EECE695f Fall 05

Homework #4

(5) 1. A Gaussian random vector $\underline{x} \in \mathbb{R}^N$ is defined as a random vector with the characteristic function

$$\varphi_{\underline{x}}(\underline{\omega}) = \exp\left(-\frac{1}{2}\underline{\omega}^T R \underline{\omega} + j\underline{\omega}^T \underline{\mu}\right)$$

where $\underline{\mu} \triangleq E\{\underline{x}\}$
$R \triangleq \mathrm{Cov}\{\underline{x}\}$

justify that

$$\underline{r} = A\underline{x} + \underline{b}$$

is also Gaussian, where
$A \in \mathbb{R}^{M \times N}$
$\underline{b} \in \mathbb{R}^N$
are deterministic.

(10) 2. When $X \sim N(0,1)$, we define

$$Q(x) = \Pr(\{X \geq x\}).$$

Using this Q-function,
Answer the following questions.

(a) When $Y \sim N(\mu, \sigma^2)$ find
$\Pr(a \leq Y < b)$

(b) When $\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \sim N(\mu_1, \mu_2; \sigma_1, \sigma_2, \rho)$,
find $\Pr(c_1 \leq aZ_1 + bZ_2 \leq c_2)$
where $c_1 < c_2$

(c) In (b), find
$\Pr(c_1 \leq Z_1 \leq c_2 \mid Z_2 = d)$

(d) Show that
$$f(r) \triangleq Q\left(\frac{r-a}{\sigma}\right) + Q\left(\frac{b-r}{\sigma}\right)$$
where $\sigma > 0$, $b > a$, is minimized at
$r = \frac{a+b}{2}$.

(e) Show that
$$E\left[Q\left(\frac{1}{\sqrt{W}}\right)\right] \geq Q\left(\frac{1}{\sqrt{E(W)}}\right)$$
if
$$\Pr\left(0 < W \leq \frac{1}{3}\right) = 1$$

(5) 3 When $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$;

we define

$$Q\left(\frac{\chi}{\sigma}, \frac{y}{\sigma}\right) \triangleq \Pr\left(\sqrt{X_1^2 + X_2^2} \geq y\right), \quad y \geq 0$$

where $\chi \triangleq \sqrt{\mu_1^2 + \mu_2^2}$.

Answer the following questions

(a) Show that, when

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

we have

$$\Pr\left(Y_1^2 + Y_2^2 \geq y^2\right) = Q\left(\frac{\chi}{\sigma}, \frac{|y|}{\sigma}\right)$$

for all $\theta \in [0, 2\pi)$

(b) Show that

$$Q\left(0, \frac{y}{\sigma}\right) = e^{-\frac{y^2}{2\sigma^2}}, \quad \text{for } y \geq 0$$

(c) Find $\Pr(X_1 \geq 0, X_2 \geq 0)$

(5) 4 When $\underline{X} \sim N(\underline{\mu}, \sigma^2 \underline{I})$ where $R \in \mathbb{R}^4$ is invertible, answer the following questions (Use the Q, Marcum's Q, or generalized Marcum's Q functions.)

(a) Find $\Pr(\|\underline{x}-\underline{a}\| \le \|\underline{x}-\underline{b}\|)$, where $\underline{a}, \underline{b} \in \mathbb{R}^4$

(b) Find $\Pr(\|\underline{X}-\underline{c}\| \le d)$, where $\underline{c} \in \mathbb{R}^4$ and $d \ge 0$

(5) 5. Suppose that $X \sim \tilde{N}(\mu; 1, 0)$ w/ μ a complex number.
Answer the following questions

(a) Find $\Pr(\mathrm{Re}\{X\} \geq x)$

(b) Find $\Pr(\{\mathrm{Re}\{X\} \geq a\} \cap \{\mathrm{Im}\{X\} \geq b\})$

(c) Find $\Pr(C_1 \leq |X| \leq C_2)$, where $0 \leq C_1 < C_2$

(5) 6. Suppose that $\underline{X} \sim \tilde{N}(\underline{\mu};\, 2\sigma^2 I,\, 0)$ where

$$\underline{\mu} \in \mathbb{C}^4$$

Answer the following questions.

(a) Find $\Pr(\|\underline{X} - \underline{a}\| \le \|\underline{X} - \underline{b}\|)$, where $\underline{a}, \underline{b} \in \mathbb{C}^4$

(b) Find $\Pr(\|\underline{X} - \underline{c}\| \le d)$, where $\underline{c} \in \mathbb{C}^4$ and $d \ge 0$

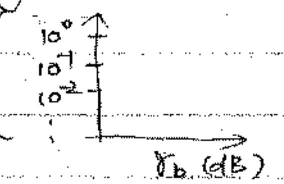
(5) 7. MATLAB has a built-in function called erfc.m defined as

$$\operatorname{erfc}(x) \triangleq \int_x^\infty \frac{2}{\sqrt{\pi}} e^{-t^2}\,dt.$$

(a) Find the relation between $Q(x)$ and $\operatorname{erfc}(x)$

(b) Suppose that γ_b represents the signal-to-noise ratio and that the average P_e of a communication system is given by

$$P_e = Q(\sqrt{2\gamma_b}), \quad \gamma_b \ge 0$$

Plot the graph of P_e versus γ_b using MATLAB. ()

Note: For x-axis use γ_b in [dB]. Let $\tilde{\gamma}_b$ be γ_b in [dB]. Then,

$$\tilde{\gamma}_b = 10 \log_{10} \gamma_b$$

Use the following MATLAB commands:

semilogy   xlabel
log10   ylabel
grid   axis