Interpolation

Interpolation understood in time domain

- There are many different ways to convert a DT signal $x[n]$ to a CT signal $x(t)$.

  Among them, a special way is to use a single interpolation function $p(t)$ as

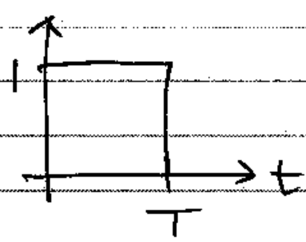
$$y(t) \triangleq \sum_{n=-\infty}^{\infty} x[n]\, p(t-nT)$$

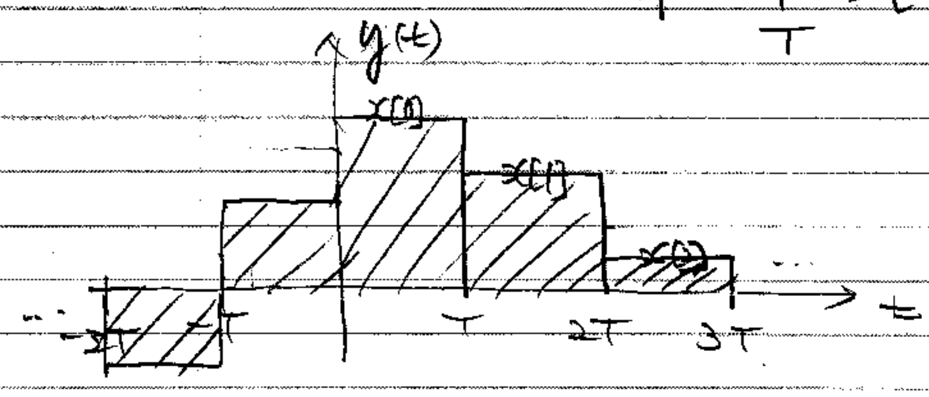
which looks like a pulse amplitude modulation.

- 2 parameters for such an interpolation are

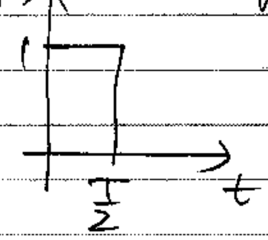
  T interpolation period and
  $p(t)$ interpolation function

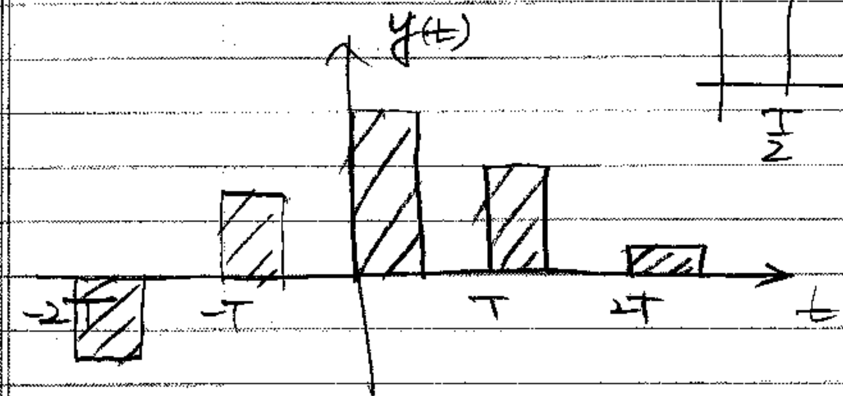
/ex 1/ a rectangular $p(t)$ w/ height 1, width T.



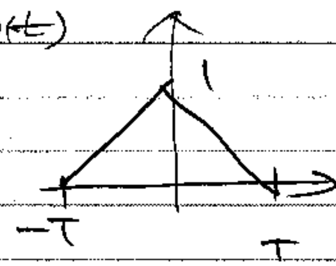


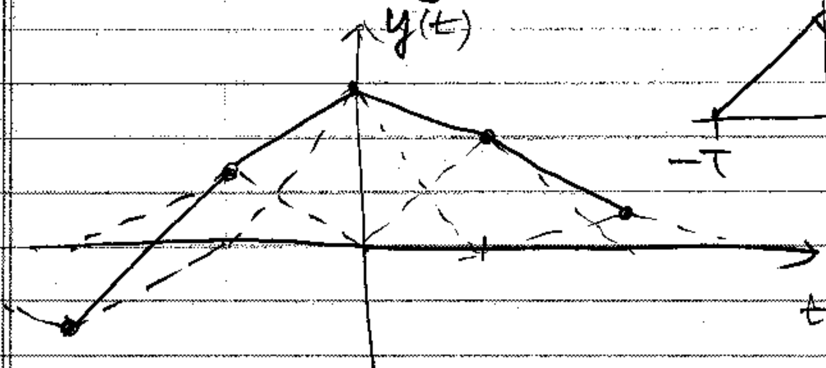
/ex2/ a rectangular p(t) w/ height 1, width $\frac{T}{2}$





/ex3/ a triangular p(t) w/ height 1, width 2T





- p(t) can be band-limited
  /ex/ p(t) can be a square-root raised cosine pulse.

○ Interpolation understood in frequency domain

Q. When $y(t) = \sum_{n=-\infty}^{\infty} x[n]\, p(t-nT)$

find **$Y_c(f)$ in terms of $X_d(f)$, T, and $P(f)$**

A. $\boxed{Y_c(f) = P(f)\, X_d(fT)}$

Sol)

$$Y_c(f) \triangleq \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n] p(t-nT) e^{-j2\pi ft} dt$$

$$= \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} p(t-nT) e^{-j2\pi ft} dt$$

