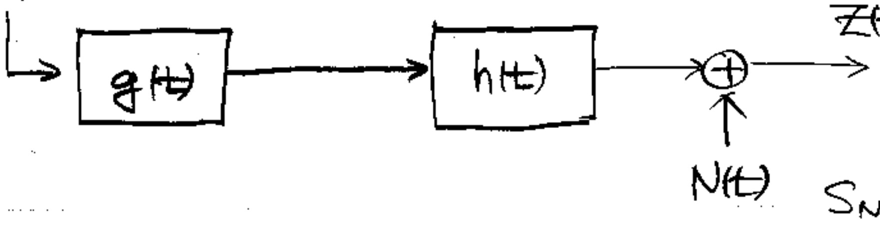
Zero-Forcing Linear Equalization (ZF-LE)

- Observation model I : CT observation

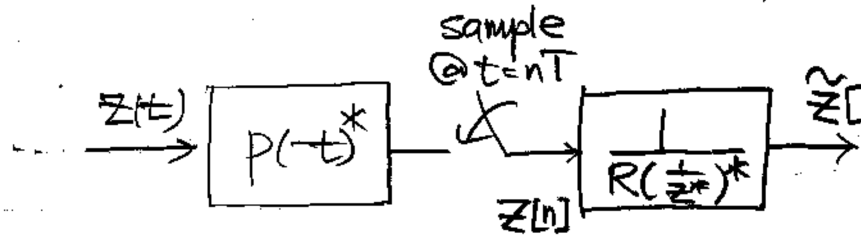
$\sqrt{2P}\sum_{m=-\infty}^{\infty} d[m]\,\delta(t-mT)$



$Z(t) = \sqrt{2P}\sum_{m=-\infty}^{\infty} d[m]\cdot p(t-nT) + N(t)$

$S_{NN}(f) = 2N_0$

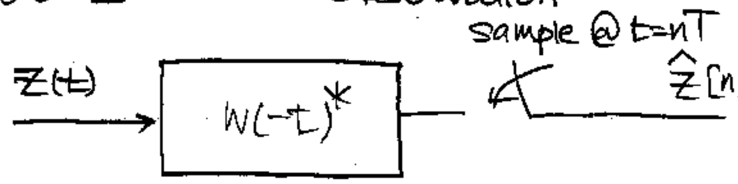
Observation model II : WMF output observation



$\tilde{Z}[n] = \sqrt{2P}\, d[n] * r[n] + \tilde{N}[n]$

$S_{\tilde{N}\tilde{N}}(e^{j2\pi f}) = 2N_0$

- Goal I : CT observation



$\hat{Z}[n] = d[n] + \underbrace{\hat{N}[n]}_{\text{Gaussian noise only}}$

The goal is to design a CT LTI system with impulse response $w(-t)^*$ s.t. in the symbol rate sampler output there is no ISI, exactly $d[n]$ and Gaussian noise component, i.e., an LTI system that forces ISI to zero.

Moreover, we want to maximize the SNR $\dfrac{E\{|d[n]|^2\}}{E\{|\hat{N}[n]|^2\}}$

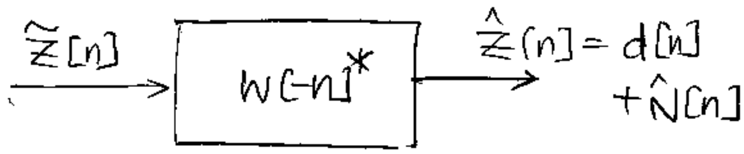
i.e., we want to minimize $E\{|\hat{N}[n]|^2\}$ (the noise variance).

