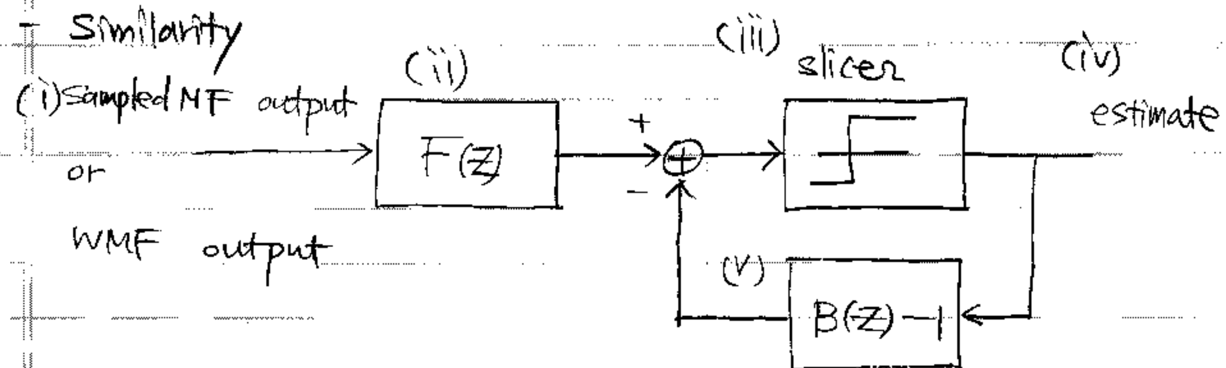
MMSE - DFE

∘ Motivation

We already learned that MMSE-LE has smaller MSE than ZF-LE. So, the idea here is to design MMSE-DFE to achieve a lower MSE than ZF-DFE.

• Similarity and dissimilarity to ZF-DFE

- Similarity



where F(z) : the feedforward filter,
B(z)−1 : the feedback filter, must be causal

After excluding the noise, the slicer input must be $d[n]$,

- Dissimilarity

a. The optimality criterion

ZF : no ISI at the slicer input & minimum MSE

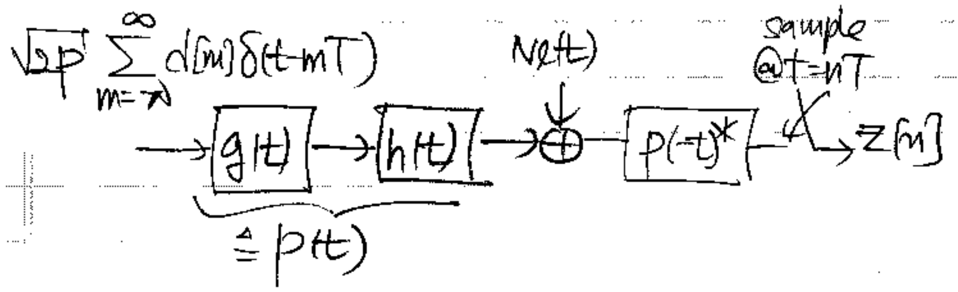
MMSE : minimum MSE at the slicer input.

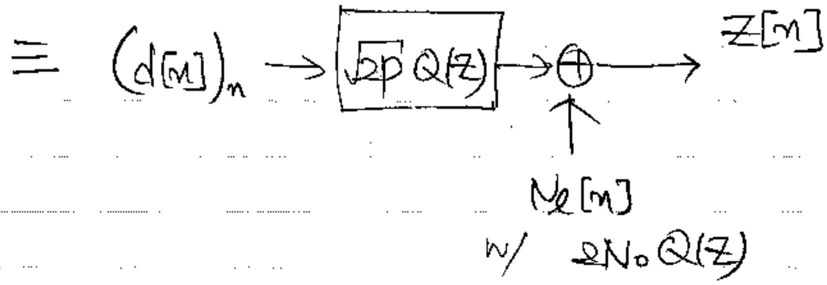
b. F(z)

ZF : must be causal → current symbol decision
MMSE : not necessarily causal → past symbol decision in general.