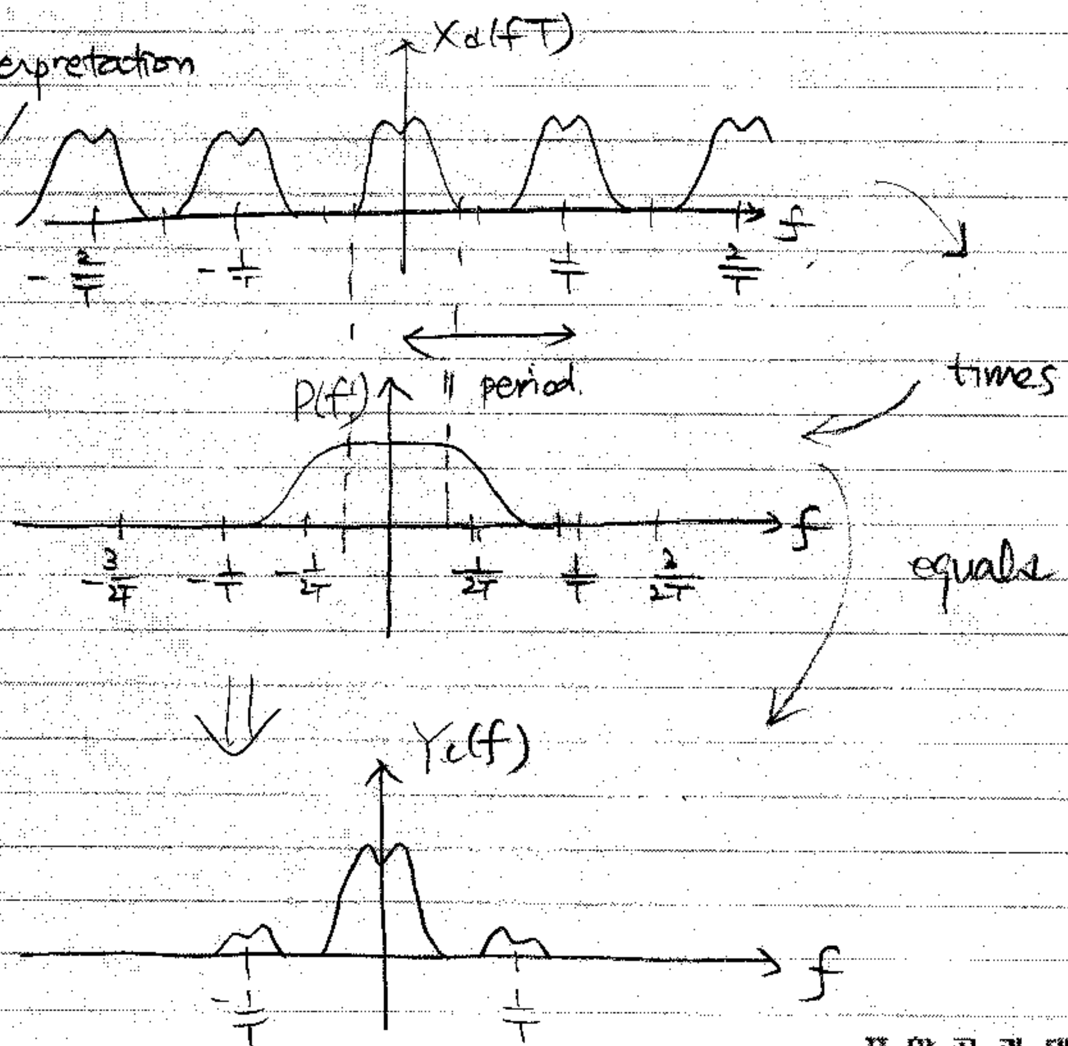
$$= \sum_{n=-\infty}^{\infty} x[n] P(f) e^{-j2\pi fnT}$$

$$= P(f) \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi fnT}$$

$$= P(f) X_d(fT)$$

- Interpretation

/ex/



bandlimited

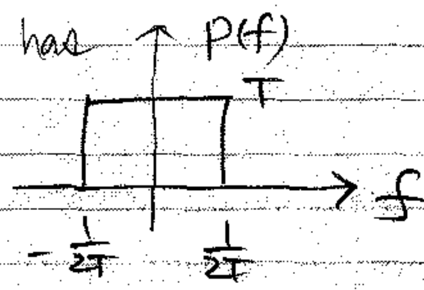
• If $x[n]$ is a sampled version of a CT signal $x(t)$ and we want $y(t)=x(t)$, then

$P(f)$ must be T for $\frac{1}{T}X_c(f)$ and
must be zero for $\frac{1}{T}X_c(f-\frac{n}{T})$ $\forall n \neq 0$

$\downarrow$

$$\because X_d(fT)=\frac{1}{T}\sum X_c\left(f-\frac{n}{T}\right) \ \& \ Y_c(f)=P(f)X_d(fT)$$

ex/ $p(t)$ has



Note if $P(f)$ is not flat for $\frac{1}{T}X_c(f)$ then there is distortion.

Note if $P(f)$ does not filter out $\frac{1}{T}X_c(f-\frac{n}{T})$ then there are high frequency terms.

○ D/A converter

Q Suppose that we want to generate

$$y(t)=\sum_{n=-\infty}^{\infty} x[n]\, p(t-nT).$$

However, if we only have an interpolator with $1/T$ & $q(t)$, what can we do?

A. We filter $x[n]$ to $\tilde{x}[n]$ such a way that

$$\tilde{X}_d(fT)Q(f)=X_d(fT)P(f).$$

$\Leftrightarrow$ We need to design a DT filter $h[n]$ such that

$$x[n] \rightarrow \boxed{h[n]} \rightarrow \tilde{x}[n]$$

with

$$\tilde{X}_d(fT) = X_d(fT)\,H(fT)$$

$$= X_d(fT)\,\underbrace{\frac{P(f)}{Q(f)}}_{\text{may not be easy to design}},$$

• Usually, a DAC upsamples the input DT signal and processes it

/ex/

$$\cdots x[-1], x[0], x[1] \cdots \rightarrow \boxed{3\times \text{ up-sample}} \rightarrow \cdots x[-1], x[-1], x[-1], x[0], x[0], x[0], x[1], x[1], x[1], \cdots$$

or $\cdots x[-1], 0, 0, x[0], 0, 0, x[1], 0, 0, \cdots$

$$\rightarrow \boxed{\text{FIR or IIR}} \rightarrow \boxed{\text{Interpolator w/ rate } 3/T \text{ and function } r(t)} \rightarrow y(t)$$

• We need to study the effect of upsampling, high-rate filtering, and interpolation on $y(t)$.

~ interpolation

Alternate derivation

$y[n] = \sum_{m=-\infty}^{\infty} x[m]\, p[n-mN]$ ← N× upsampling

$\leftrightarrow Y_d(f) = \sum_n y[n] e^{-j2\pi f n}$

$= \sum_m x[m] P_d(f) e^{-j2\pi f m N}$

$= P_d(f) X_d(fN)$

Upsampling

- N-times up-sampling of a DT signal $x[n]$ is given by

$$\cdots, 0, x[-2], \underbrace{0, 0, \cdots, 0}_{N-1 \text{ zeros}}, x[-1], \underbrace{0, 0, \cdots, 0}_{N-1 \text{ zeros}}, x[0], 0, 0, \cdots$$

$$0, x[1], 0, 0, \cdots, 0, x[2], 0, 0, \cdots, 0, x[3], 0, 0, \cdots, 0, \cdots$$

- Let $y[n] = \begin{cases} 0, & n \neq \text{integer multiple of } N \\ x[\frac{n}{N}], & n = \text{integer multiple of } N \end{cases}$