Thus, the optimization problem is formulated as

$$\underset{w(-t)^*}{\text{minimize}} \quad E\{|\hat{z}[n]-d[n]|^2\}$$

$$\text{subject to} \quad \hat{z}[n]=d[n]+\hat{N}[n].$$

Goal II : WMF output observation



$$\underset{w[-n]^*}{\text{minimize}} \quad E\{|\hat{z}[n]-d[n]|^2\}$$

$$\text{subject to} \quad \hat{z}[n]=d[n]+\hat{N}[n]$$

⇒ In both cases, we want to design an LTI system that minimizes the mean of the square estimation error $\hat{z}[n]-d[n]$ in estimating $d[n]$ by $\hat{z}[n]$
under the constraint that the signal component is **$d[n]$ without generating ISI.**

Note that under the constraint, the estimation error reduces to $\hat{z}[n]-d[n]=\hat{N}[n]$, the Gaussian noise only.

⇒ **the ZF-LE**

cf). We will also study the equalizer that has

$$\underset{w(-t)^* \text{ or } w[-n]^*}{\text{minimize}} \quad E\{|\hat{z}[n]-d[n]|^2\}$$

as the optimization problem without the ZF constraint. This equalizer is called the MMSE-LE.

○ ZF-LE

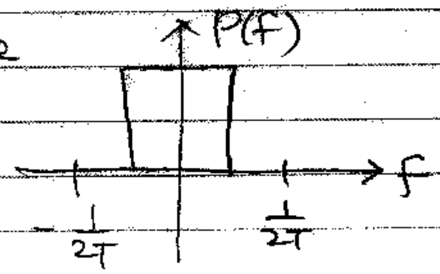
- Q. Examine the existence & the uniqueness of the ZF-LE.

A. A ZF-LE does not always exist.

ex/

$$Z(t) = \sqrt{E_p} \sum_{m=-\infty}^{\infty} d[m]\, p(t-mT) + N(t)$$

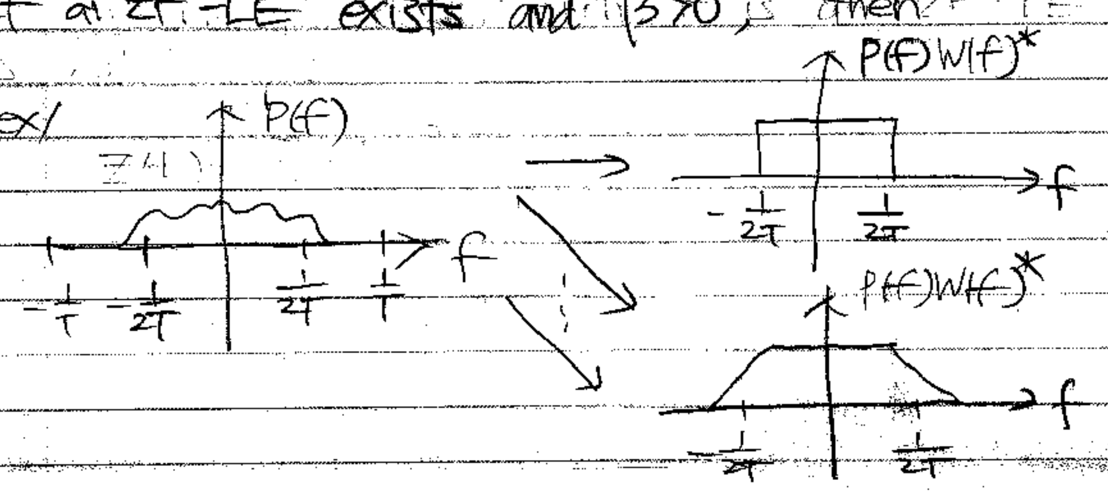
where



"$\beta < 0$" is a trivial sufficient condition for non-existence of the ZF-LE.

If a ZF-LE exists and $\beta > 0$, then

ex/

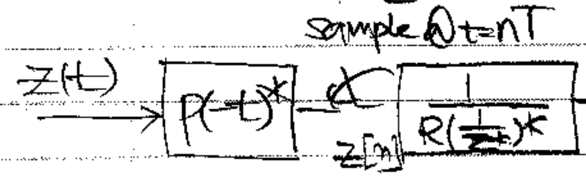


there exist **non-unique feasible solutions** that eliminate the ISI. Among those feasible solutions, we want to find the one that maximizes the SNR.

