MLSD (Forney vs. Ungerböck)

○ Optimal detection of PAM data symbols in ISI

- There can be many different optimality criteria. Among them we learn

  (i) maximum likelihood sequence detection **(MLSD) by Forney**

  (ii) **MLSD by Ungerböck**

  in this lecture note.

- The main idea of the MLSD is to group all the data symbols into a **single data vector**. For example, if M $|A|$-ary data symbols $\{d[m]\}_{m=0}^{M-1}$ are transmitted using a PAM scheme, then we define

$$\underline{d} \triangleq \begin{bmatrix} d[0] \\ d[1] \\ \vdots \\ d[M-1] \end{bmatrix}$$

as an $|A|^M$-ary symbol to be detected.

ex) $A = \{+1, -1\}$, $A = \{e^{j\frac{\pi}{4}}, e^{j\frac{3\pi}{4}}, e^{j\frac{5\pi}{4}}, e^{j\frac{7\pi}{4}}\}$

This $|A|^M$-ary symbol is detected using the **ML criterion** in an MLSD.

○ Received continuous-time signal

- The signal models in Forney's and Ungerböck's approaches are the same and given by

$$Z(t) = \sqrt{2P} \sum_{m=0}^{M-1} d[m]\, p(t - mT) + N(t)$$

where $z(t)$ is the complex baseband equivalent of the passband received signal, and $N(t)$ is a proper-complex additive **WHITE Gaussian** noise.
The common assumption is that **$p(t)$ and P are known.**

Forney's approach (Fischer pp. 108–112)

- In Forney's approach, the continuous-time received signal is converted to a discrete-time signal using the whitened matched filter (**WMF**).

$$z(t) \longrightarrow \boxed{p(-t)^*} \xrightarrow{\substack{\text{Sample} \\ @t=nT}} \boxed{\frac{1}{R\left(\frac{1}{z^*}\right)^*}} \longrightarrow \tilde{z}[n]$$

- The discrete-time signal $\{\tilde{z}[n]\}_n$ is a sufficient statistic and has a very nice property

$$\tilde{z}[n] = \sqrt{2P} \sum_{m=0}^{M-1} d[m]\, r[n-m] + \tilde{N}[n]$$

where (i) $r[n]$ is **causal** and minimum phase.
(ii) $\tilde{N}[n]$ is white.

$r[n] \simeq 0$ for $n<0,\ n>P$

- From (ii), the MLSD criterion becomes

$$\hat{\underline{d}} = \arg\min_{\underline{d} \in |A|^M} \sum_{n=0}^{M-1+P} \left| \tilde{z}[n] - \sqrt{2P} \sum_{m=0}^{M-1} d[m]\, r[n-m] \right|^2$$

At first glance, to make this decision, we need to evaluate the objective function $|A|^M$ times, where $|A|^M$ grows exponentially.

- From (i) and the special PAM structure, this **exponential complexity** can be reduced.

If $r[n]$ has only a few non-zero terms.

Suppose $r[n] = 0$, $n<0$, $n>p$, so that $r[n]$ has only at most **$p+1$ terms**. Then, ~~to~~ generate a formula

$$\sqrt{2P}\sum_{m=0}^{M-1} d[m]\, r[n-m] \quad \text{for } 0 \le n \le M-1+p$$

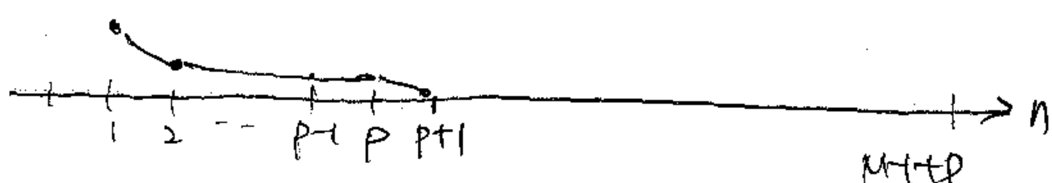
we can use a trellis. In other words, a data sequence $\{d[m]\}_{m=0}^{M-1}$ can be represented as a path on the trellis.

