Review of handout #22

- Random field $c(\xi, t)$

$$Y(t) = \int_{-\infty}^{\infty} c(\xi, t)\, X(t-\underset{\substack{\uparrow \\ \text{delay}}}{\xi})\, d\xi$$

- WSSUS

(i) WSS
Given ξ, $c(\xi, t)$ is a WSS random process

(ii) US
Given $\xi_1 \neq \xi_2$, two random processes $c(\xi_1, t)$ and $c(\xi_2, t)$ are uncorrelated

(i) and (ii) implies

space (delay) time

$$E\{c(\xi_1, t_1)^* c(\xi_2, t_2)\} \triangleq \underbrace{\phi_c(\xi_1, t_2-t_1)\, \delta(\xi_1-\xi_2)}$$

called space-time correlation function

- TD

$$R_{YY}(t_1, t_2) \triangleq E\{Y(t_1)^* Y(t_2)\} = E\left\{\left(\int_{-\infty}^{\infty} c(\xi_1, t_1) X(t_1-\xi_1) d\xi_1\right)^* \left(\int_{-\infty}^{\infty} c(\xi_2, t_2) X(t_2-\xi_2) d\xi_2\right)\right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{c(\xi_1, t_1)^* c(\xi_2, t_2)\} E\{X(t_1-\xi_1)^* X(t_2-\xi_2)\}\, d\xi_1 d\xi_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_c(\xi_1, t_2-t_1)\, \delta(\xi_1-\xi_2)\, R_{XX}(t_1-\xi_1, t_2-\xi_2)\, d\xi_1 d\xi_2$$

$$= \int_{-\infty}^{\infty} \phi_c(\xi_1, t_2 - t_1)\, R_{XX}(t_1 - \xi_1, t_2 - \xi_1)\, d\xi_1$$

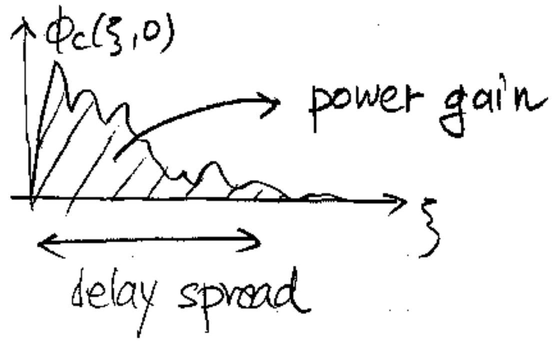
$$R_{YY}(t, t+\Delta t) = \int_{-\infty}^{\infty} \phi_c(\xi_1, \Delta t)\, R_{XX}(t - \xi_1, t + \Delta t - \xi_1)\, d\xi_1$$

$$\begin{aligned}
\mathcal{R}_{YY}(\Delta t) &= A\{R_{YY}(t, t+\Delta t)\} \\
&= \int_{-\infty}^{\infty} \phi_c(\xi_1, \Delta t)\, A\{R_{XX}(t - \xi_1, t + \Delta t - \xi_1)\}\, d\xi_1 \\
&= \int_{-\infty}^{\infty} \phi_c(\xi_1, \Delta t)\, \mathcal{R}_{XX}(\Delta t)\, d\xi_1 \\
&= \left[\int_{-\infty}^{\infty} \phi_c(\xi, \Delta t)\, d\xi\right] \times \mathcal{R}_{XX}(\Delta t)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\underbrace{A\left[E\{|Y(t)|^2\}\right]}_{\text{output power}} &= \mathcal{R}_{YY}(0) \\
&= \left(\int_{-\infty}^{\infty} \phi_c(\xi, 0)\, d\xi\right) \mathcal{R}_{XX}(0) \\
&= \underbrace{\int_{-\infty}^{\infty} E\{|c(\xi, t)|^2\}\, d\xi}_{\text{power gain}} \times \underbrace{A\left[E\{|X(t)|^2\}\right]}_{\text{input power}}
\end{aligned}$$

where $\phi_c(\xi, 0) \triangleq E\{|c(\xi, t)|^2\}$ is called the multipath intensity profile (MIP) or power delay profile.



