BW, then frequency flat. Otherwise, " selective!

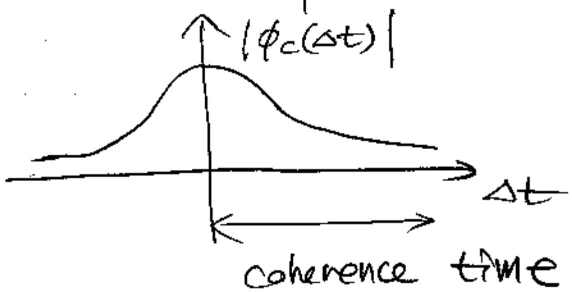
O Random Multiplicative Channel

- $c(\xi, t) = \alpha(t)\,\delta(\xi)$
- $E\{\alpha(t)\} = 0, \quad \forall t$

TD $E\{\alpha(t)^* \alpha(t+\tau)\} = \phi_c(\tau)$, ∵ WSS

or

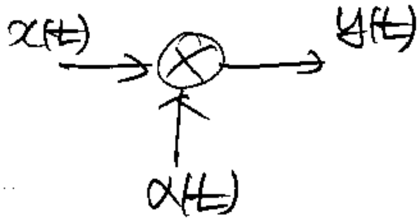
$$E\{\alpha(t_1)^* \alpha(t_2)\} = \phi_c(t_2 - t_1) = \underbrace{\phi_c(\Delta t)}_{\text{spaced time correlation function}}$$



∥ autocorrelation function of the multiplicative process.

- FD autocorrelation $\xleftrightarrow{\mathcal{F}}$ power spectrum

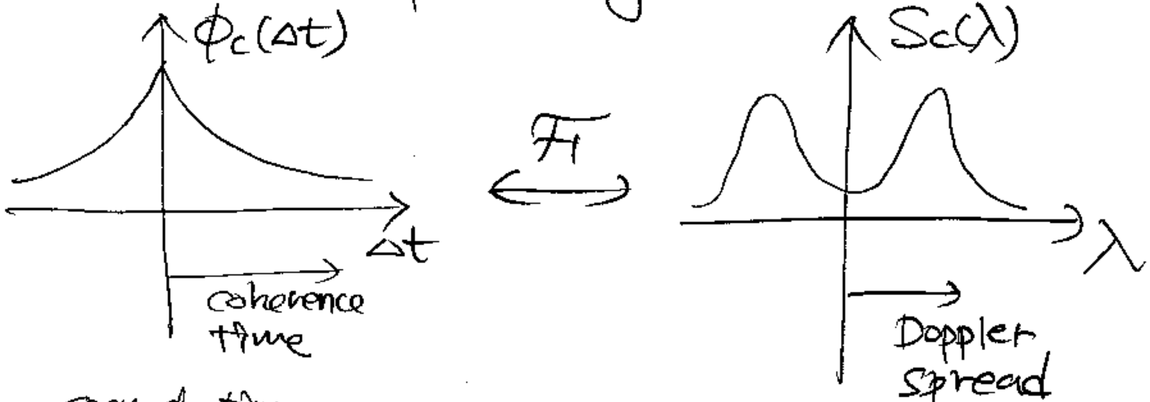
$$S_c(\lambda) = \mathcal{F}\{\phi_c(\Delta t)\} = \int_{-\infty}^{\infty} \phi_c(\Delta t)\, e^{-j2\pi\lambda\Delta t}\, d\Delta t$$



$$\begin{aligned} S_{YY}(f) &= \mathcal{F}\left(A\left[E\{Y(t_1)^* Y(t_2)\}\right]\right) \\ &= \mathcal{F}\left(A\left[\phi_c(t_2 - t_1) R_{XX}(t_1, t_2)\right]\right) \\ &= \mathcal{F}\{\phi_c(t_2 - t_1)\, \bar{R}_{XX}(t_2 - t_1)\} \\ &= \underbrace{\mathcal{F}\{\phi_c(\Delta t)\}}_{\text{approximate using deltas}} * S_{XX}(f) \end{aligned}$$