- There is **a trick** to find the ZF-LE using the fact that the WMF generates a sufficient statistic. (See Barry, Lee & Messerschmitt 8.1.2)

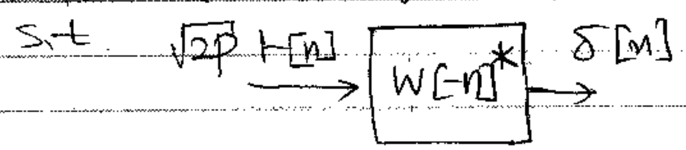
ZF-LE w/ WMF front-end.

- Suppose that $\rho[n] \triangleq \rho(nT)$ has the Z-transform which is well approximated by a rational transfer function $Q(Z)$. We also assume that $R(Z)$ is the minimum phase response that leads to $Q(Z) = R(Z)\, R(\frac{1}{Z^*})^*$.



$\tilde{Z}[n] = \sqrt{2p}\, d[n] * r[n] + \tilde{N}[n]$

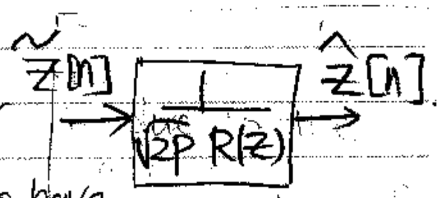
- Note that we want to find the response $W[-n]^*$ s.t.



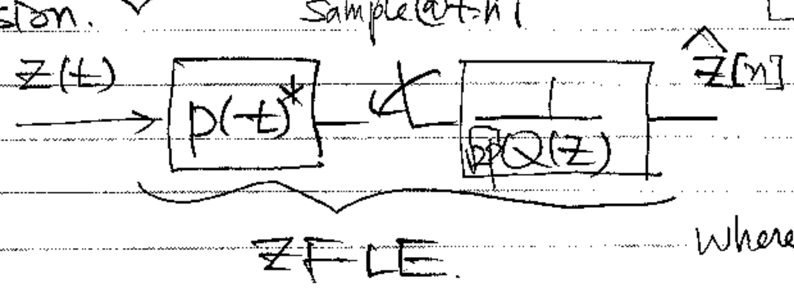
Then, we want to find among these responses the one that minimizes the MSE.

Surprisingly, the feasible solution is unique in this case, i.e., $W(\frac{1}{Z^*})^* = \frac{1}{\sqrt{2p}\, R(Z)}$ is **the only** Z-transform that eliminates the ISI.

Thus, this is the ZF-LE !!!



- Conclusion. Combined w/ whitening filter, we have



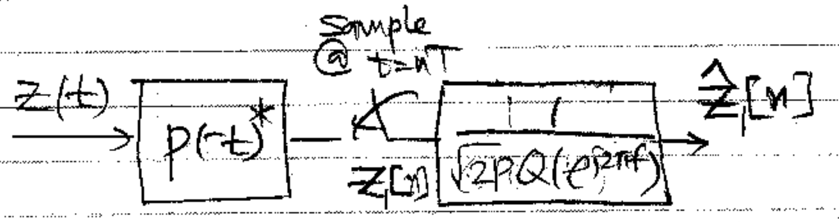
where $Q(Z) = R(Z)\, R(\frac{1}{Z^*})^*$

> $\sum_{n=1}^{N} a_n x_n = \sum_{n=1}^{N} b_n x_n \quad \forall x_n$
>
> $\Rightarrow a_n = b_n$
>
> $\therefore W_1(f)^* = W_2(f)^*$ unique.

○ ZF-LE (CT case)

Q. Hinted from the "ZF-LE w/ WMF front-end", guess the CT ZF-LE that processes $z(t)$.

A.