- FD

$$S_{rr}(\lambda) = \mathcal{F}\{R_{rr}(\Delta t)\} \triangleq \int_{-\infty}^{\infty} R_{rr}(\Delta t) e^{-j2\pi\lambda\Delta t} d\Delta t$$

$$= \mathcal{F}\left\{ \int_{-\infty}^{\infty} \phi_c(\xi, \Delta t) d\xi \right\} * \mathcal{F}\{R_{xx}(\Delta t)\}$$

$$\underbrace{\phantom{S_{rr}(\lambda)}}_{\text{output PSD}} = \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_c(\xi, \Delta t) e^{-j2\pi\lambda\Delta t} d\Delta t\, d\xi}_{\text{Doppler PSD} \triangleq S_c(\lambda)} * \underbrace{S_{xx}(\lambda)}_{\text{input PSD}}$$

- Spaced-frequency Spaced-time correlation function

Consider double Fourier transform of space-time correlation function.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\phi_c(\xi_1, \Delta t)\delta(\xi_1 - \xi_2)}_{E\{c(\xi_1, t)^* c(\xi_2, t+\Delta t)\}} e^{j2\pi(f_1\xi_1 - f_2\xi_2)} d\xi_1 d\xi_2$$

$$= \int_{-\infty}^{\infty} \phi_c(\xi_1, \Delta t) e^{-j2\pi\xi_1(f_2 - f_1)} d\xi_1$$

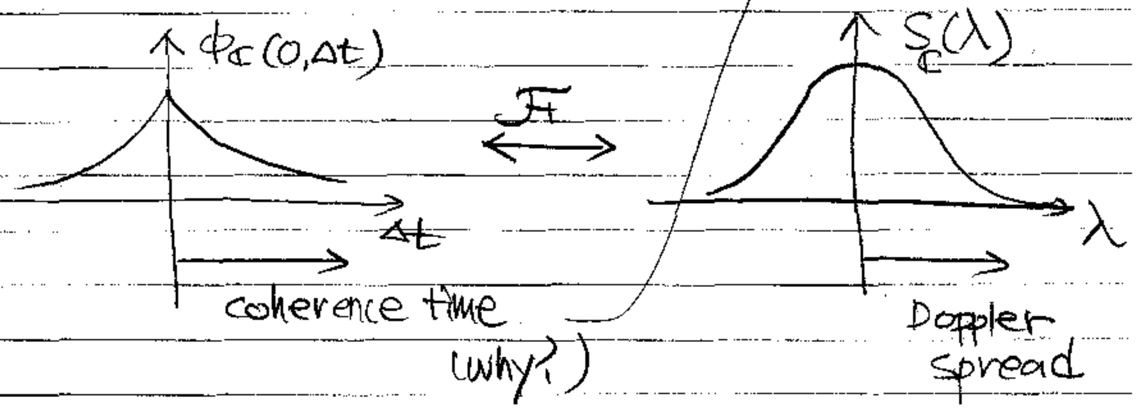
$$\triangleq \underbrace{\phi_C(\Delta f, \Delta t)}_{\substack{\text{spaced-frequency} \\ \text{spaced-time}}} \text{ correlation function} \qquad \text{where } \Delta f = f_2 - f_1$$

$$\| \quad E\left[\left(\int_{-\infty}^{\infty} c(\xi_1, t_1) e^{-j2\pi f_1 \xi_1} d\xi_1\right)^* \left(\int_{-\infty}^{\infty} c(\xi_2, t_2) e^{-j2\pi f_2 \xi_2} d\xi_2\right)\right] \triangleq E\left[C(f_1, t_1)^* C(f_2, t_2)\right]$$

- Since $\phi_c(\xi_1, \Delta t)\delta(\xi_1 - \xi_2) \leftrightarrow \phi_C(\Delta f, \Delta t)$, let's represent the derived entities in terms of $\phi_C(\Delta f, \Delta t)$

(i) Doppler PSD

$$S_C(\lambda) = \mathcal{F}_t\{\underbrace{\phi_C(0,\Delta t)}_{\text{spaced-time correlation function}}\} = \mathcal{F}_t\{E[C(f,t_1)^* C(f,t_2)]\}$$



(ii) Multipath Intensity Profile

$$\phi_C(\xi,0) = \int_{-\infty}^{\infty} \phi_C(\Delta f,0)\, e^{j2\pi\xi\Delta f}\, d\Delta f$$

$$= \mathcal{F}_t^{-1}\{\underbrace{\phi_C(\Delta f,0)}_{\text{spaced-frequency correlation function}}\} = \mathcal{F}_t^{-1}\{E[C(f_1,t)^* C(f_2,t)]\}$$