$$S_{xx}(\lambda) \to \boxed{S_c(\lambda)} \to S_{rr}(\lambda)$$

$S_{xx}(\lambda)$ is convolved w/ $S_c(\lambda)$
$S_{xx}(\lambda)$ is spread by $S_c(\lambda)$



spaced-time correlation function

Doppler power spectrum

○ Channel correlation property 2 (Freq.-Domain)

zero mean, proper-complex

- For WSSUS channel model,

$$\phi_c(\xi;\Delta t) \triangleq E\{c(\xi,t)^* c(\xi,t+\Delta t)\}$$

completely characterizes the channel, because

- $$\phi_c(\xi_1,\xi_2;\Delta t) = \phi_c(\xi_1,\Delta t)\,\delta(\xi_1-\xi_2).$$

- Now consider a 2-D Fourier transform of $\phi_c(\xi_1,\xi_2;\Delta t)$ as

$$\Phi_c(f_1,f_2;\Delta t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_c(\xi_1,\xi_2;\Delta t)\, e^{j2\pi(f_1\xi_1 - f_2\xi_2)} d\xi_1 d\xi_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_c(\xi_1;\Delta t)\,\delta(\xi_1-\xi_2)\, e^{j2\pi(\ )} d\xi_1 d\xi_2$$

$$= \int_{-\infty}^{\infty} \phi_c(\xi_1;\Delta t)\, e^{-j2\pi(f_2-f_1)\xi_1} d\xi_1$$

$$\triangleq \Phi_c(f_2-f_1;\Delta t)$$

This shows

$$\Phi_c(f_2-f_1;t_2-t_1)$$

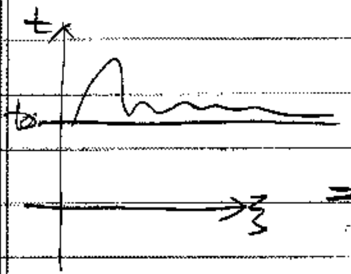
completely characterizes the channel.

called **the spaced-frequency, spaced-time correlation function**.

- Note that

$$\Phi_c(f_1,f_2;\Delta t) = E\left[\left(\int_{-\infty}^{\infty} c(\xi_1,t)\, e^{-j2\pi f_1\xi_1} d\xi_1\right)^* \left(\int_{-\infty}^{\infty} c(\xi_2,t+\Delta t)\, e^{-j2\pi f_2\xi_2} d\xi_2\right)\right]$$

Since $\int_{-\infty}^{\infty} c(\xi, t_0)\, e^{-j2\pi f \xi}\, d\xi \triangleq C(f, t_0)$.



and $y(t_0) = \int_{-\infty}^{\infty} c(\xi, t_0)\, x(t_0 - \xi)\, d\xi$

$= c(\xi, t_0) * x(\xi)\big|_{\xi = t_0} = \int_{-\infty}^{\infty} C(f, t_0)\, X(f)\, e^{+j2\pi f t_0}\, df$,

$C(f_0, t_0)$ means the weighting of the signal $X(f)\, e^{j2\pi f t_0}$ at $f = f_0$.

As time (t_0) changes (t_0+1), the weighting at f_0 changes.

Therefore,

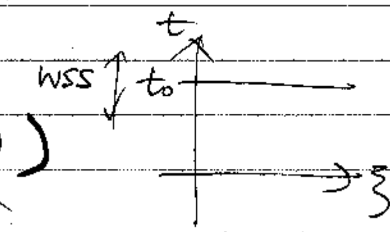
$$\phi_C(\Delta f; \Delta t)$$

means the correlation b/w $C(f, t)$ and $C(f+\Delta f, t+\Delta t)$.

