Review of Handout #13

- The Most popular OFDM signal model

$$x(t) = \sum_{k=1}^{K} X_k e^{j2\pi f_k t}, \quad 0 \le t < T_0$$

where

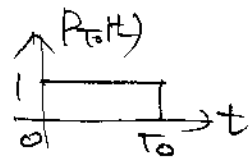
$$f_{k+1} - f_k = \frac{1}{T_0}, \quad \forall k$$

← To keep the orthogonality

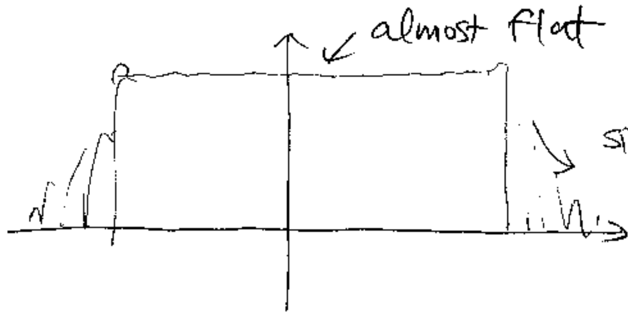
⇒ No PSD because $x(t)$ is not a power signal

- Slight modification of the most popular model

$$x(t) = \sum_{k=1}^{K} \Big(\sum_{m=-\infty}^{\infty} \underbrace{X_k[m]}_{\text{assume uncorrelated w/ unit power}} P_{T_0}(t - mT_0) \Big) e^{j2\pi f_k t}, \quad -\infty < t < \infty$$

where  and $f_{k+1} - f_k = \frac{1}{T_0}, \ \forall k$

$$\Rightarrow S_{XX}(f) = \sum_{k=1}^{K} \frac{1}{T_0} \left| P_{T_0}(f - f_k) \right|^2 \propto \sum_{k=1}^{K} \frac{1}{T_0} \operatorname{sinc}^2\big((f - f_k) T_0\big)$$

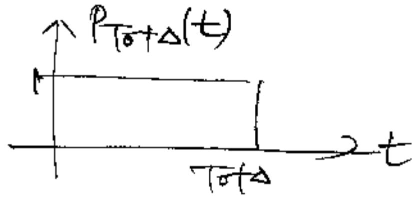


- More modification to adopt CP (Cyclic Prefix)

$$X(t) = \sum_{k=1}^{K} \left(\sum_{m=-\infty}^{\infty} X_k[m]\, P_{T_0+\Delta}\big(t - m(T_0+\Delta)\big) \right) e^{j2\pi f_k t},$$

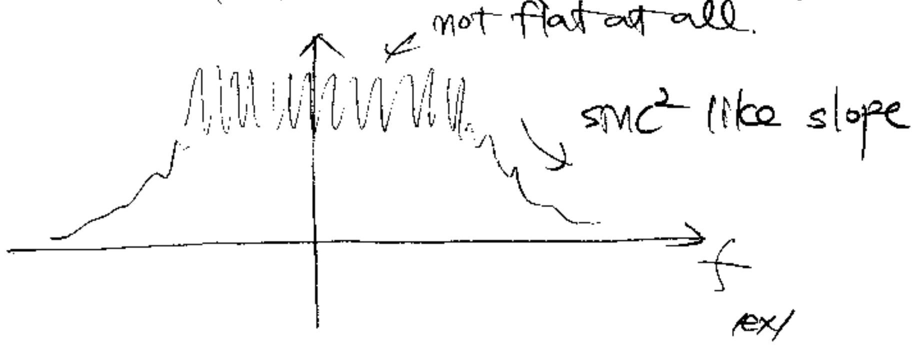
$$-\infty < t < \infty$$

where



and $f_{k+1} - f_k = \frac{1}{T_0}, \ \forall k$

$$\Rightarrow S_{xx}(f) \propto \sum_{k=1}^{K} \frac{1}{T_0}\, \mathrm{sinc}^2\big((f-f_k)(T_0+\Delta)\big)$$