Q Find $Y_d(f)$ in terms of $X_d(f)$ and N

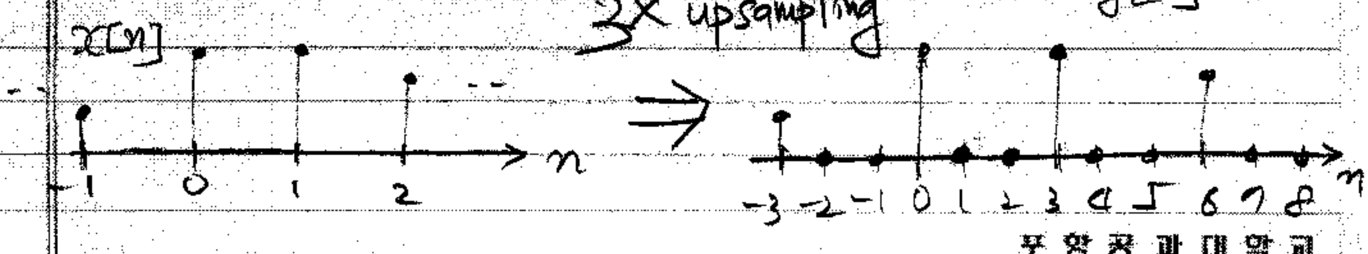
A. $\boxed{Y_d(f) = X_d(fN)}$

sol) $Y_d(f) = \sum_{n=-\infty}^{\infty} y[n] e^{-j2\pi f n}$

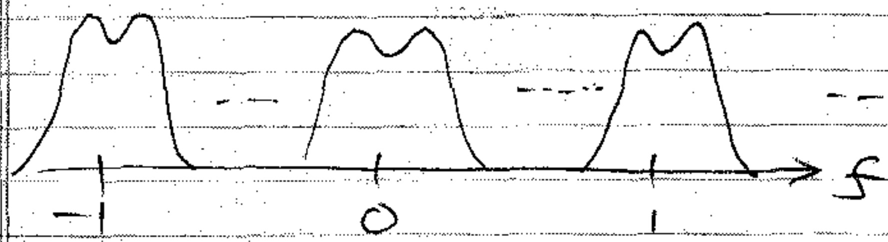
$= \sum_{m=-\infty}^{\infty} x[m] e^{-j2\pi f N m}$

$= X_d(fN)$

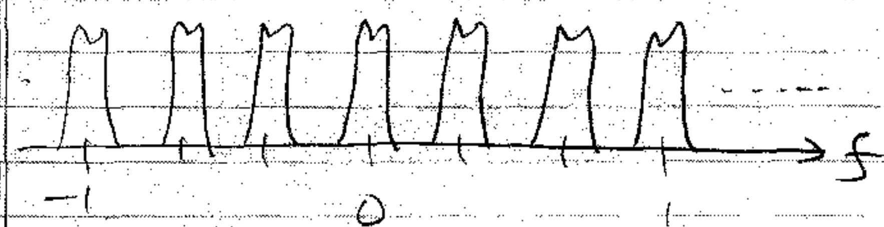
- time-domain



- frequency-domain



↓ 3x upsampling w/ zero insertion

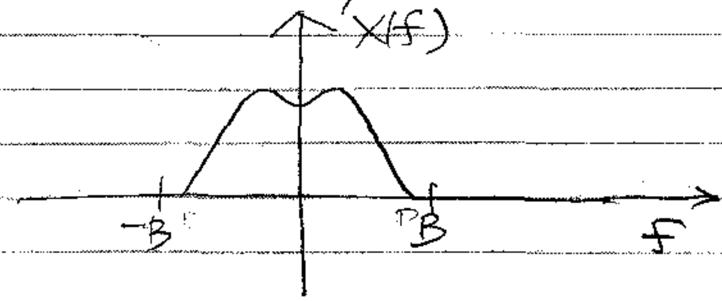


- Usually after upsampling low-pass, band-pass, or high-pass filtering in DT domain is performed w/ other digital processing such as equalization

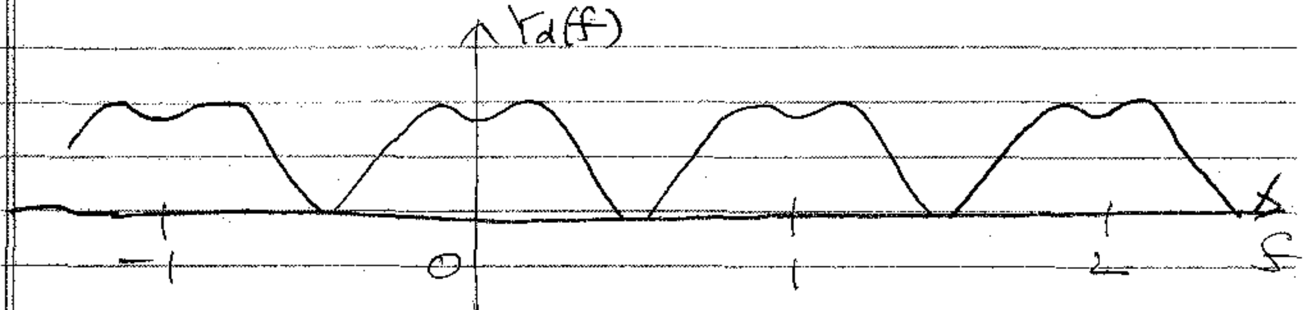
- The processed upsampled signal is often D/A converted using an N-times fast DAC or D/A converted using 1-times DAC after downsampling.

○ Examples

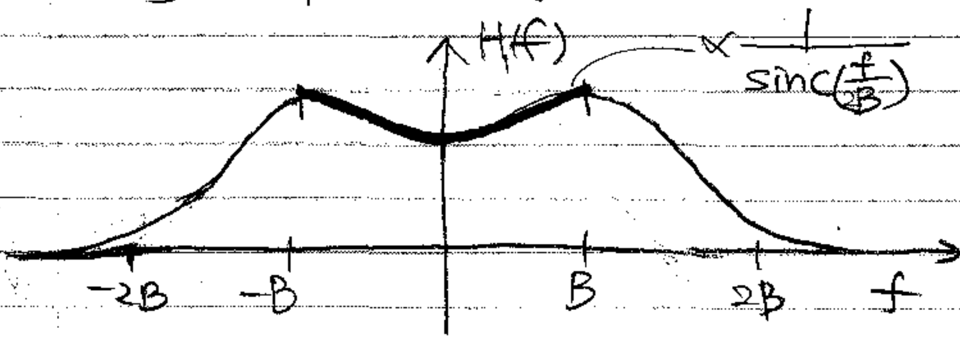
- Suppose that we want to generate a CT signal $x(t)$ whose frequency response is given by



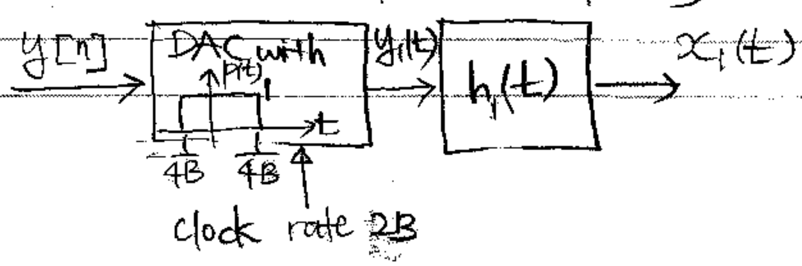
- Assume that we have a DT system that generates a DT signal $y[n]$ whose frequency response is given by