spaced-frequency

spaced-time

- Consider $\tau = 0$ and define



$$\phi_C(\Delta f) \triangleq \phi_C(\Delta f; 0)$$

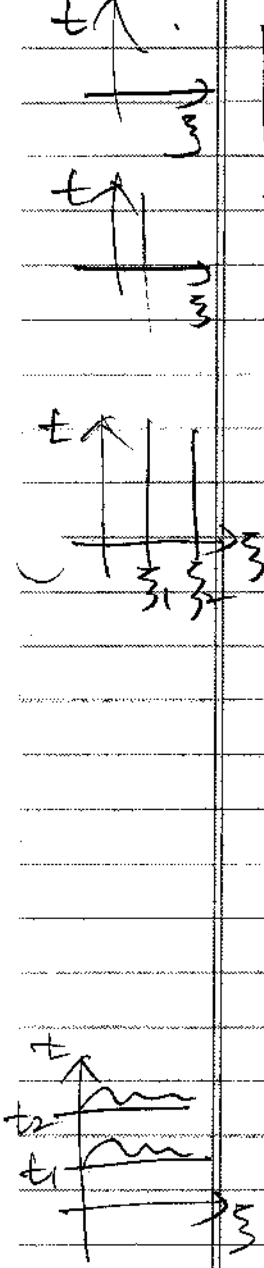
tells us **frequency coherence** of the channel.

at $t_1 = t_2$ ($= t_0$)

correlation b/w the components at f and $f + \Delta f$.

Channel correlation property 3 (frequency Domain)

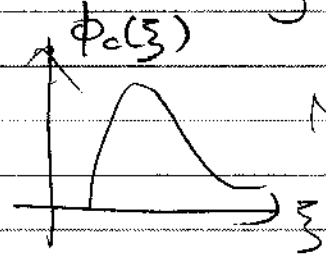
- Review of what we have done so far.



- $C(\xi,t)$ modeled as a zero-mean proper-complex Gaussian random field.
- $C(\xi_0;t)$'s are modeled as jointly WSS uncorrelated random processes
- $E[C(\xi_1,t_1)^* C(\xi_2,t_2)] = \phi_c(\xi_1, t_2-t_1)\delta(\xi_1-\xi_2)$

  $\underbrace{\phi_c(\xi,\tau)}_{\text{auto-correlation function at } \xi.}$ characterizes the property