∘ IFFT-based OFDM

$\cdots \mid d[-1] \mid d[0] \mid d[1] \mid d[2] \mid \cdots$ → serial to parallel

ex/

$\cdots \mid X_1[-1] \mid X_1[0] \mid X_1[1] \mid X_1[2] \mid \cdots$

$\cdots \mid X_2[-1] \mid X_2[0] \mid X_2[1] \mid X_2[2] \mid \cdots$

$\vdots$

$\cdots \mid X_K[-1] \mid X_K[0] \mid X_K[1] \mid X_K[2] \mid \cdots$

$$\cdots, \begin{bmatrix} X_1[-1] \\ X_2[-1] \\ \vdots \\ X_K[-1] \end{bmatrix}, \begin{bmatrix} X_1[0] \\ X_2[0] \\ \vdots \\ X_K[0] \end{bmatrix}, \cdots$$

↓ IFFT

$$\underline{Y}[m] = \text{IFFT}\ \underline{X}[m]$$

$$= X_1 \begin{bmatrix} \frac{1}{\sqrt{K}} e^{j\frac{2\pi}{K}0\cdot 0} \\ \frac{1}{\sqrt{K}} e^{j\frac{2\pi}{K}0\cdot 1} \\ \vdots \\ \frac{1}{\sqrt{K}} e^{j\frac{2\pi}{K}0\cdot(K-1)} \end{bmatrix} + \cdots + X_K \begin{bmatrix} \frac{1}{\sqrt{K}} e^{j\frac{2\pi}{K}(K-1)\cdot 0} \\ \frac{1}{\sqrt{K}} e^{j\frac{2\pi}{K}(K-1)\cdot 1} \\ \vdots \\ \frac{1}{\sqrt{K}} e^{j\frac{2\pi}{K}(K-1)(K-1)} \end{bmatrix}$$

$\downarrow$ Add CP

$$\begin{bmatrix} \left.\begin{matrix} Y_{K-L+1}[m] \\ \vdots \\ Y_K[m] \end{matrix}\right\} L \\ \left.\begin{matrix} Y_1[m] \\ Y_2[m] \\ \vdots \\ \boxed{\begin{matrix} Y_{K-L+1}[m] \\ \vdots \\ Y_K[m] \end{matrix}} \end{matrix}\right\} K \end{bmatrix}$$

$\downarrow$ Parallel to Serial Conversion

$$\ldots, \begin{bmatrix} \hat{Y}_1[-1] \\ \hat{Y}_2[-1] \\ \vdots \\ \hat{Y}_{K+L}[-1] \end{bmatrix}, \begin{bmatrix} \hat{Y}_1[0] \\ \hat{Y}_2[0] \\ \vdots \\ \hat{Y}_{K+L}[0] \end{bmatrix}, \begin{bmatrix} \hat{Y}_1[1] \\ \hat{Y}_2[1] \\ \vdots \\ \hat{Y}_{K+L}[1] \end{bmatrix}, \ldots \rightarrow \boxed{\hat{Y}_1[-1]}\boxed{\hat{Y}_2[-1]}\cdots\boxed{\hat{Y}_{K+L}[-1]}\boxed{\hat{Y}_1[0]}\cdots$$

$\downarrow$ linear modulation

$$- \quad x(t) = \sum_{m=-\infty}^{\infty} \left(\sum_{k=1}^{K+L} y_k[m]\, p(t-(k+(K+L)m)T_c) \right)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{k=1}^{K+L} \left(\sum_{k'=1}^{K} x_{k'}[m] \frac{1}{\sqrt{K}} e^{j\frac{2\pi}{K}(k-L-1)(k'-1)} \times p(t-(k+(K+L)m)T_c) \right)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{k'=1}^{K} x_{k'}[m] \underbrace{\left(\sum_{k=1}^{K+L} \frac{1}{\sqrt{K}} e^{j\frac{2\pi}{K}(k-L-1)(k'-1)} \times p(t-(k+(K+L)m)T_c) \right)}_{\triangleq\, s_{k'}(t-m(K+L)T_c)}$$

$$= \sum_{k'=1}^{K} \underbrace{\left(\sum_{m=-\infty}^{\infty} x_{k'}[m]\, s_{k'}(t-mT_b) \right)}_{\text{linearly modulated signal w/ Tx pulse } s_{k'}(t) \text{ \& symbol Time } T_b.}$$