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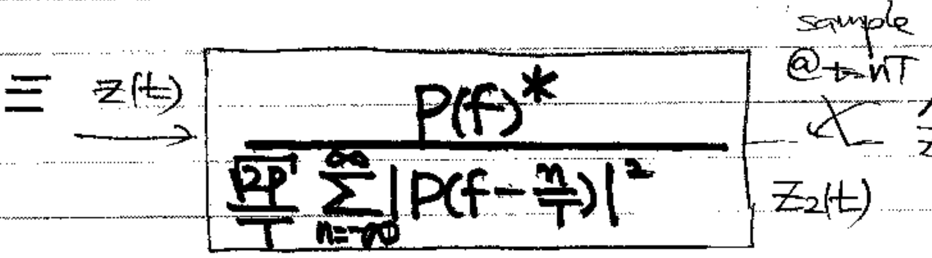


We show that the DTFT of $\hat{z}_1[n]$ is the same as the DTFT of $\hat{z}_2[n]$ in the below.

> For simplicity we use $Q(f)$ instead of $Q(e^{j2\pi f})$

First,

$$Z_{1,d}(fT) \doteq \frac{1}{T}\sum_{n=-\infty}^{\infty} Z\left(f-\frac{n}{T}\right) P\left(f-\frac{n}{T}\right)^*$$

$$\Rightarrow \hat{Z}_{1,d}(f) = \frac{Z_{1,d}(f)}{\sqrt{2P}\,Q(f)} = \frac{\frac{1}{T}\sum_{n=-\infty}^{\infty} Z\left(\frac{f-n}{T}\right) P\left(\frac{f-n}{T}\right)^*}{\frac{\sqrt{2P}}{T}\sum_{n=-\infty}^{\infty}\left|P\left(\frac{f-n}{T}\right)\right|^2}$$

Second,

$$Z_{2,c}(f) = \frac{Z(f)P(f)^*}{\frac{\sqrt{2P}}{T}\sum_{n=-\infty}^{\infty}\left|P\left(f-\frac{n}{T}\right)\right|^2}$$

$$\Rightarrow \hat{Z}_{2,d}(fT) = \frac{1}{T}\sum_{n'=-\infty}^{\infty} Z_{2,c}\left(f-\frac{n'}{T}\right)$$

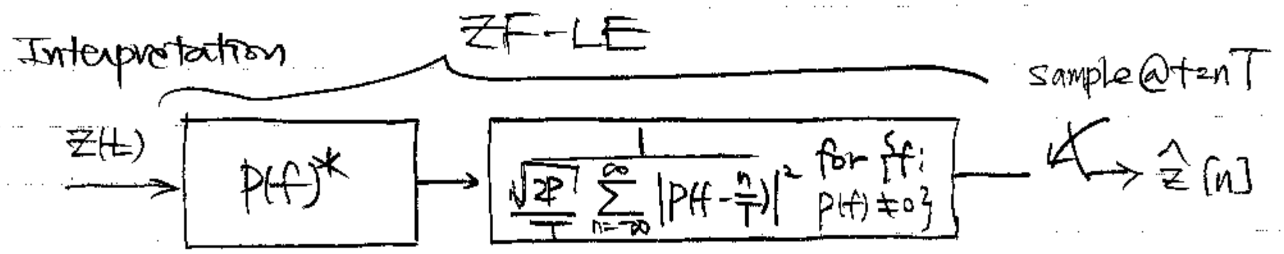
$$= \frac{1}{T}\sum_{n'=-\infty}^{\infty} \frac{Z(f-\frac{n'}{T})\,P(f-\frac{n'}{T})^*}{\frac{\sqrt{2P}}{T}\sum_{n=-\infty}^{\infty} \left|P(f-\frac{n'}{T}-\frac{n}{T})\right|^2}$$