  ~~$\phi_c(\xi,0)$ tells us the power at each~~ ξ



MIP

spaced-frequency
spaced-time
correlation function

- $E[C(f_1,t_1)^* C(f_2,t_2)] = \phi_C(f_1, f_2; t_2-t_1)$

$$\triangleq \phi_C(\Delta f; \Delta t)$$

$$= \mathcal{F}_{\xi}\{\phi_c(\xi,\Delta t)\}$$

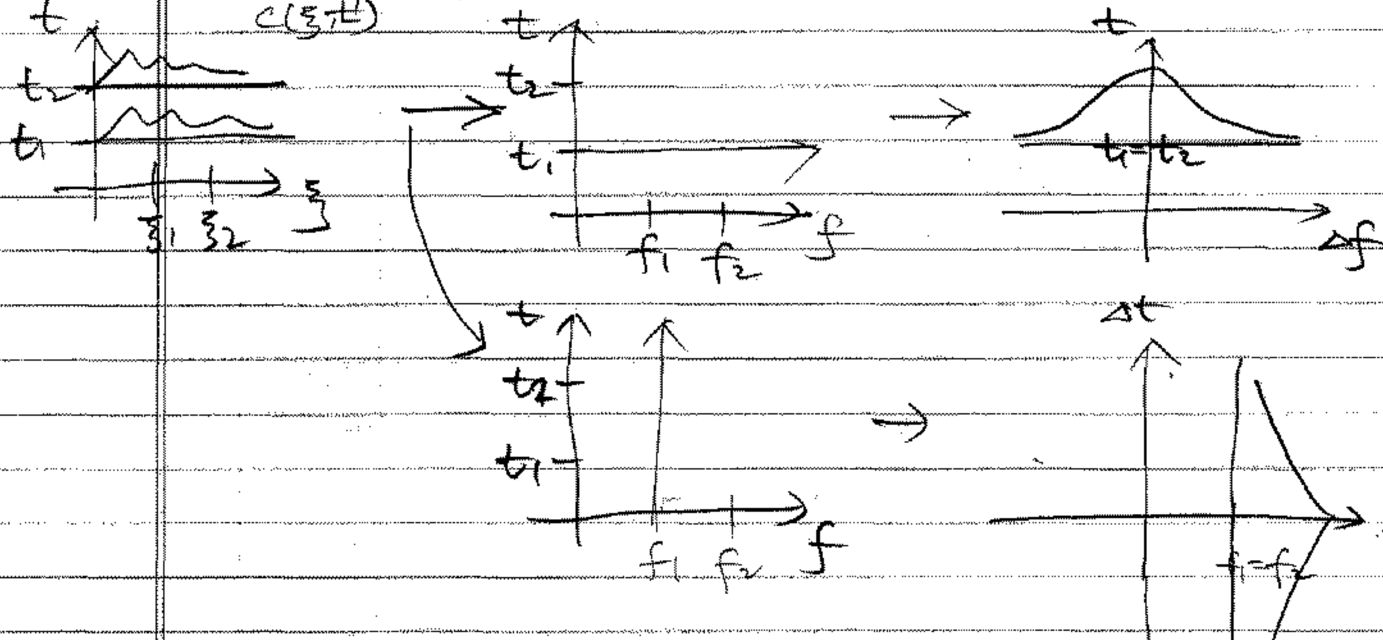
$t_1=t_2$

$$\phi_C(\Delta f; 0) = \mathcal{F}\{\phi_c(\xi)\}$$

coherence bandwidth

MIP ; delay spread

• Now, we focus on $f_1 = f_2$ case of the spaced-frequency spaced-time correlation function



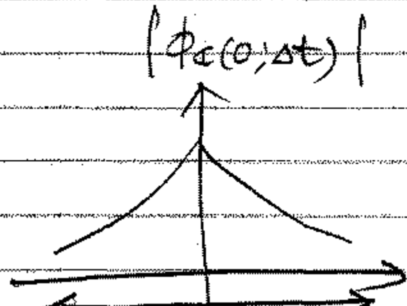
Define

$$S_C(\Delta f;\lambda)=\int_{-\infty}^{\infty}\phi_C(\Delta f;\Delta t)\,e^{-j2\pi\lambda\Delta t}\,d\Delta t$$

Set $\Delta f = 0$ and define

$$S_C(\lambda) \triangleq S_C(0,\lambda)$$

$$= \int_{-\infty}^{\infty} \underbrace{\phi_C(0;\Delta t)}_{\text{autocorrelation of a frequency component at different time instants } t \text{ \& } t+\Delta t} e^{-j2\pi\lambda\Delta t}\, d\Delta t$$

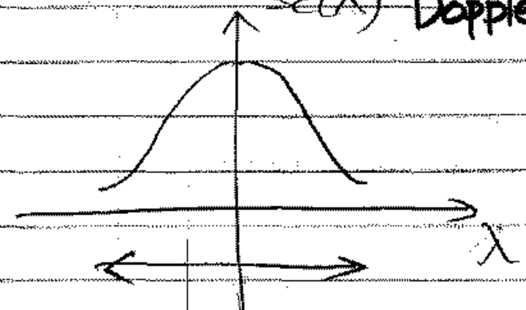


coherence time $\simeq \frac{1}{B_d}$.

spaced-time autocorrelation function

$S_C(\lambda)$ is called the **Doppler power spectrum** of the channel.

$S_c(\lambda)$ Doppler PS.



