Complex baseband representation of real-valued bandpass signals

Title dissection

- **representation**

$$\begin{cases} x(t) : \text{a time function} \\ X(f) \triangleq \mathcal{F}\{x(t)\} : \text{a frequency domain representation of } x(t) \end{cases}$$

- complex baseband signal $\overset{?}{=}$ real bandpass signal

  in what sense?

- why **real bandpass** signals & systems do we deal with?

- What is the advantage of handling **complex baseband** representation of real bandpass signals and systems?

- Next topic:
  complex baseband representation of real-valued bandpass **random processes**.

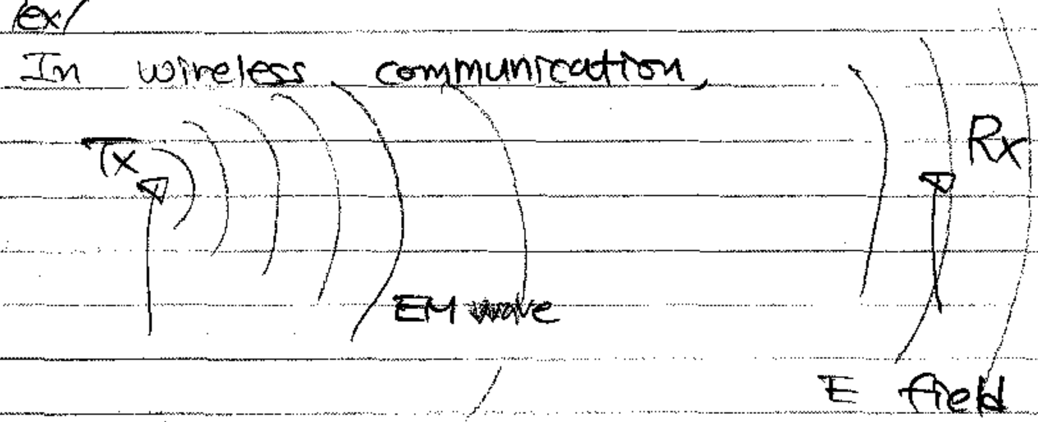
○ Why real bandpass signals?

- Electrical signals are **real**-valued.

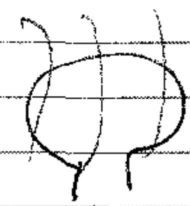
    current
    voltage

ex/

- In wireless communication,

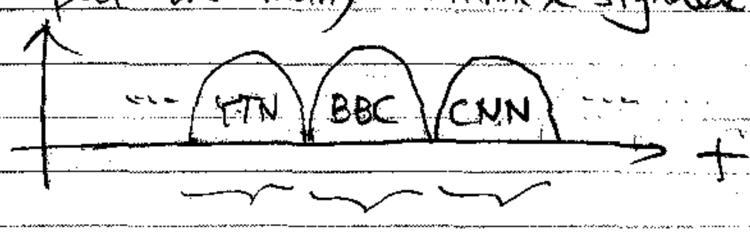


E field generates current in antenna.



B field generates current in antenna.

- In many systems, spectrum is a valuable resource. For example, in a cable for CATV, we want to put as many channel signals as possible.



Each signal is a **bandpass** signal.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○ What is the **advantage** of handling complex baseband signals instead of real bandpass signals?

- 2 real numbers are equivalent to 1 complex number.

  There are many cases where handling a complex number is much **easier** than " two real numbers.

- In real bandpass signaling, it can be shown that two real baseband signals contain "all" the information of the real bandpass signal. f_c

$x(t) \equiv x_c(t), x_s(t) \equiv z(t) \triangleq x_c(t) + j x_s(t)$

one real bandpass **two real baseband** **one complex baseband**

$+\alpha$

① The main result.