- Let $\tilde{s}[n] \triangleq \sqrt{2P}\sum_{m=0}^{M-1} d[m]\, r[n-m]$.

Then,

the contribution of $d[0]$ is

the contribution of $d[1]$ is



⋮

and

the contribution of $d[M-1]$ is

Thus, $\tilde{s}[0]$ is determined by $d[0]$,

$\tilde{s}[1]$ " " " $d[1], d[0]$

$\tilde{S}[p+1]$ is determined by $d[p], \cdots, d[0]$

$\tilde{S}[p+2]$ 〃 〃 〃 $d[p+1], \cdots, d[1], \cancel{d[0]}$

$\vdots$

$\underbrace{\tilde{S}[n]}_{\text{current signal value}}$ 〃 〃 〃 $\underbrace{d[n-1]}_{\text{current data symbol}}, \underbrace{d[n-2], \cdots, d[n-p-1]}_{\text{past } p \text{ data symbols}},$

$\vdots$

$\tilde{S}[M+1]$ 〃 〃 〃 $d[M], \overbrace{d[M-1], \cdots d[M-p-2]}^{p}$

$\tilde{S}[M+2]$ 〃 〃 〃 $d[M-1], \cdots d[M-p-1]$ X

$\tilde{S}[M+3]$ 〃 〃 〃 $p-1$ $\underbrace{d[M-1], \cdots d[M-p]}_{p-2}$

$\vdots$

$\tilde{S}[M+p-1]$ 〃 〃 〃 $d[M-1]$

We choose for $\tilde{S}[n]$,

- current state $\begin{bmatrix} d[n-2] \\ d[n-3] \\ \vdots \\ d[n-p-1] \end{bmatrix}$
- input $d[n-1]$
- output $r[0]\, d[n-1] + \begin{bmatrix} r[1] \\ r[2] \\ \vdots \\ r[p] \end{bmatrix}^T \begin{bmatrix} d[n-2] \\ \vdots \\ d[n-p-1] \end{bmatrix}$

Then, a data sequence $\{d[m]\}_{m=0}^{M-1}$ can be represented as a path on a trellis with

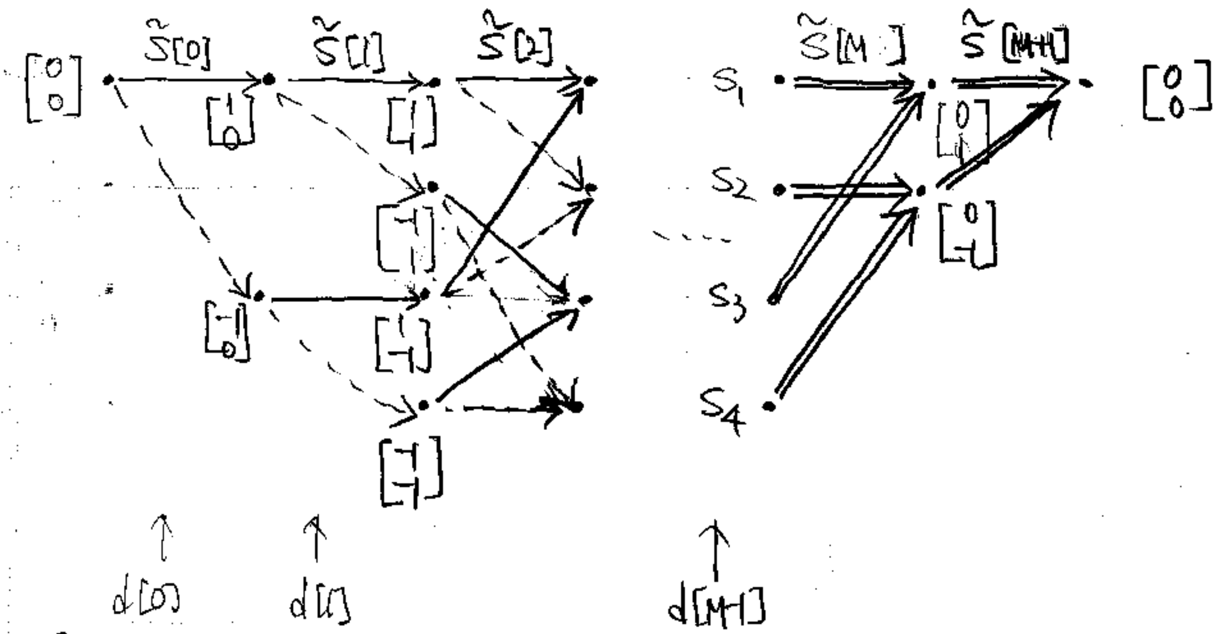
- \# of states $|A|^P$ where $|A|$ is the cardinality of the data symbol alphabet
- \# of state transition $M+P$