Doppler spread $\simeq B_d$

- Why called Doppler spread?

ex/ $c(\xi,t) = \alpha(t)\,\delta(\xi)$

$$\phi_c(f_1,f_2;\Delta t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[c(\xi_1,t)^* c(\xi_2,t+\Delta t)\right] e^{j2\pi(f_1\xi_1 - f_2\xi_2)}\, d\xi_1 d\xi_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[\alpha(t)\overset{*}{\alpha}(t+\Delta t)\right] \delta(\xi_1-\xi_2)\, e^{j2\pi(\quad)}\, d\xi_1 d\xi_2$$

$$= \int_{-\infty}^{\infty} E\left[\alpha(t)\overset{*}{\alpha}(t+\Delta t)\right] e^{j2\pi\Delta f\xi_1}\, d\xi_1$$

$$= \underbrace{E\left[\alpha(t)^*\alpha(t+\Delta t)\right]}_{\phi_c(\Delta t)}\,\delta(\Delta f)$$

$$S_c(\lambda) = \mathcal{F}\{\phi_c(\Delta t)\}$$

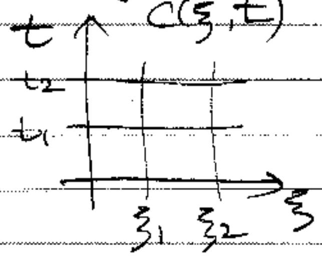
○ Scattering function.

2-D Fourier transform of spaced-freq spaced-time correlation function $\phi_c(\Delta f;\Delta t)$

$$S(\tau;\lambda) \triangleq 2D\,\mathcal{F}\{\phi_c(\Delta f;\Delta t)\}$$

Summary

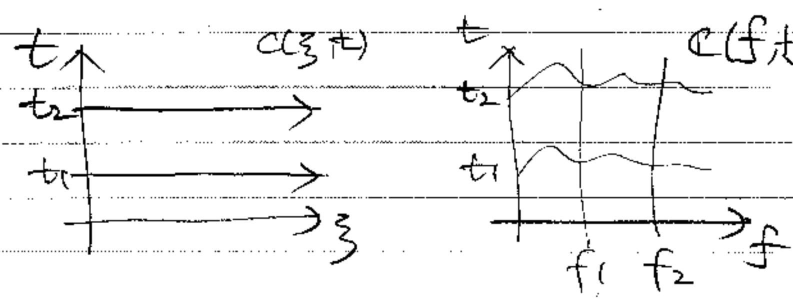
space-time correlation function

(i) 

$$E\left[C(\xi_1, t_1)\overset{*}{C}(\xi_2, t_2)\right]$$

$$= \phi_C(\xi_1; \underbrace{t_2 - t_1}_{\Delta t})\,\delta(\xi_1 - \xi_2)$$

$\because$ WSSUS.

(ii) 

$$E\left[C(f_1, t_1)^* C(f_2, t_2)\right] \triangleq DFT\{\ \ \}$$

spaced-freq spaced-time correlation ft. $= \Phi_C(\Delta f; \Delta t)$

$\because$ WSSUS.

(iii) $\phi_C(\Delta f; 0)$ spaced-freq. correlation ft.
→ coherence bandwidth

$\updownarrow$ FT

$\phi_C(\Delta\xi)$ Multipath Intensity profile (MIP)
→ delay spread

(iv) $\phi_C(0; \Delta t)$ spaced-time correlation ft
→ coherence time

$\updownarrow$ FT

$S_C(\lambda)$ Doppler power spectrum
→ doppler spread

Wireless Channels : Scattering function

○ Doubly spread channel (PZB 8-2.3)

- A **Doubly spread channel** is a WSSUS channel that spreads the signal transmitted through the channel in time and frequency.

- A **singly spread degenerate channel** is a WSSUS channel that spreads the signal transmitted through the channel only in one domain.