Given
$x(t)$, a real-valued "bandpass" signal with finite energy and

$X(f) \triangleq \mathcal{F}\{x(t)\}$ that has a finite support $f \in [0, B]$.
(= **bandlimited**)

We choose f_c and θ, $f_c \in (\frac{B}{2}, B)$
$\theta \in [0, 2\pi)$

Then, there **exists** a **unique pair** $x_c(t)$ and $x_s(t)$ of real-valued band-limited "baseband" signals such that

$$x(t) \overset{?}{=} x_c(t)\cos(2\pi f_c t + \theta) - x_s(t)\sin(2\pi f_c t + \theta)$$

$$= \mathrm{Re}\left\{ \left(x_c(t) + j\, x_s(t)\right) e^{j(2\pi f_c t + \theta)} \right\}$$

where $j \triangleq \sqrt{-1}$.

- The complex baseband signal $x_c(t) + j x_s(t)$, f_c, and θ contains all the information carried by $x(t)$.

- $e^{j\phi} = \cos\phi + j\sin\phi$ (Euler's identity)

- $\overset{?}{=}$: $\int_{-\infty}^{\infty} \left| x(t) - \left(x_c(t)\cos 2\pi f_c t - x_s(t)\sin 2\pi f_c t \right) \right|^2 dt = 0$

mean-square sense.
= almost everywhere ?!

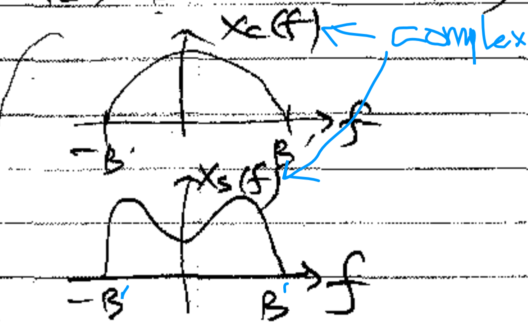
- $x_c(t)$: in-phase component
$x_s(t)$: quadrature component $\frac{2\pi}{4} \equiv 90°$

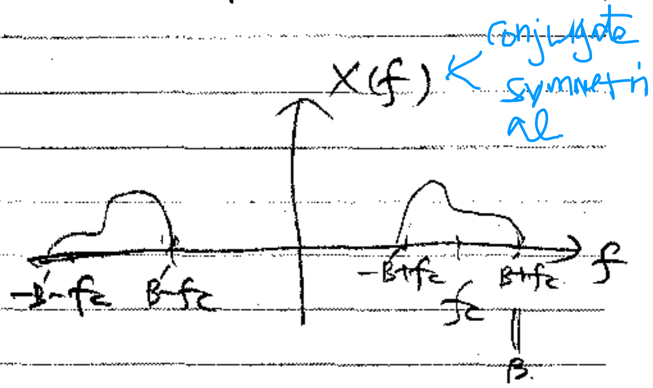
○ Derivation

• It is obvious that

$x_c(t)$ & $x_s(t)$ are baseband, bandlimited real-valued signals. $f_c > B'$ $\theta \in [0, 2\pi)$

$\Longrightarrow$ $x(t) \triangleq x_c(t)\cos(2\pi f_c t + \theta) - x_s(t)\sin(2\pi f_c t + \theta)$ is a real-valued bandpass signal





$\left.\begin{array}{l} f_c + B' = B \\ f_c > B' \end{array}\right\} \Rightarrow f_c \in \left(\frac{B}{2}, B\right)$

• The main result says that a kind of the converse is also true.

• How can we prove?

In the time domain, we have no idea.

↓

We tackle the problem in the frequency domain

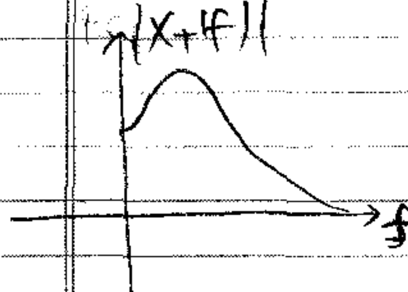
- Since $x(t)$ is real, its Fourier transform $X(f)$ has conjugate symmetry, i.e.,

$$X(f) = X(-f)^*$$

why?