, and that we have a CT system ~~h1(t)~~ whose frequency response is given by

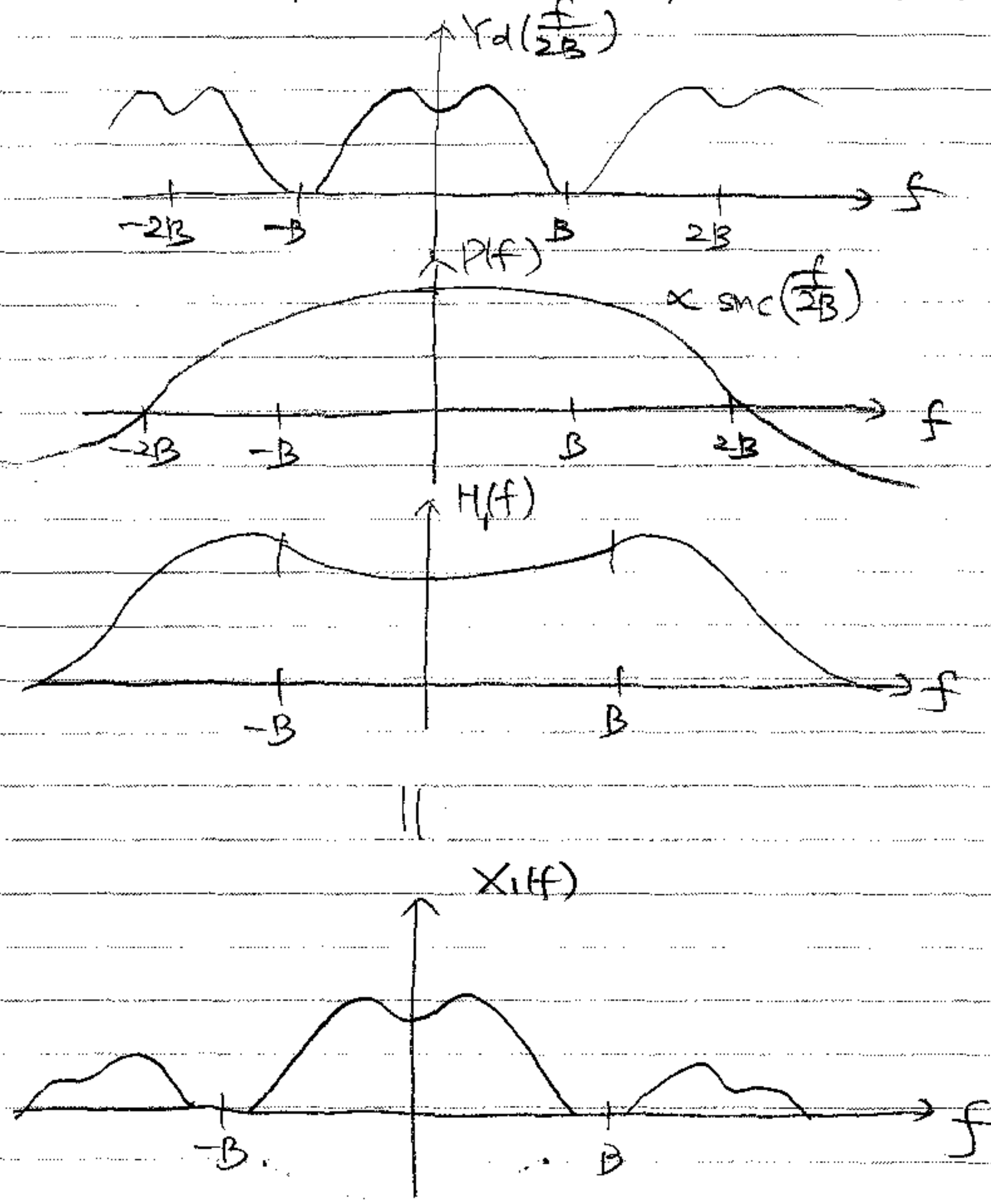


- Q1. Sketch the output frequency response of

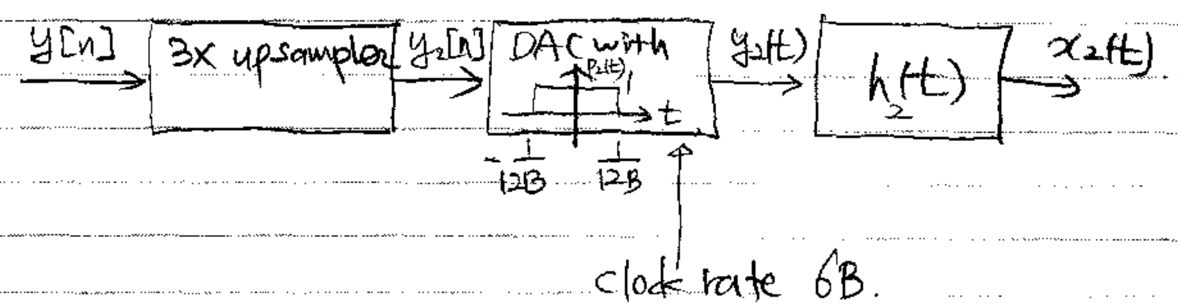


A.