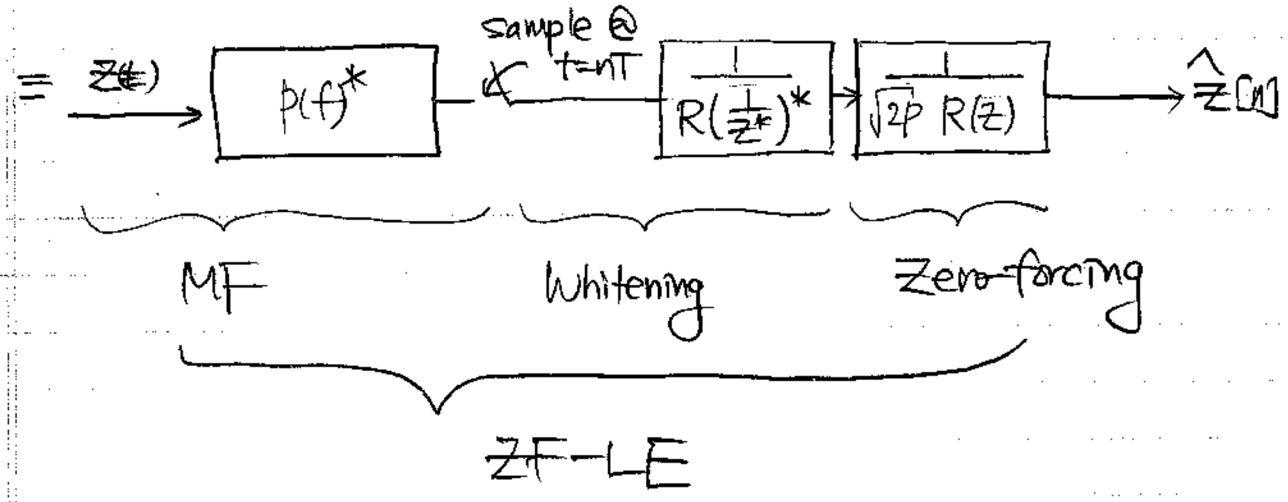
← $= \sum_{n=-\infty}^{\infty} |P(f-\frac{n}{T})|^2$: a periodic function w/ period $\frac{1}{T}$.

$$= \frac{\sum_{n=-\infty}^{\infty} Z(f-\frac{n}{T})\,P(f-\frac{n}{T})^*}{\sqrt{2P}\sum_{n=-\infty}^{\infty} |P(f-\frac{n}{T})|^2}$$

$\therefore\ \hat{Z}_{2d}(f) = \hat{Z}_{1,d}(f)$.

· Interpretation





MMSE - LE

Minimum Mean Squared Error (MMSE) Linear Equalizer (LE)

- Review

Q. Given the observation vector $\underline{Y}$, find the estimator $\hat{\underline{\theta}}(\underline{Y})$ that minimizes the mean square of the estimation error $E\{\|\hat{\underline{\theta}}(\underline{Y})-\underline{\Theta}\|^2\}$ subject to linearity constraint, i.e.,

$$\min_{W} E\{\|W\underline{Y}-\underline{\Theta}\|^2\}$$

A. By orthogonality principle, we have

$$E\{(\underbrace{W\underline{Y}-\underline{\Theta}}_{\text{estimation error}})\underbrace{\underline{Y}^T}_{\text{observation}}\}=\underline{0}$$

← any entry in estimation error ⊥ any entry in observation

which implies

$$W R_{\underline{Y}\underline{Y}} = R_{\underline{\Theta}\underline{Y}}$$

$$\therefore W = R_{\underline{Y}\underline{Y}} R_{\underline{\Theta}\underline{Y}}^{-1}$$

- Application to MMSE-LE.

$\tilde{z}[n] = \sqrt{2p}\, d[n]*r[n] + \tilde{N}[n]$ → $W[n]^*$ → $\hat{z}[n]$

$$\Rightarrow E\{(\underbrace{\hat{z}[m]-d[m]}_{\text{estimation error}})^* \underbrace{\tilde{z}[n+m]}_{\text{observation}}\}=0, \quad \forall m,n.$$

This must hold for all m & n+m because the orthogonality must hold for any entry in estimation error and any entry in observation

- Assumption $\phi_{dd}[n] = E\{d[m]^* d[m+n]\} = \delta[n], \forall n.$

$\Leftarrow$ zero-mean unit-variance uncorrelated symbols.