where $T_b = (K+L)T_c$

$$\Rightarrow S_{xx}(f) = \sum_{k'=1}^{K} \frac{1}{T_b} |S_{k'}(f)|^2$$

assuming uncorrelated data seq. w/ unit power

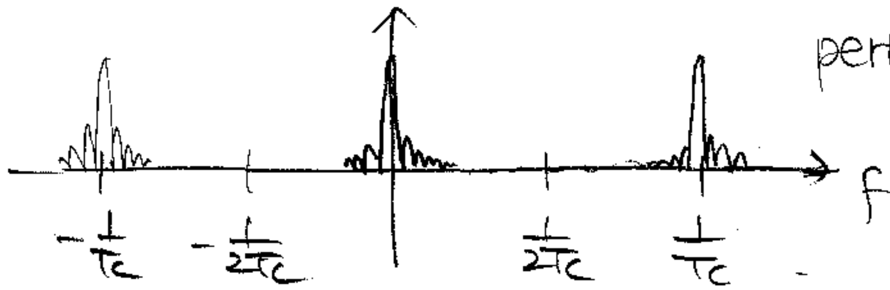
$$= \sum_{k'=1}^{K} \frac{1}{T_b} |P(f)|^2 \left| \sum_{k=1}^{K+L} \frac{1}{\sqrt{K}} e^{j\frac{2\pi}{K}(k-L-1)(k'-1)} \times e^{-j2\pi f k T_c} \right|^2$$

$$\times\ e^{-j2\pi k(fT_c - \frac{(k'-1)}{K})}$$

$$\triangleq \sum_{k'=1}^{K} \frac{1}{T_b} |P(f)|^2\, W_D\!\left(f - \frac{(k'-1)}{KT_c}\right)$$

DTFT of a finite-support DT signal evaluated at fT_c

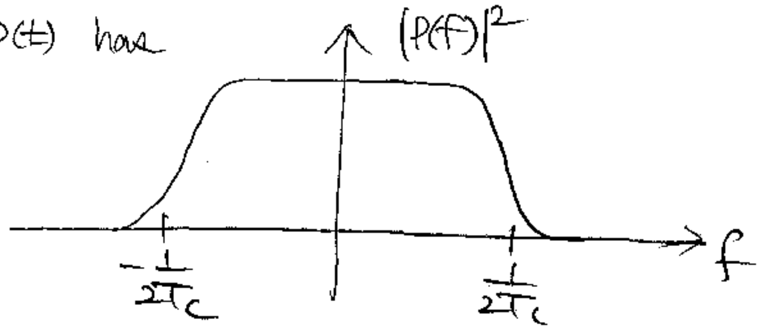
where
$$W_D(f) \triangleq \left| \sum_{k=1}^{K+L} \frac{1}{\sqrt{K}} e^{-j2\pi f k T_c} \right|^2$$
$$= \left| \frac{\sin(\pi f (K+L) T_c)}{\sin(\pi f T_c)} \right|^2$$