$$X_1(f) = H_1(f)\, Y_1(f) = H_1(f)\, Y_d\left(\frac{f}{2B}\right) P(f)$$

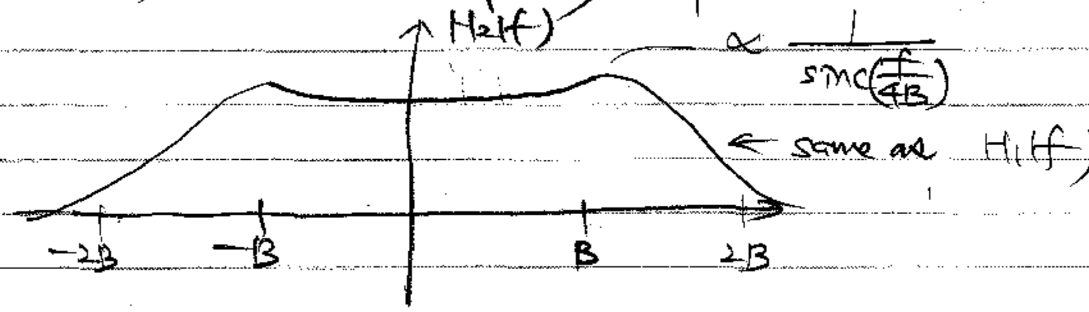


· Q2. Sketch the output frequency response of



where

$h_2(t)$ has the frequency response



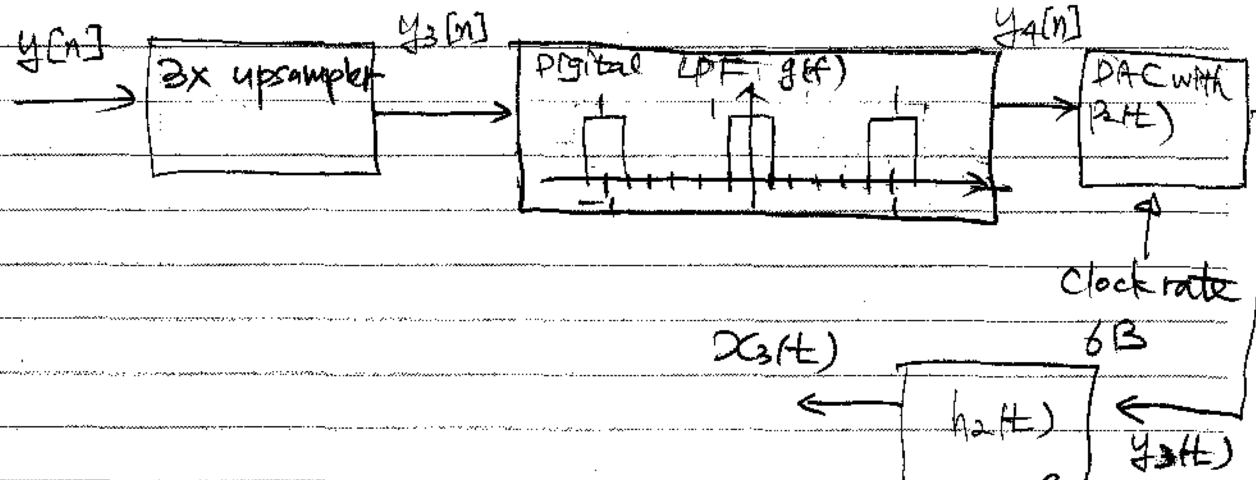
A. $X_2(f) = H_2(f)\,Y_{2,c}(f) = H_2(f)\,P_2(f)\,Y_{2,d}(\frac{f}{6B})$

$$= H_2(f)P_2(f)\,Y_d(\frac{3f}{6B})$$

$$= H_1(f)\,P_1(f)\,Y_d(\frac{f}{2B})$$

$$= X_1(f)$$

Q3. Sketch the output frequency response of

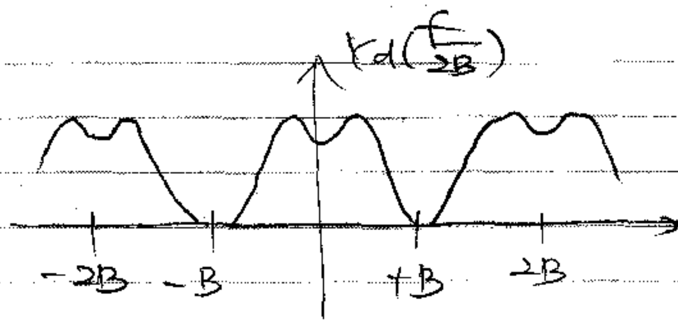


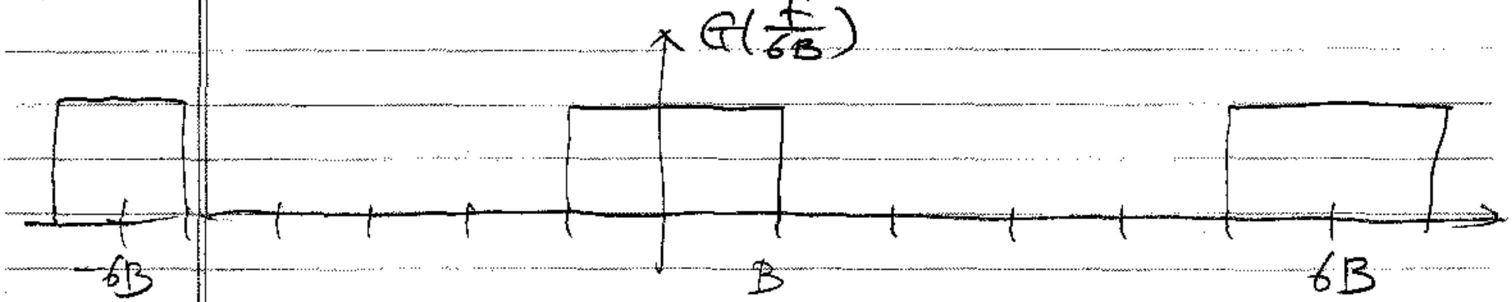
A. $X_3(f) = H_2(f)\,Y_{3,c}(f) = H_2(f)\,P_2(f)\,Y_{4,d}(\frac{f}{6B})$

$$= H_2(f)\,P_2(f)\,G(\frac{f}{6B})\,Y_3(\frac{f}{6B})$$

$$= H_2(f)\,P_2(f)\,G(\frac{f}{6B})\,Y_d(\frac{3f}{6B})$$

$$= X_1(f)$$





○ Downsampling = Decimation

Q. Given $x[n]$, we define $y[n]$ as

$$y[n] = x[nM]$$

Find the relation b/w $X(f)$ & $Y_d(f)$

A. $$Y_d(f) = \sum_{n=-\infty}^{\infty} y[n] e^{-j2\pi f n}$$ ⟨ How?

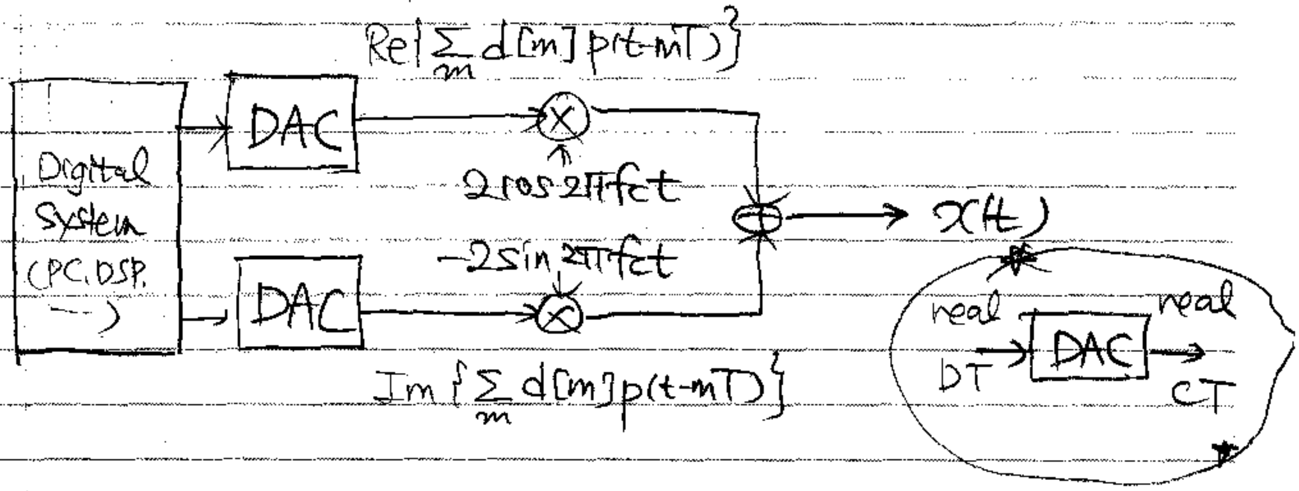
$$= \frac{1}{M} \sum_{n=-\infty}^{\infty} X_d\left(\frac{f-n}{M}\right)$$