$\xrightarrow{z} \Phi_{dd}(z) = 1, \forall z$

- Derivation

(i) $\phi_{\tilde{z}\tilde{z}}[n] \triangleq E\{\tilde{z}[m]^* \tilde{z}[m+n]\}$

$$= E\{(\sqrt{2P}\, d[m] * r[m] + \tilde{N}[m])^* (\sqrt{2P}\, d[m'] * r[m'] + \tilde{N}[m'])\big|_{m'=m+n}\}$$

$\updownarrow$

$$\Phi_{\tilde{z}\tilde{z}}(z) = 2P\, \Phi_{dd}(z) R(z) R\left(\frac{1}{z^*}\right)^* + 2N_0$$

$$= 2P\, Q(z) + 2N_0$$

(ii) $\phi_{d\tilde{z}}[n] \triangleq E\{d[m]^* \tilde{z}[m+n]\}$

$$= E\{d[m]^* (\sqrt{2P}\, d[m'] * r[m'] + \tilde{N}[m'])\big|_{m'=m+n}\}$$

$$= \sqrt{2P}\, \phi_{dd}[n] * r[n]$$

$\updownarrow$

$$\Phi_{d\tilde{z}}(z) = \sqrt{2P}\, R(z)$$

(iii) $\phi_{\hat{z}\tilde{z}}[n] \triangleq E\{(\tilde{z}[m] * w[-m]^*)^* \tilde{z}[m']\big|_{m'=m+n}\}$

$$= \phi_{\tilde{z}\tilde{z}}[n] * w[n]$$

$\updownarrow$

$$\Phi_{\hat{z}\tilde{z}}(z) = \Phi_{\tilde{z}\tilde{z}}(z)\, W(z) = (2PQ(z) + 2N_0) \times W(z)$$

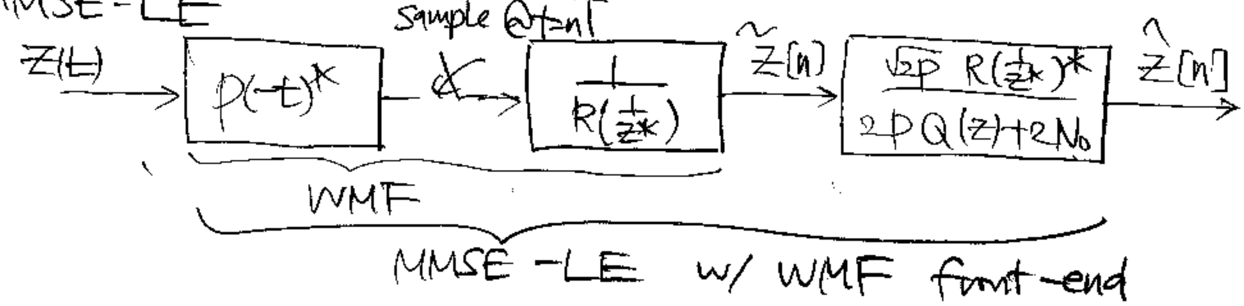
(i),(ii),(iii)

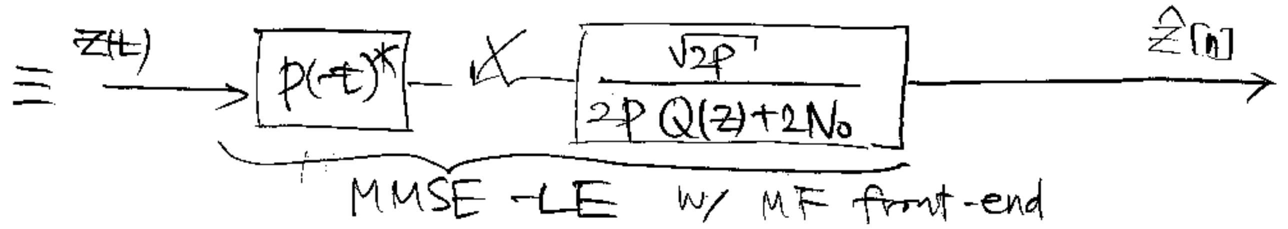
$$\Rightarrow E\{(\hat{z}[m]-d[m])^* \tilde{z}[m+n]\}=0, \ \forall m,n$$