periodic w/ period $\frac{1}{T_c}$

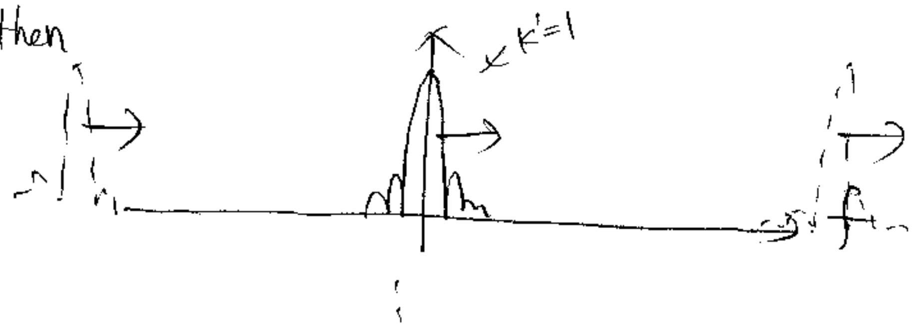
o Null sub-carriers

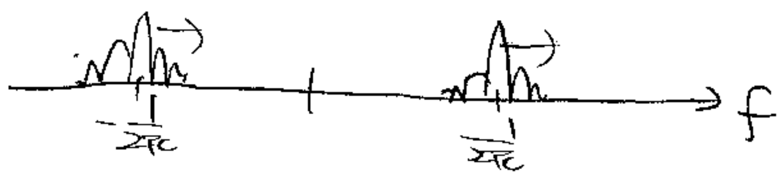
If $p(t)$ have



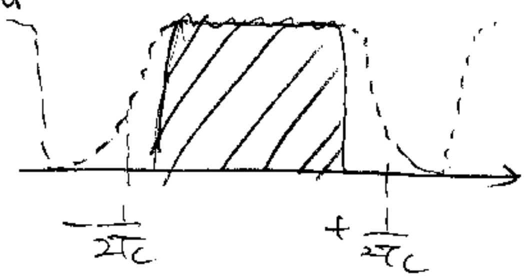
and uncorrelated $X(m)$,

then

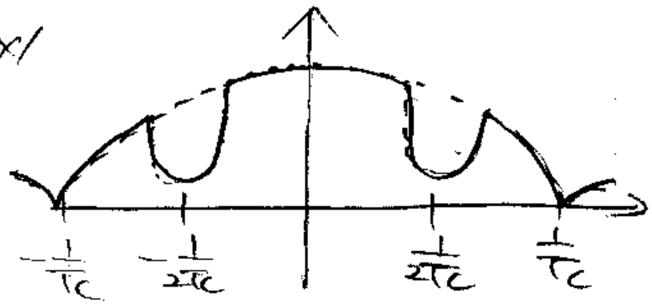




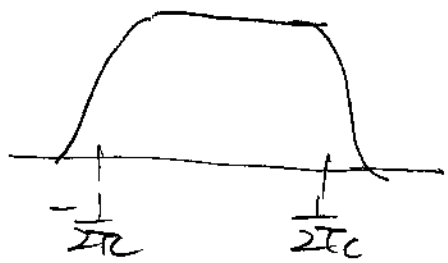
To make this spectrum as similar as that of modified w/ rectangular ..., we put null sub-carriers so that



/ex/



→ filter



⇒ K : # of FFT points ⇒ K−N subcarriers

N : # of null subcarriers.

Review of PAM & the optimal continuous-time front end.

○ Review of PAM (received signal model)

① Real baseband signal model

$$z(t) = \sqrt{P} \sum_{m=-\infty}^{\infty} d[m]\, p(t-mT) + N(t)$$

where P is the signal power

$E\{d[m]\}=0$, $E\{d[m]^2\}=1$,

$\|p(t)\|^2 = T$,

$N(t)$ is a real-valued AWGN w/ $S_{NN}(f) = \frac{N_0}{2}$

② Real passband signal model

$$z(t) = \mathrm{Re}\left\{ \sqrt{2P} \sum_{m=-\infty}^{\infty} d[m]\, p(t-mT)\, e^{j2\pi f_c t} \right\} + N(t)$$

where P is the signal power ⋯ (*)

$E\{d[m]\}=0$, $\left\langle \begin{array}{l} E\{d[m]^2\}=1 \quad \text{(for BPSK)} \\ E\{d[m]^2\}=0,\ E\{|d[m]|^2\}=\cancel{0}\ 1 \end{array} \right.$ ⋯ (**)

$\|p(t)\|^2 = T$, ⋯ (***)

$N(t)$ is a real-valued AWGN w/ $S_{NN}(f) = \frac{N_0}{2}$.