○ Digital IF upconversion

- Suppose that we want to generate a real-valued bandpass signal

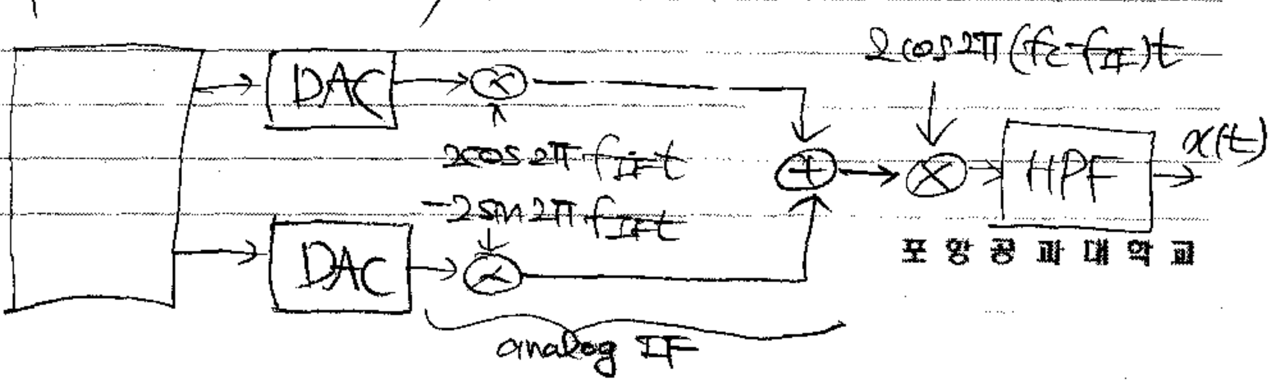
$$x(t) = \mathrm{Re}\left\{ \sum_{m=-\infty}^{\infty} d[m]\, p(t-mT)\, e^{j2\pi f_c t} \right\}$$

- The conceptually straightforward way to generate this signal is ~~to~~ use the following system.



However, it is not easy to avoid
(i) phase noise of high f_c LO
(ii) imperfect gain in $\boxed{\text{DAC}}$ – $\boxed{\text{mixer}}$ chains for I-branch & Q-branch.

- We learned that, in this course, multi-stage up-conversion may be useful.



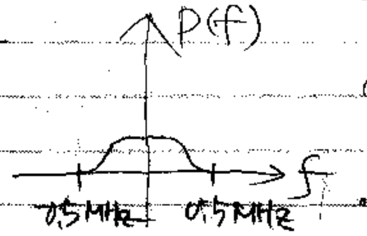
- The trend in these days, is to change the analog IF stage to a **digital IF** stage because the analog IF stage still requires perfect balance b/w I & Q branches. Fortunately, **very high-rate DAC**s are now available.

- Suppose that we want to generate a CT IF signal with bandwidth 1 MHz by using a DAC with conversion rate 200MHz & maximum output bandwidth 100MHz.

The IF signal is modeled as

$$x(t) = \mathrm{Re}\left\{ \sum_m d[m]\, p(t-mT)\, e^{j2\pi f_{IF} t} \right\}$$

where

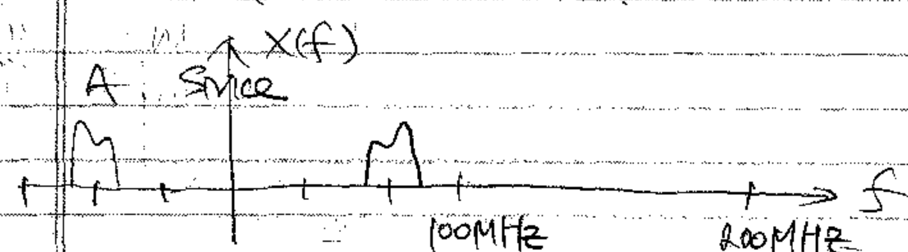


and $f_{IF} \simeq 100$ MHz

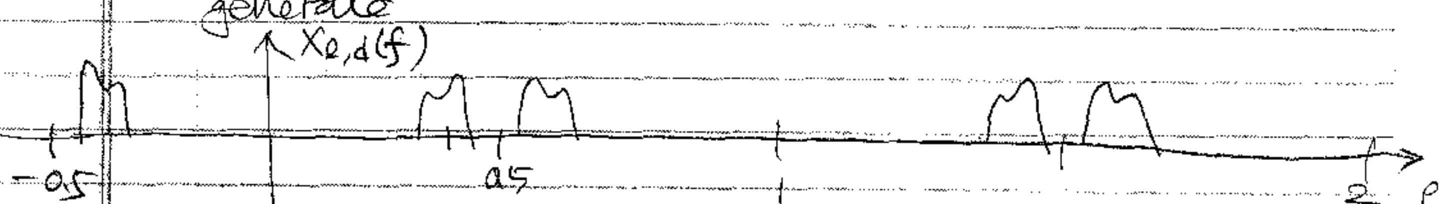
Answer the following questions.

Q1. Design a digital system whose output is fed to the DAC.

A. Since



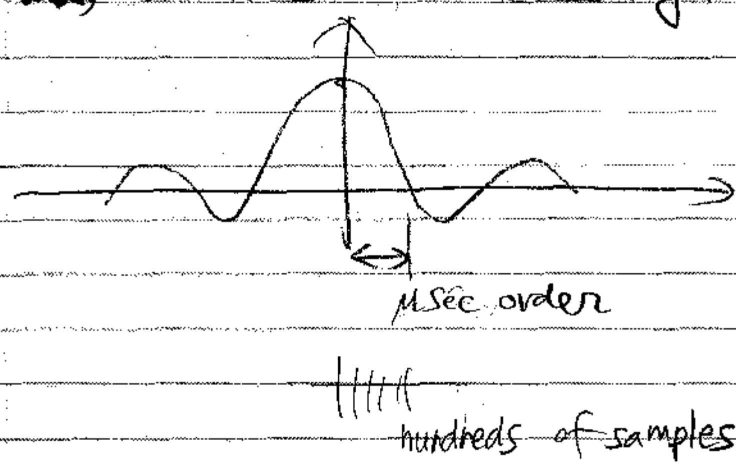
is the desired frequency response, we need to generate



Thus, we need to design a digital system that generate a sampled version of $x(t)$ at rate 200 MHz.