Thus, $X_+(f) \triangleq \begin{cases} 2X(f), & \text{for } f>0 \\ 0, & \text{for } f<0 \end{cases}$

contains all the information in $X(f)$. This can be easily shown as

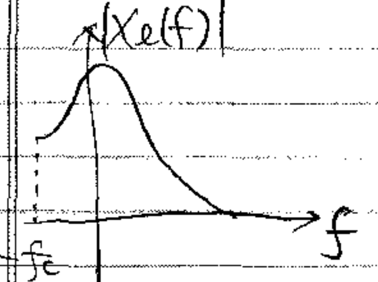


$$X(f) = \frac{X_+(f) + X_+(-f)^*}{2} \quad \cdots (*)$$

$$= \begin{cases} X(f), & \text{for } f>0 \\ X(-f)^*, & \text{for } f<0 \end{cases}$$

Def. $x_+(t) \triangleq \mathcal{F}^{-1}\{X_+(f)\}$ is called the **pre-envelope** (**analytic signal**) of $x(t)$.

- Using $f_c \in (\frac{B}{2}, B)$ and $\theta \in [0, 2\pi)$, we can define



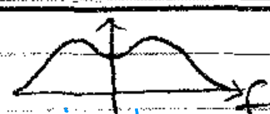
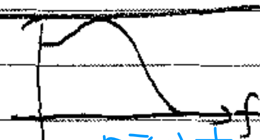
$$X_e(f) \triangleq X_+(f+f_c)\, e^{-j\theta} \quad \cdots (**)$$

This also contains all the information in $X(f)$. (Given f_c & θ)

Def. $x_e(t) \triangleq \mathcal{F}^{-1}\{X_e(f)\}$ is called **the complex-envelope of $x(t)$** with f_c & θ.

(In most cases, θ is set to zero.)

-

	$x(t)$	$x_+(t)$	$x_\ell(t)$
real / complex	real	complex	complex
baseband / bandpass	bandpass	one-sided	baseband
energy $\int_{-\infty}^{\infty} \lvert\cdot\rvert^2 dt$	E	$2E$	$2E$
spectrum*	 conjugate symmetrical	 right sided	

*only magnitudes are plotted.

- From (*) and (**), we have

$$x(t) = \frac{x_+(t) + x_+(t)^*}{2} = \mathrm{Re}\{x_+(t)\} \quad \cdots (*)'$$

and

$$x_+(t) = x_\ell(t)\, e^{j(2\pi f_c t + \theta)} \quad \cdots (**)''$$

Thus,

$$x(t) = \mathrm{Re}\{ x_\ell(t)\, e^{j(2\pi f_c t + \theta)} \} \quad \cdots (***)$$