③ Complex baseband signal model I

$$z_\ell(t) = \sqrt{2P} \sum_{m=-\infty}^{\infty} d[m]\, p(t-mT) + N_\ell(t)$$

where (*), (**), (***), and

$N_\ell(t)$ is a proper-complex AWGN w/ $S_{N_\ell N_\ell}(f) = 2N_0$

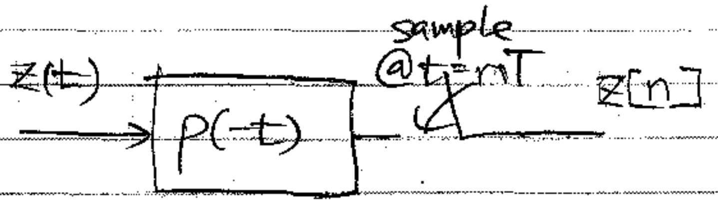
③' Complex baseband signal model II

$$z_\ell(t) = \sqrt{P} \sum_{m=-\infty}^{\infty} d[m]\, p(t-mT) + N_\ell(t)$$

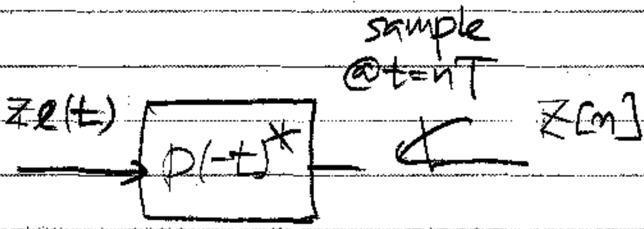
where (*), (**) (***), and
$N_\ell(t)$ is a proper-complex AWGN w/ $S_{N_\ell N_\ell}(f) = N_0$

○ Review of PAM (optimal receiver front end)