○ Relation b/w correlation functions

- $\phi_c(\xi_1, \Delta t)\,\delta(\xi_1-\xi_2)$ **space-time correlation function**

  $\Updownarrow$ 2-D Fourier transform pair w/ $\xi_1 f_1$, $\xi_2 f_2$

  $\phi_c(\Delta f; \Delta t)$ **spaced-frequency spaced-time correlation function**

  $\Updownarrow$ 2-D Fourier transform pair w/ $\Delta f \tau$, $\Delta t \lambda$

  $S(\tau; \lambda)$ **scattering function**

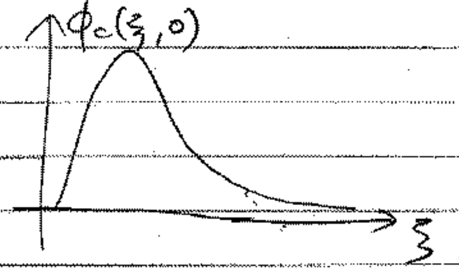
  $\Updownarrow$ 1-D Fourier transform pair w/ $\Delta t \lambda$

  $\phi_c(\tau, \Delta t)$

  1-D Fourier transform pair w/ $\Delta f \tau$

③ Scattering function

(i) The MIP was defined as $\phi_c(\xi, 0)$ and the delay spread was defined as the length of the interval that contains the support of $\phi_c(\xi,0)$.



The related definitions are **the mean delay** and **the mean-square delay spread** defined as

$$m_R = \frac{\int_{-\infty}^{\infty} \xi \phi_c(\xi,0) d\xi}{\int_{-\infty}^{\infty} \phi_c(\xi,0) d\xi}$$

and

$$L = \frac{\int_{-\infty}^{\infty} \xi^2 \phi_c(\xi,0) d\xi}{\int_{-\infty}^{\infty} \phi_c(\xi,0) d\xi} - m_R^2$$

respectively.

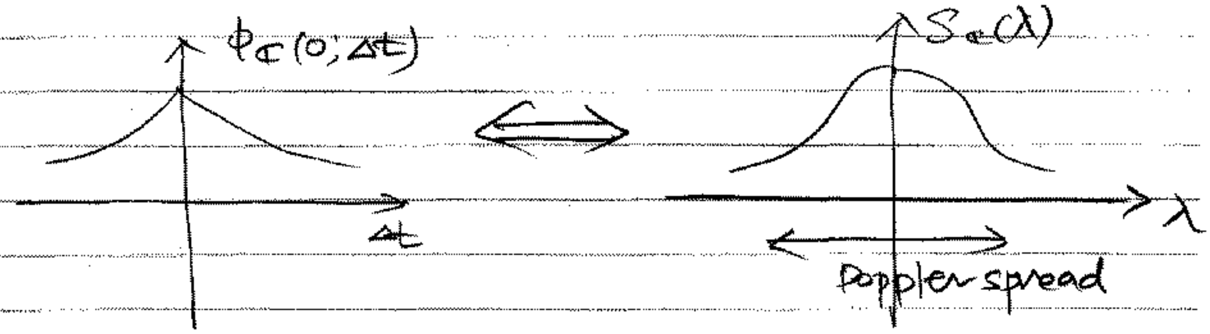
In terms of the scattering functions,

$$m_R = \frac{\int_{-\infty}^{\infty} \xi \int_{-\infty}^{\infty} S(\xi,\lambda) e^{j2\pi \Delta t \lambda} d\lambda \Big|_{\Delta t=0} d\xi}{\quad}$$

$$= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau S(\tau,\lambda) d\lambda d\tau}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\tau,\lambda) d\lambda d\tau}$$

and
$$L = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau^2 S(\tau,\lambda) d\lambda d\tau}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\tau,\lambda) d\lambda d\tau} - m_R^2$$

(ii) The Doppler spread was defined in terms of $S_C(\lambda)$, which is the Fourier transform of $\phi_C(0, \Delta t)$



The related definitions are **the mean Doppler spread** and **the mean-square Doppler spread** defined as

$$m_R = \frac{\int_{-\infty}^{\infty} \lambda S_C(\lambda) d\lambda}{\int_{-\infty}^{\infty} S_C(\lambda) d\lambda}$$

$$= \frac{\int_{-\infty}^{\infty} \lambda \int_{-\infty}^{\infty} \phi_C(\Delta f;\Delta t) e^{-j2\pi\lambda\Delta t} d\Delta t \Big|_{\Delta f=0} d\lambda}{\cdots}$$