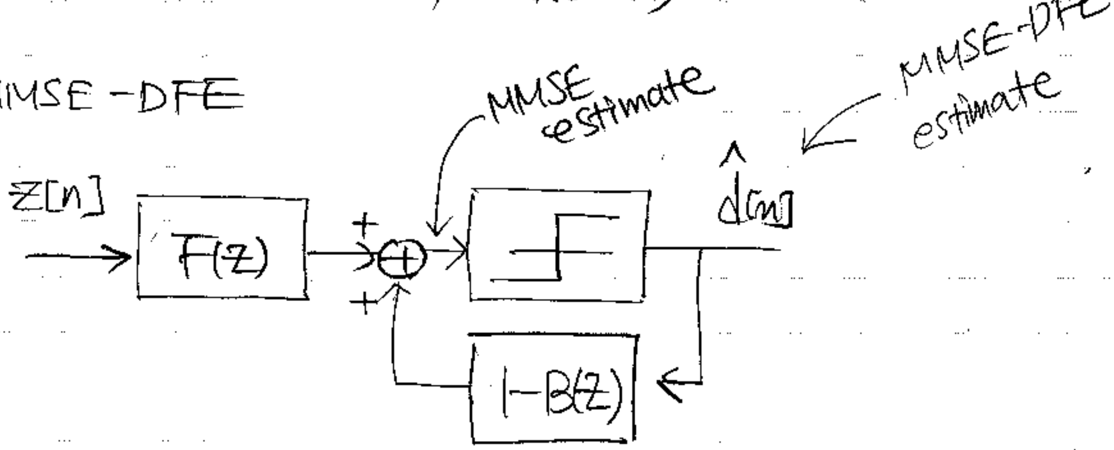
O MMSE - DFE derivation from Sampled MF front end

- Sampled MF front end





- MMSE - DFE



Objective : To find the feedforward filter $F(z)$ & the feedback filter $1-B(z)$ that jointly minimizes the mean squared error b/w the MMSE estimate as the input to the slicer and the true value of the data symbol $d[n]$

Constraints: subject to causal & monic $B(z)$, so that decision is causally fed back to subtract some of the ISI at time n caused by previous data symbols & the current decision is not subtracted at all.

$$
\begin{aligned}
(\quad 1-B(z) &= 1-(b[0]+b[1]z^{-1}+\cdots) \\
&= (1-b[0]) - b[1]z^{-1} - b[2]z^{-2} + \cdots \\
&= \quad 0 \quad - b[1]z^{-1} - b[2]z^{-2} + \cdots
\end{aligned}
$$

- Approach.

estimation error

$$\min_{F(z),B(z)} E\{\|e[n]\|^2\}$$

s.t. monic & causal $B(z)$

$$\equiv \begin{cases} \min\limits_{B(z)} \min\limits_{F(z)} E\{\|e[n]\|^2\} \\ \\ \text{s.t. monic \& causal } B(z) \end{cases}$$

Thus, we first assume that a monic & causal $B(z)$ is given & fixed, and find $F(z)$ that minimizes ~~the~~ MSE.

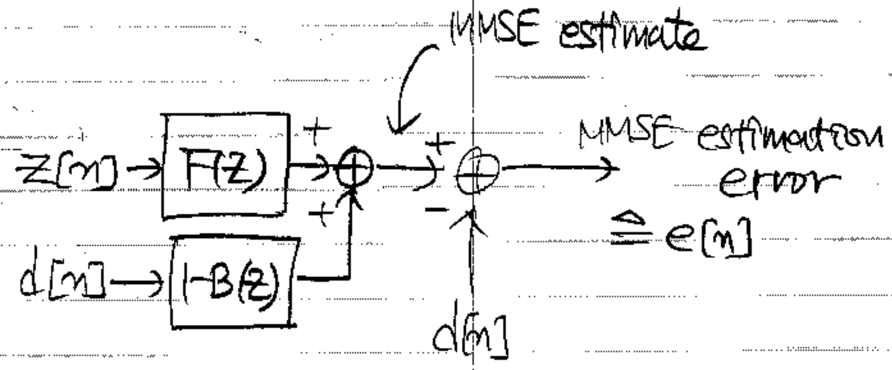
Second, we find $B(z)$ that minimizes the MSE under the monic & causal $B(z)$ constraints.

- Inner Minimization

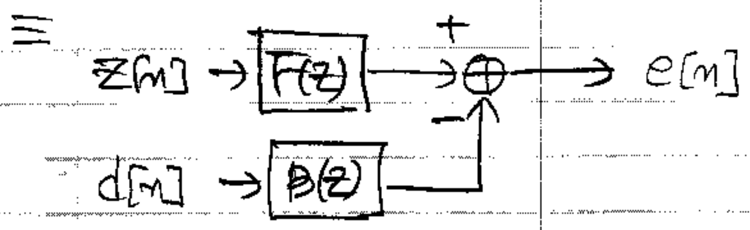
orthogonality principle,

Given a fixed, monic, and causal $B(z)$, by the the MMSE estimation error must be orthogonal to observation.