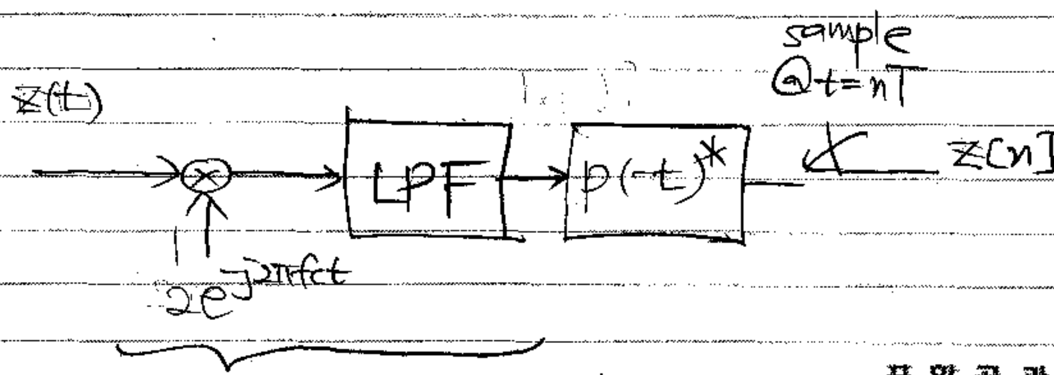
① Real baseband case



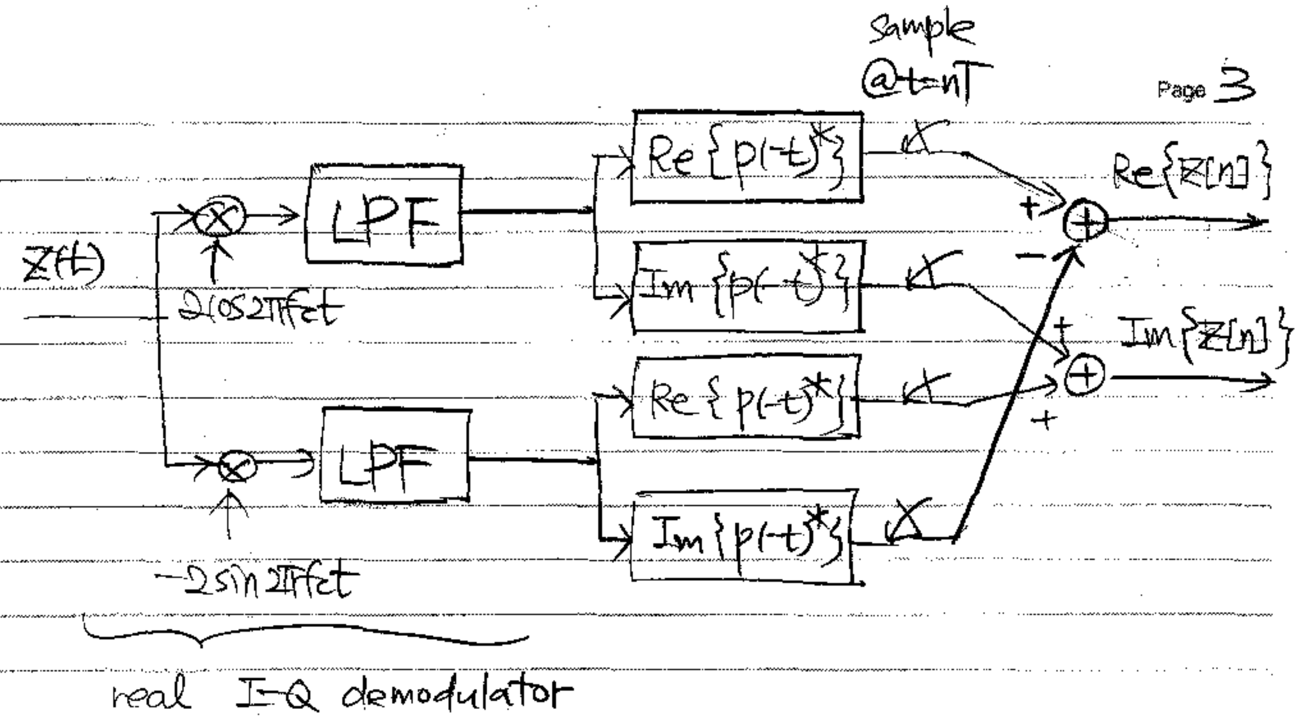
③' Complex baseband case



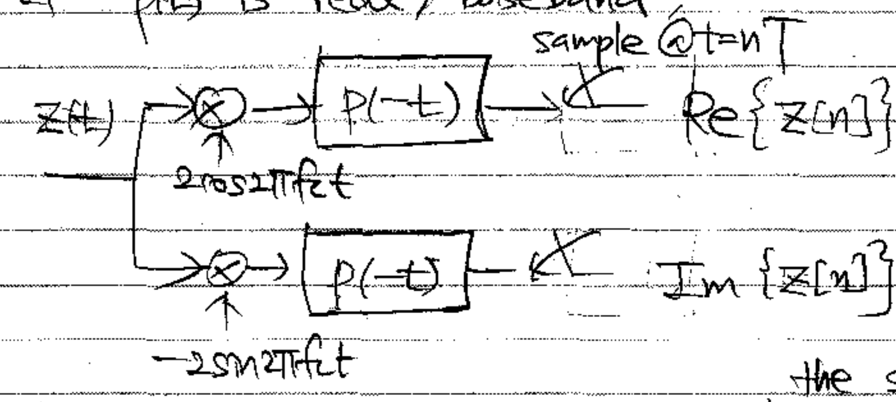
② Real passband case



complex I-Q demodulator



- If p(t) is real, baseband



the sufficient statistic

○ Review of the properties of $\{Z[n]\}_{n=-\infty}^{\infty}$.

- $Z[n] = \sqrt{2P} \sum_{m=-\infty}^{\infty} d[m] \int_{-\infty}^{\infty} p(t-mT)\, p(t-nT)^* dt$

$$+ \int_{-\infty}^{\infty} N_e(t)\, p(t-nT)^* dt$$

- Signal component

If we define

$$\bar{p}(t) \triangleq p(t) * p(-t)^* = p(-t)^* * p(t)$$

then, from

$$\tilde{p}(t) = \int_{-\infty}^{\infty} p(t-\tau)\, p^*(-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} p(\tau+t)\, p^*(\tau)\, d\tau$$

$$\Rightarrow \int_{-\infty}^{\infty} p(t-mT)\, p(t-nT)^* dt = \tilde{p}((n-m)T),$$

we obtain

$$z[n] = \sqrt{2P} \sum_{m=-\infty}^{\infty} d[m]\, \tilde{p}((n-m)T) + N_e[n].$$

$$= \sqrt{2P}\, d[n]\, \tilde{p}(0) + \underbrace{\sqrt{2P} \sum_{m \neq n} d[m]\, \tilde{p}((n-m)T)}_{ISI} + N_e[n]$$

- Gaussian noise component

$$R_{N_eN_e}[n] \triangleq E\{N_e[m]\, N_e^*[n+m]\}$$

$$= E\left\{\left(\int_{-\infty}^{\infty} N_e(t)\, p(t-mT)^* dt\right)^* \int_{-\infty}^{\infty} N_e(t)\, p(t-(m+n)T)\, dt\right\}$$

$$= 2N_0 \int_{-\infty}^{\infty} p(t-mT)\, p(t-(m+n)T)\, dt$$

$$= \underbrace{2N_0\, \tilde{p}(nT)}$$

In general, this sequence is not proportional to $\delta[n]$. $\therefore$ The noise sequence is colored in general.