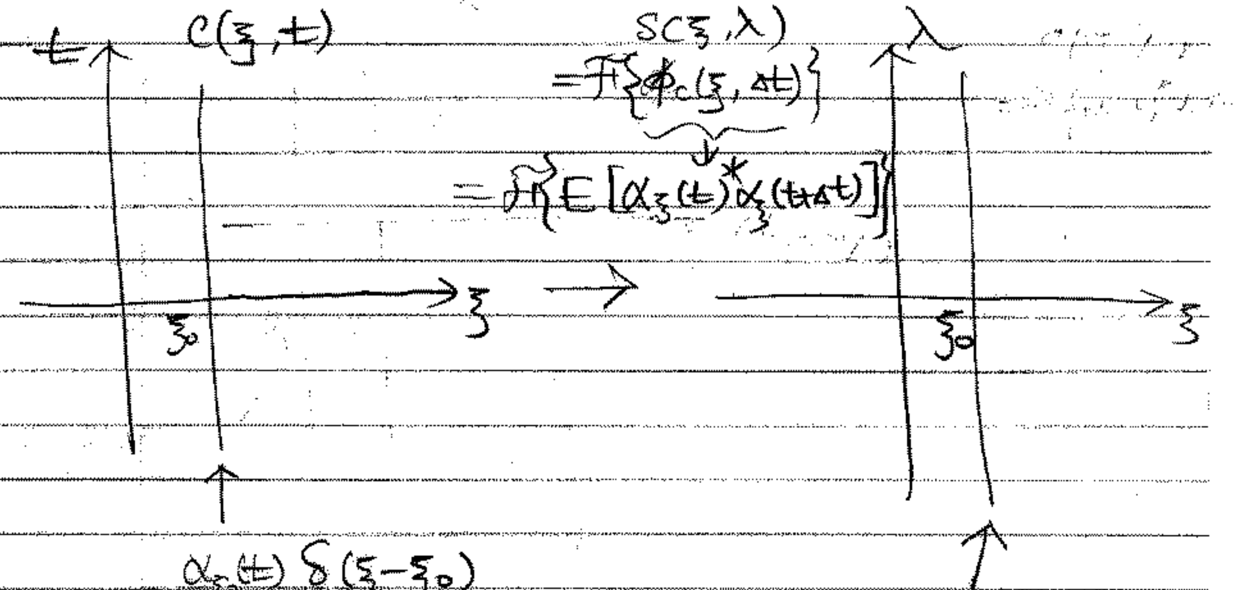
$$= \frac{\int_{-\infty}^{\infty} \lambda \int_{-\infty}^{\infty} S(\tau,\lambda) e^{j2\pi\tau\Delta f} d\tau \Big|_{\Delta f=0} d\lambda}{\cdots}$$

$$= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lambda S(\tau,\lambda) d\tau d\lambda}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\tau,\lambda) d\tau d\lambda}$$

$$B = \frac{\int_{-\infty}^{\infty} \lambda^2 S_C(\lambda) d\lambda}{\int_{-\infty}^{\infty} S_C(\lambda) d\lambda} - m_R^2$$

$$= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lambda^2 S(\tau,\lambda) d\lambda d\tau}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\tau,\lambda) d\lambda d\tau} - m_R^2$$

- Thus, the scattering function $S(\xi,\lambda)$ gives us the information about the relation b/w the propagation delay and associated multiplicative process' PSD.



$$S(\xi,\lambda) = \mathcal{F}\{\phi_c(\xi,\Delta t)\} = \mathcal{F}\{E[\alpha_\xi^*(t)\alpha_\xi(t+\Delta t)]\}$$

$\alpha_{\xi_0}(t)\,\delta(\xi-\xi_0)$

contains the information of the power spectrum of $\alpha_{\xi_0}(t)$.

Since $\alpha_{\xi_0}(t)$ is strictly stationary, $\alpha_{\xi_0}(t)\delta(t-\xi_0)$ has the PSD that is the convolution of $S(\xi_0,\lambda)$ and the PSD of $\delta(t-\xi_0)$.