Q2. Compared to the bandwidth 1 MHz of the signal, is 200 MHz sampling rate too high?

A. Yes. We **sample more than 100 times per symbol interval**, which is not necessary.



Q3 Then, how can we reduce the complexity?

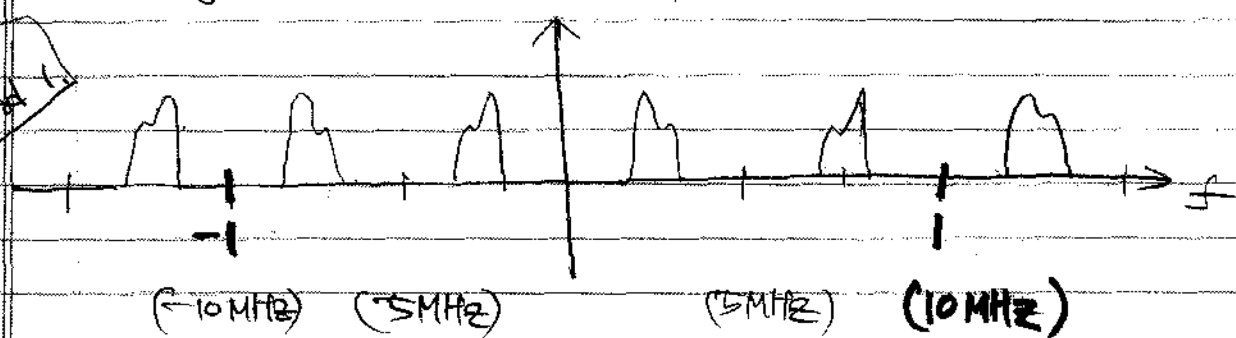
A. By upsampling & filtering followed by upsampling & filtering.

For simplicity, we consider IF upconverter that use 4x and 5x upsamplers. (We may also use multiple upsamplers) and highpass filters.

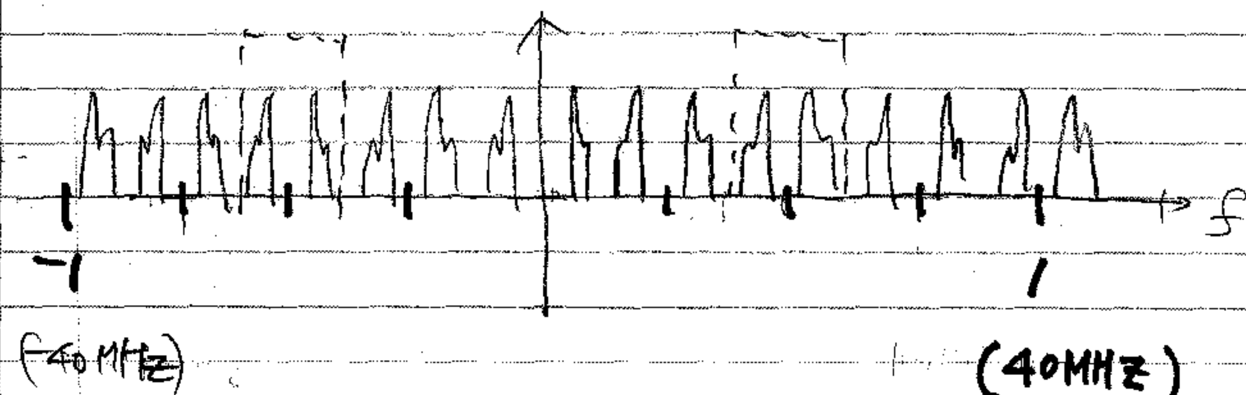
(1) First, we generate the sampled version of $\mathrm{Re}\{x_e(t)\, e^{j2\pi(7.5\ \mathrm{MHz})t}\}$

at rate 10 MHz. (So, the complexity reduces to 1/20.)
Then, the frequency response of this DT signal is given by

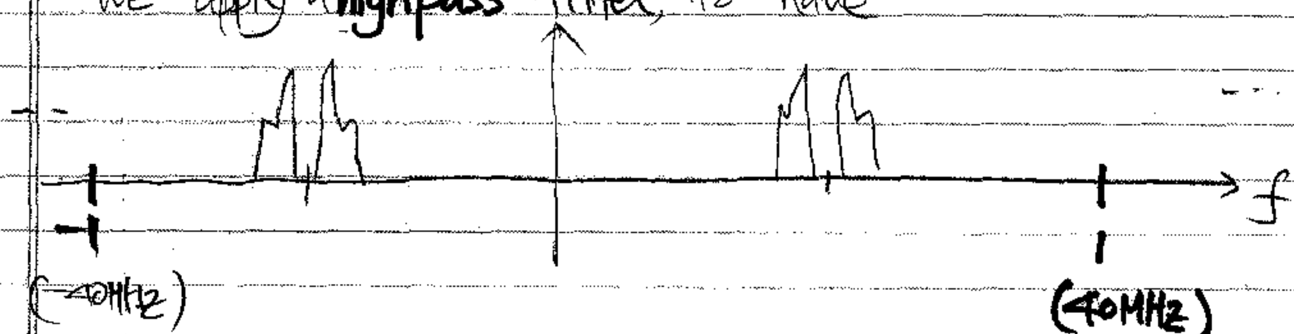
periodic w/ period 1



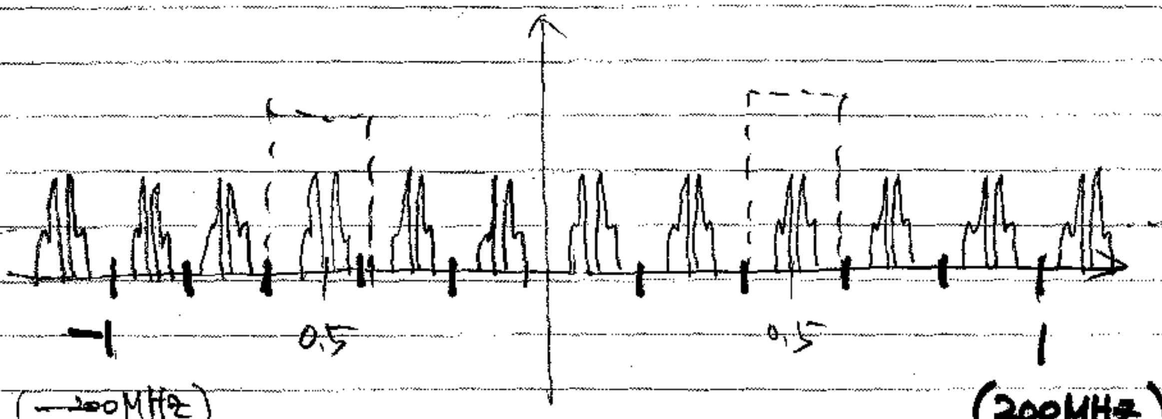
If we **4X upsample** this signal, then the frequency response is given by