If we define

$$x_c(t) \triangleq \mathrm{Re}\{x_\ell(t)\} \quad \&$$

$$x_s(t) \triangleq \mathrm{Im}\{x_\ell(t)\},$$

then (***) can be written as

$$= \mathrm{Re}\{ (x_c(t) + j\, x_s(t))\, e^{j(2\pi f_c t + \theta)} \}$$

$$= x_c(t) \cos(2\pi f_c t + \theta) - x_s(t) \sin(2\pi f_c t + \theta)$$

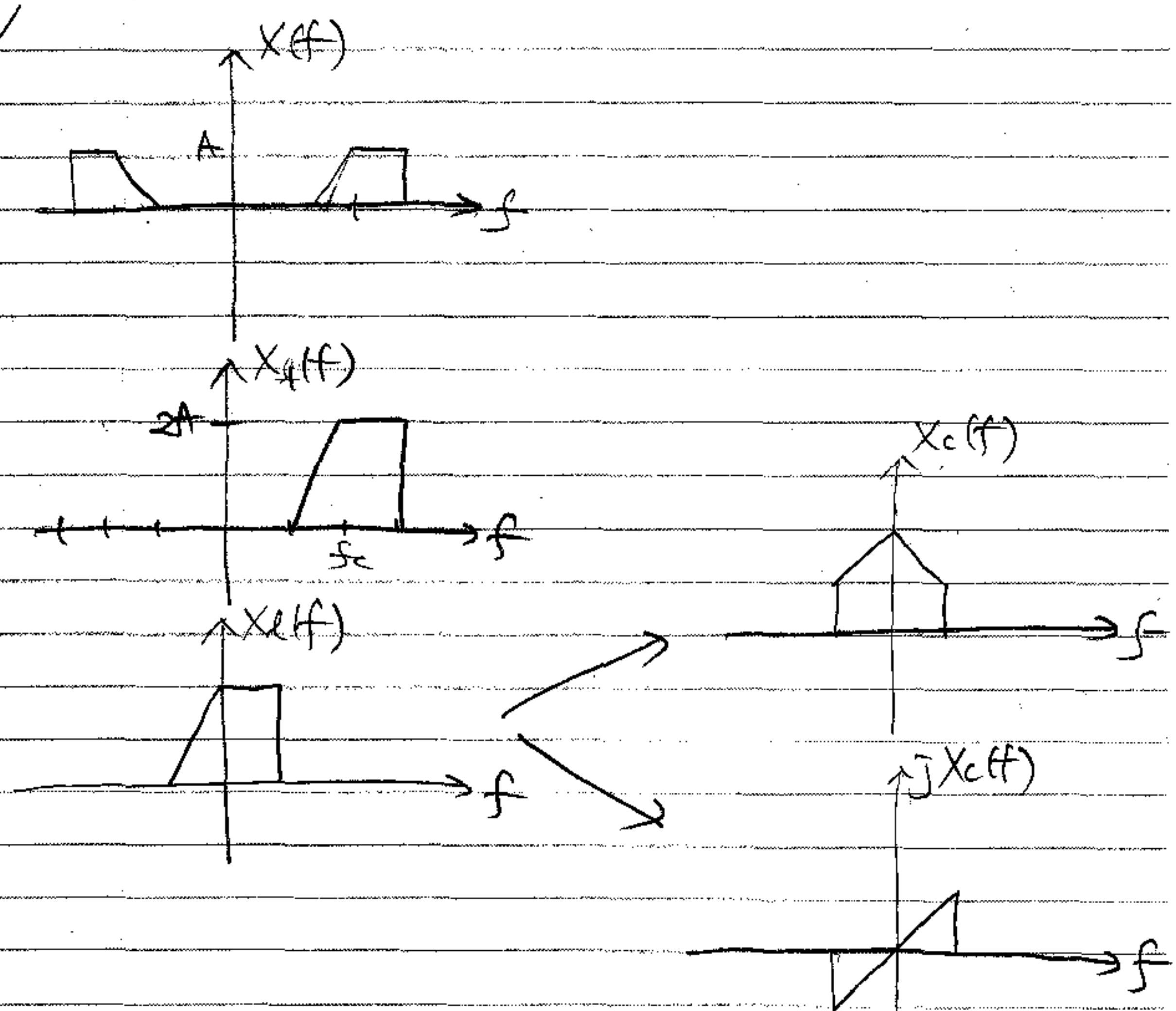
- $$x_c(t) = \frac{x_\ell(t) + x_\ell(t)^*}{2} \overset{\mathcal{F}}{\longleftrightarrow} X_c(f) = \frac{X_\ell(f) + X_\ell(-f)^*}{2}$$

$$x_s(t) = \frac{x_\ell(t) - x_\ell(t)^*}{2j} \overset{\mathcal{F}}{\longleftrightarrow} X_s(f) = \frac{X_\ell(f) - X_\ell(-f)^*}{2j}$$

$$X_\ell(f) = \underbrace{X_c(f)}_{\text{conjugate symmetrical}} + j\underbrace{X_s(f)}_{\text{anti-conjugate symmetrical.}}$$

- If $X(f)$ is real, then $X_c(f)$ is the even part of $X_\ell(f)$ and $jX_s(f)$ is the odd part of $X_\ell(f)$.