- A very special case where $p(t)$ is a square-root Nyquist pulse.

If $\tilde{p}(nT) \propto \delta_{n,0}$

$$\equiv \sum_{m=-\infty}^{\infty} \left| P(f+\tfrac{m}{T}) \right|^2 = \text{a constant } \forall f$$

← called the **folded spectrum**

then
(i) There is no ISI in the signal component and
(ii) The noise sequence is white.

Summary

- For a PAM, the matched filter matched to the symbol waveform and the following baud rate sampling can generate a sequence that is sufficient.

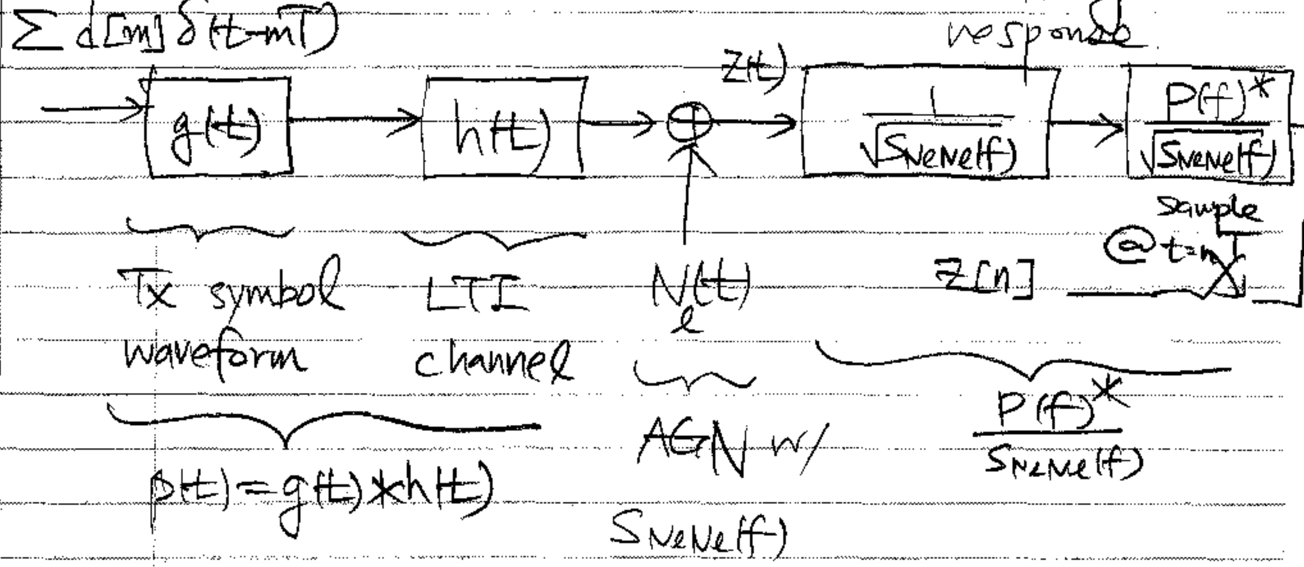
- The sequence contains the ISI and a colored noise component unless the symbol waveform is a square-root Nyquist pulse.

- For the matched filtering, we need to know $p(t)$, which may not be realistic in general because $p(t)$ is usually the convolution of the transmitted symbol waveform $g(t)$ and the channel impulse response $h(t)$, i.e.,

$$p(t) = g(t) * h(t)$$

and $h(t)$ may not be known to the receiver.

- This result is from the AWGN assumption on **N(t)** or **N$_e$(t)**. If it is **colored**, then the noise must be **whitened first**, if it is possible, Then, the Rx filter needs to match to $p(t) * \underbrace{h_w(t)}_{\text{whitening filter response}}$



In this case, the data sequence have the desired waveform $\mathcal{F}^{-1}\left\{\frac{|P(f)|^2}{S_{N_eN_e}(f)}\right\}$ and the noise PSD $\propto |P(f)|^2$.

Note that if the noise is white than we have $\mathcal{F}^{-1}\{|P(f)|^2\}$ and $|P(f)|^2$, respectively.

- In the next handout, we only consider the case where the AGN N(t) or N$_e$(t) is white, so that the desired waveform and the noise autocorrelation function both of them are proportional to $\tilde{p}(nT)$.