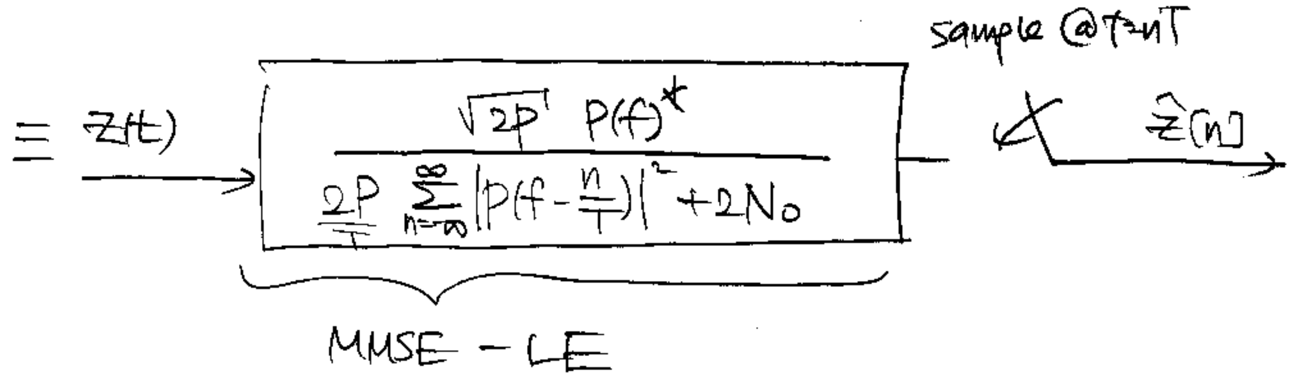
$$\Leftrightarrow \left(2P\,Q(z)+2N_0\right) W(z) = \sqrt{2P}\,R(z)$$

$$\therefore W\left(\tfrac{1}{z^*}\right)^* = \frac{\sqrt{2P}\,R\left(\frac{1}{z^*}\right)^*}{2P\,Q(z)+2N_0} \quad \because Q\left(\tfrac{1}{z^*}\right)^* = Q(z)$$