Ex /



○ In-phase / Quadrature vs Amplitude / phase

- $x(t) = Re\{ x_\ell(t) e^{j(2\pi f_c t + \theta)} \}$

$\Rightarrow$ $x_\ell(t)$ contains all the information in $x(t)$

f_c

θ

- Cartesian coordinate

$$x_\ell(t) = Re\{x_\ell(t)\} + Im\{x_\ell(t)\}$$
$$= \underbrace{x_c(t)}_{\textbf{In-phase component}} + j\underbrace{x_s(t)}_{\textbf{Quadrature component}}$$

- Polar coordinate

$$x_\ell(t) = \underbrace{a(t)}_{\textbf{amplitude}} e^{j\underbrace{\theta(t)}_{\cancel{\text{phase}}\ \text{angle}}}$$

$\Rightarrow$ Any modulation can be considered as a combination of amplitude modulation and angle modulation

- $x_\ell(t) = \hat{a}(t) e^{j\hat{\theta}(t)}$

where $\hat{a}(t) \triangleq \sqrt{x_c(t)^2 + x_s(t)^2} = |x_\ell(t)|$ ← magnitude

$\hat{\theta}(t) \triangleq \tan^{-1} \dfrac{x_s(t)}{x_c(t)} = \angle x_\ell(t)$ ← phase

called **the envelope of $x(t)$**

I-Q modulator

Q How do we generate a bandpass signal?

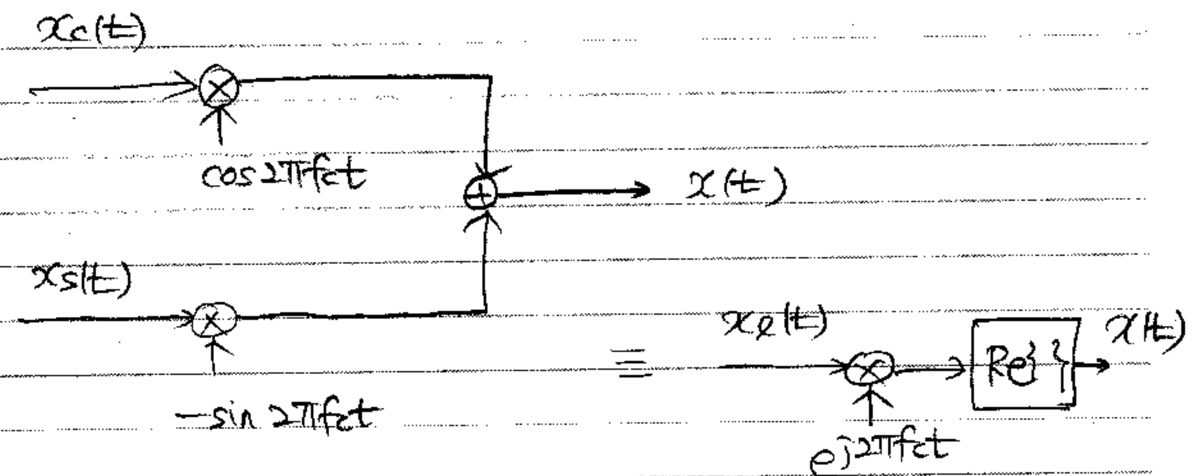
- So far, we studied the Rx (receiver) part. What about the Tx (transmitter) part?

- From

$$x(t) = \mathrm{Re}\{x_\ell(t)e^{j2\pi f_c t}\}$$

$$= x_c(t)\cos 2\pi f_c t - x_s(t)\sin 2\pi f_c t,$$

we use the **I-Q modulator**



to generate a bandpass signal

complex I-Q modulator

- However, again, to have $\cos 2\pi f_c t$ & $\sin 2\pi f_c t$ with a very high f_c is not easy.