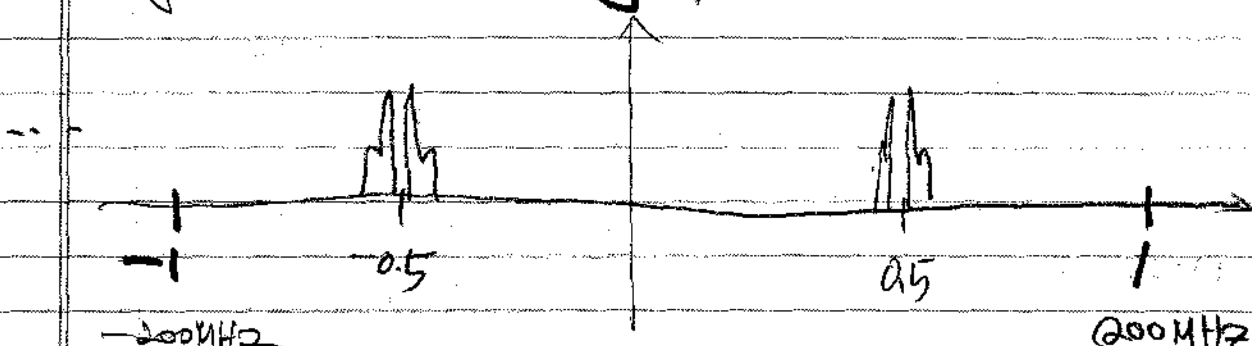
We apply a **highpass** filter, to have



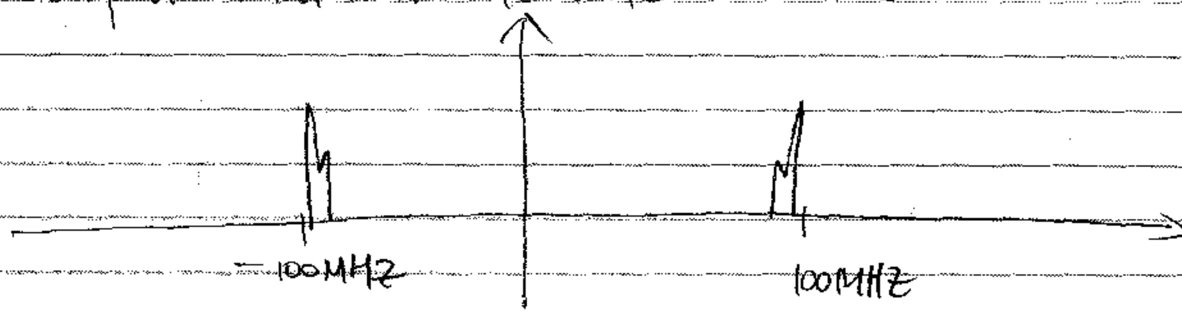
We now **5X upsample** the signal to have



Again, we use a **high pass** filter to have



We D/A convert this discrete-time signal using the **DAC** w/ conversion rate 200 MHz & output bandwidth 100MHz to have

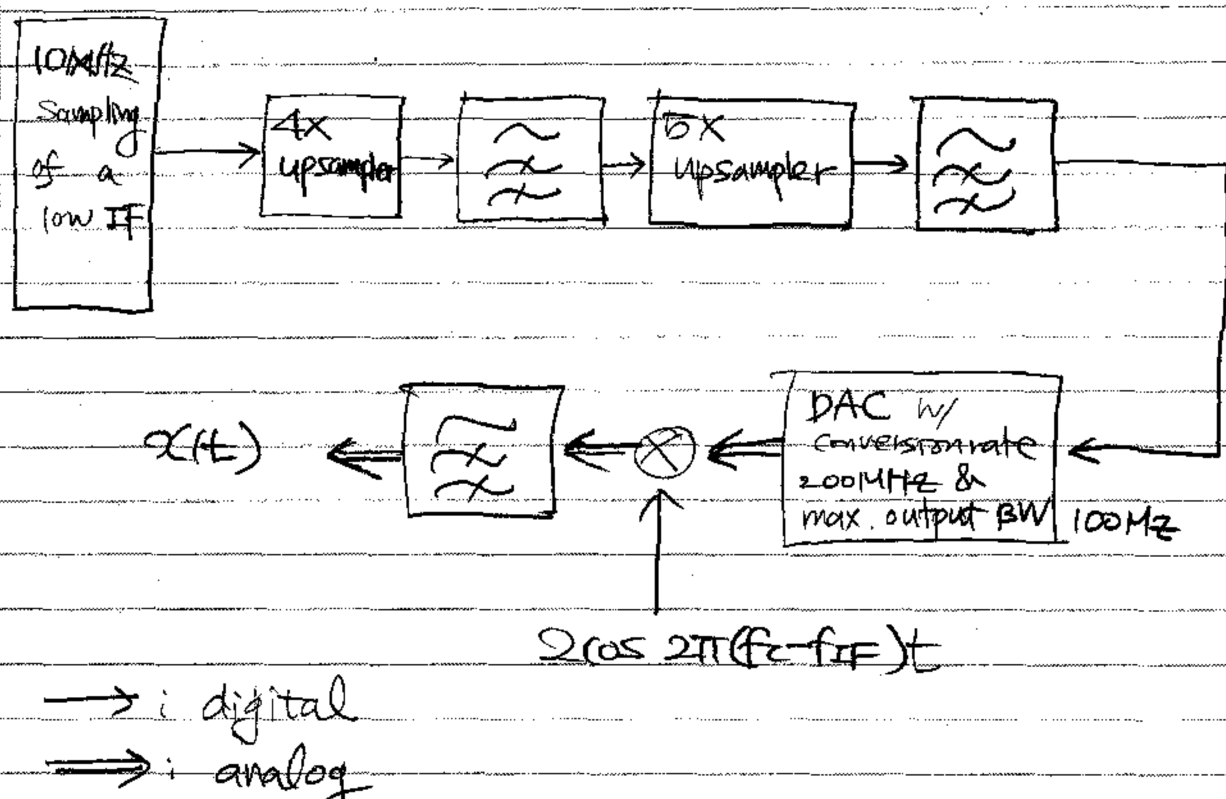


Then, we **mix** this signal w/ $2\cos 2\pi(f_c - f_{IF})t$, and filter out the low frequency component. (**highpass**)

※ Caution.
The selection of
- upsampling rates
- band / low / high pass filtering
- # of upsamplers .. etc

depends on the difficulty of implementing individual components. This example is just an example assuming ideal LPF in the DAC for the conversion.

· Summary



· Note that
(i) no analog I-Q modulator is used
(ii) the DAC may upsample again internally.
(iii) $\underset{\text{of 1MHz signal}}{10} \times 4 \times 5 = \underset{\text{MHz}}{200}$

· A Digital IF w/ down-samplers can be implemented for receivers !! Do it yourself !!!