∘ MMSE-LE







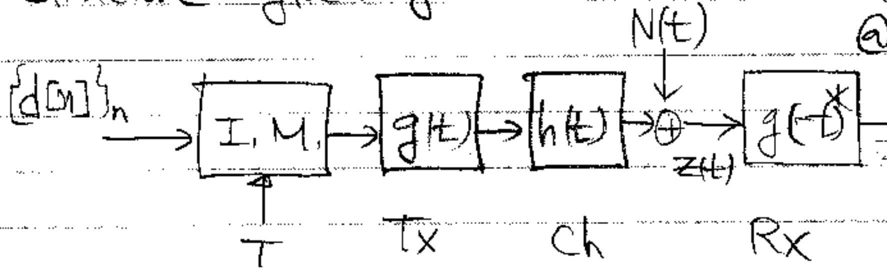
∘ Remarks

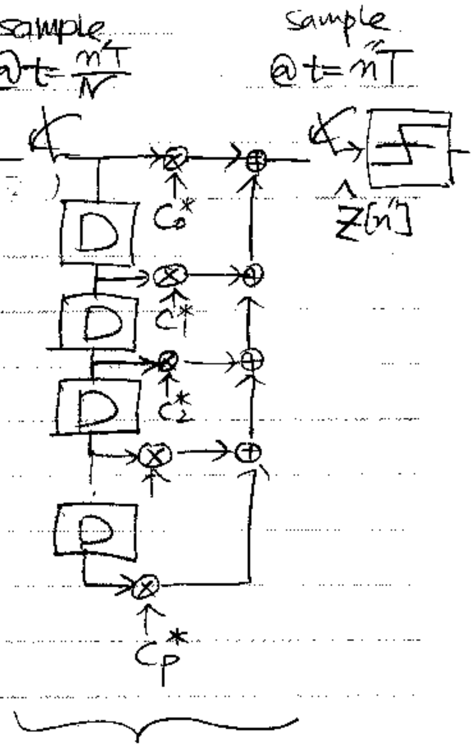
(i) MMSE-LE converges to ZF-LE, as $N_0 \to 0$

(ii) MMSE-LE converges to MF, as $N_0 \to \infty$

○ General structure of a linear equalizer (LE)

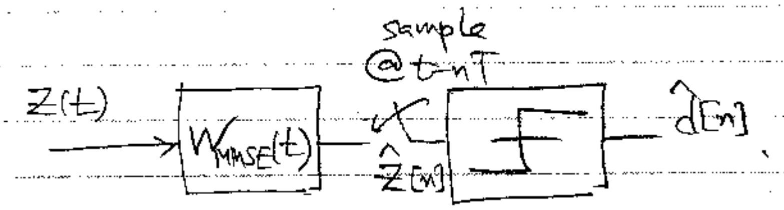
- I believe all of you have some sort of a priori knowledge about linear equalizers. For example, a **fractionally-spaced** linear **equalizer** has the structure given by



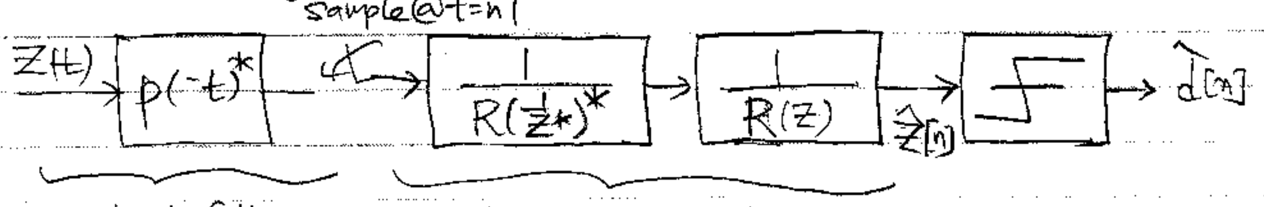


a transversal filter
a tapped delay line
an FIR filter

- You have also seen a continuous-time linear MMSE equalizer given by



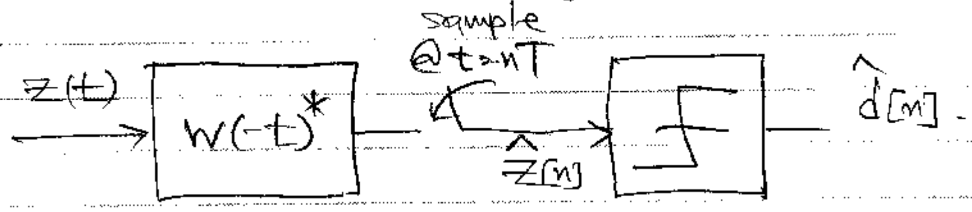
- You have also seen a discrete-time zero-forcing (ZF) linear equalizer given by



matched filter receiver where $p(t)=g(t)*h(t)$

discrete-time linear ZF equalizer

- An inspection of these diagrams will give you an idea that **every linear equalizer** that feeds a decision statistic to a slicer at a time for each data symbol
has **the common structure** given by



because in every case $\hat{z}[n]$ can be written as

$$\hat{z}[n] = \int_{-\infty}^{\infty} z(t)\, w(t-nT)^* \, dt.$$

with some properly chosen $w(t)$.