Since



$\equiv$



the orthogonality principle leads to

$$\underbrace{\Phi_{ze}(z)}_{\substack{\text{ZTF of the} \\ \text{cross correlation} \\ \text{function b/w} \\ z[n] \text{ \& } e[n]}} = \underbrace{\Phi_{zz}(z)}_{\substack{\text{ZTF of the} \\ \text{auto correlation} \\ \text{ft of } z[n]}} F(z) - \underbrace{\Phi_{zd}(z)}_{\substack{\text{ZTF of the} \\ \text{cross correlation} \\ \text{function b/w} \\ z[n] \text{ \& } d[n]}} B(z) = 0. \quad \cdots (*)$$

Note that

$$\Phi_{zz}(z) = \left(\sqrt{2P}\, Q(z)\right)^2 + 2N_0 Q(z)$$

$$= Q(z)\left(2P Q(z) + 2N_0\right).$$

Thus, (*) leads to the feedforward filter given by

$$F(z) = \frac{\Phi_{zd}(z)}{\Phi_{zz}(z)} B(z) = \frac{\sqrt{2P}\, \cancel{Q(z)}\, B(z)}{\cancel{Q(z)}(\qquad)}$$

$$= \frac{B(z)}{\left(\sqrt{2p}\,Q(z) + \frac{2N_0}{\sqrt{2p}}\right)} \triangleq \frac{B(z)}{Q'(z)}$$

Note that the feedforward filter $F(z)$ is uniquely determined, by the orthogonality principle, in terms of $B(z)$ & $Q'(z)$.

– Outer Minimization

Since

$$E(z) = Z(z)F(z) - D(z)B(z)$$

$$= Z(z)\frac{B(z)}{\sqrt{2p}\,Q(z) + \frac{2N_0}{\sqrt{2p}}} - D(z)B(z)$$

$$= B(z)\left(\frac{Z(z)}{Q'(z)} - D(z)\right) \triangleq B(z)\,E'(z),$$

we have

$$\Phi_{e'e'}(z) = \frac{\Phi_{zz}(z)}{(Q'(z))^2} - 2\,\frac{\Phi_{zd}(z)}{Q'(z)} + \Phi_{dd}(z)$$

$$= \frac{\sqrt{2p}\,Q(z)\,Q'(z)}{(Q'(z))^2} - 2\,\frac{\sqrt{2p}\,\Phi_{dd}(z)\,Q(z)}{Q'(z)} + \Phi_{dd}(z)$$

assume white

$$= \frac{\sqrt{2p}\,Q(z) - 2\sqrt{2p}\,Q(z) + Q'(z)}{Q'(z)}$$

$$= \frac{\sqrt{2p}\,Q(z) - \cancel{2\sqrt{2p}\,Q(z)} + \cancel{\sqrt{2p}\,Q(z)} + \frac{2N_0}{\sqrt{2p}}}{Q'(z)}$$

$$= \frac{2N_0}{\sqrt{2p}}\,\frac{1}{Q'(z)}$$

Thus,

$$\Phi_{ee}(z) = B(z)B\left(\frac{1}{z^*}\right)^* \Phi_{dd'}(z) = B(z)B\left(\frac{1}{z^*}\right)^* \frac{2N_0}{\sqrt{2P}}\frac{1}{Q'(z)}$$

$$= B(z)B\left(\frac{1}{z^*}\right)^* \left(\frac{2N_0}{\sqrt{2P}} \frac{1}{\sqrt{2P}Q(z) + \frac{2N_0}{\sqrt{2P}}}\right)$$

- Here we use the fact that the estimation error sequence must be white. Otherwise, the estimation error for the previous symbols can be use to reduce the estimation error for the current symbol.

Therefore,

$$\Phi_{ee}(z) = C, \quad \forall z.$$

This implies

$$B(z)B\left(\frac{1}{z^*}\right)^* \frac{2N_0}{\sqrt{2P}}\frac{1}{Q'(z)} = C \quad \forall z$$

Thus, we need to factor $\frac{\sqrt{2P}}{2N_0}Q'(z)$ as

$$\frac{\sqrt{2P}}{2N_0}Q'(z) = \frac{\sqrt{2P}}{2N_0}\left\{\sqrt{2P}Q(z) + \frac{2N_0}{\sqrt{2P}}\right\} = G(z)G\left(\frac{1}{z^*}\right)^*$$

and use a scaled version of $G(z)$ as $B(z)$.

However, B(z) must be monic & causal. Therefore, G(z) must contain all poles of Q'(z) inside the unit circle.

There are non-unique G(z) that satisfies this condition. Let g[0] be its 0-th term. Then,

$$B(z) = \frac{G(z)}{g[0]} \Rightarrow \Phi_{ee}(z) = \frac{1}{|g[0]|^2}, \quad \forall z$$

Therefore, among causal $G(z)$, we want $G(z)$ that has the maximum value of $|g[0]|^2$. Therefore,

$G(z)$ must be the <u>minimum phase</u>, i.e., $G(z)$ is chosen to contain all poles & zeros of $Q'(z)$ inside the unit circle.

- $$F_{opt}(z) = \frac{\frac{G(z)}{g[0]}}{Q'(z)} = \frac{\cancel{G(z)}}{g[0]} \cdot \frac{1}{\frac{2N_0}{\sqrt{2P}} \cancel{G(z)} G\left(\frac{1}{z^*}\right)^*}$$

$$= \frac{\sqrt{2P}}{g[0]\, 2N_0\; G\left(\frac{1}{z^*}\right)^*}$$

all poles & zero are outside the unit circle

$\Rightarrow$ need to delay the impulse response to realize.