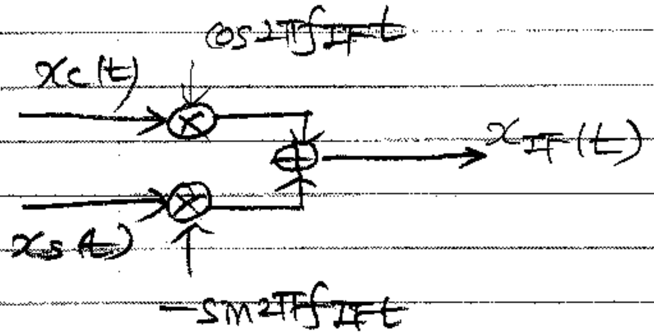
So, we use multiple stages to **up-convert** the baseband signal.

Q There is a necessary condition on f_c & B s.t. $x_c(t)$ & $x_s(t)$ be inphase and quadrature components of $x(t)$. What is this condition?

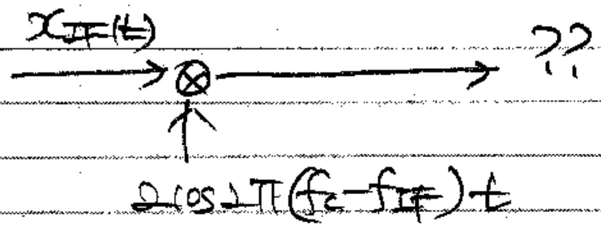
A $f_c \geq B$.

IF to RF up-conversion

- 1st stage : Quadrature modulator w/ f_{IF}



- 2nd stage: **up-conversion mixer**



- Every mixer w/ $\cos 2\pi f_0 t$ generates an image.
In this case,

$$x_{IF}(t) \times 2\cos 2\pi(f_c - f_{IF})t$$

$$= \frac{x_e(t)e^{j2\pi f_{IF}t} + x_e^*(t)e^{-j2\pi f_{IF}t}}{2} \times 2\,\frac{e^{j2\pi(f_c - f_{IF})t} + e^{-j2\pi(f_c - f_{IF})t}}{2}$$

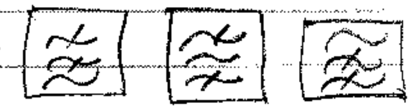
$$= \underbrace{Re\{x_e(t)e^{j2\pi f_c t}\}}_{\text{upper side band}} + \underbrace{Re\{x_e^*(t)e^{j2\pi(f_c - 2f_{IF})t}\}}_{\text{lower side band}}$$

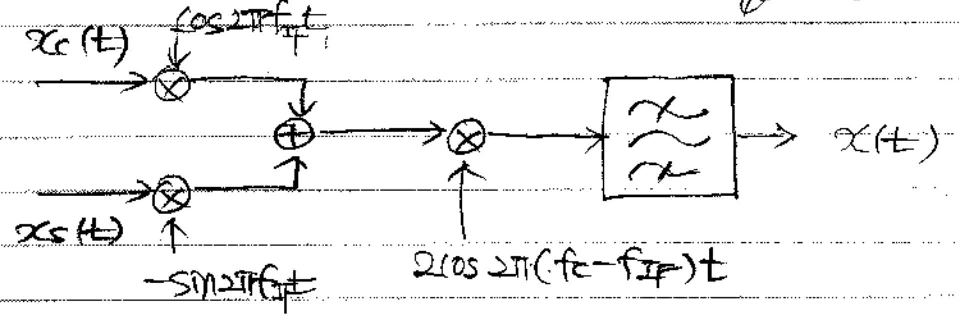
desired   image

Thus, we need a highpass filter.

- The full story !

cf)





- This part is often processed in digital, called **digital IF**. Then, D/A converted.

About digital IF, we will study later in this course.