  because $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \to \begin{bmatrix} d[0] \\ 0 \\ \vdots \\ 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} d[M-1] \\ d[M-2] \\ \vdots \\ d[M-2-P] \end{bmatrix} \to \cdots \begin{bmatrix} 0 \\ \vdots \\ d[M-1] \end{bmatrix}$

- \# of output symbols $M+P$

  $\tilde{S}[0]$ - - - - - $\tilde{S}[M-1+P]$

/ex/ $P=2$, $A=\{+1,-1\}$ $\Rightarrow$ \# of states $2^2=4$



$\longrightarrow$ input 1
$\dashrightarrow$ input -1
$\Longrightarrow$ input 0

· So, the MLSD reduces to find the path that generates an output sequence $\{\tilde{S}[n]\}_{n=0}^{M+P-1}$ whose distance from the observation sequence $\{\tilde{Z}[n]\}_{n=0}^{M+P-1}$ is minimized.

One important property of the Euclidean distance metric is that

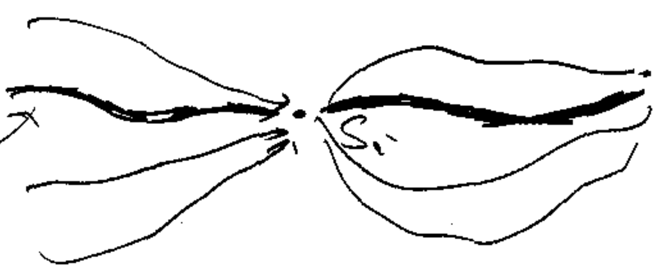
$$d\left((\tilde{S}[n])_{n=0}^{M+P-1}, (\tilde{Z}[n])_{n=0}^{M+P-1} \right)$$

$$= \sum_{n=0}^{M+P-1} \underbrace{\left| \tilde{S}[n] - \tilde{Z}[n] \right|^2}_{\geq 0} \Rightarrow \text{monotonically increasing}$$

Thus, the optimal path that has the state S_i up to time m has the minimum value of

$$\sum_{n=0}^{m} \left| \tilde{S}[n] - \tilde{Z}[n] \right|^2.$$

optimal path



different paths that crosses S_i at m.

So, at each state transition, we need to keep the paths that give the minimum distance up to that state.

- Define

$$J = \sum_{n=0}^{M+P-1} \left| \tilde{Z}[n] - \tilde{S}[n] \right|^2$$

Then, an update formula becomes

$$J_{n'} \triangleq \sum_{n=0}^{n'} \left| \tilde{Z}[n] - \tilde{S}[n] \right|^2$$

$$= J_{n'-1} + \left| \tilde{Z}[n'] - \tilde{S}[n'] \right|^2$$

i.e.,

$$J_{n'} \triangleq \sum_{n=0}^{n'} \left| \tilde{z}[n] - \sqrt{2P} \sum_{m=0}^{M-1} d[m] r[n-m] \right|^2$$

$$= J_{n'-1} + \left| \tilde{z}[n'] - \sqrt{2P} \sum_{m=n'-p}^{n'} d[m] r[n'-m] \right|^2$$

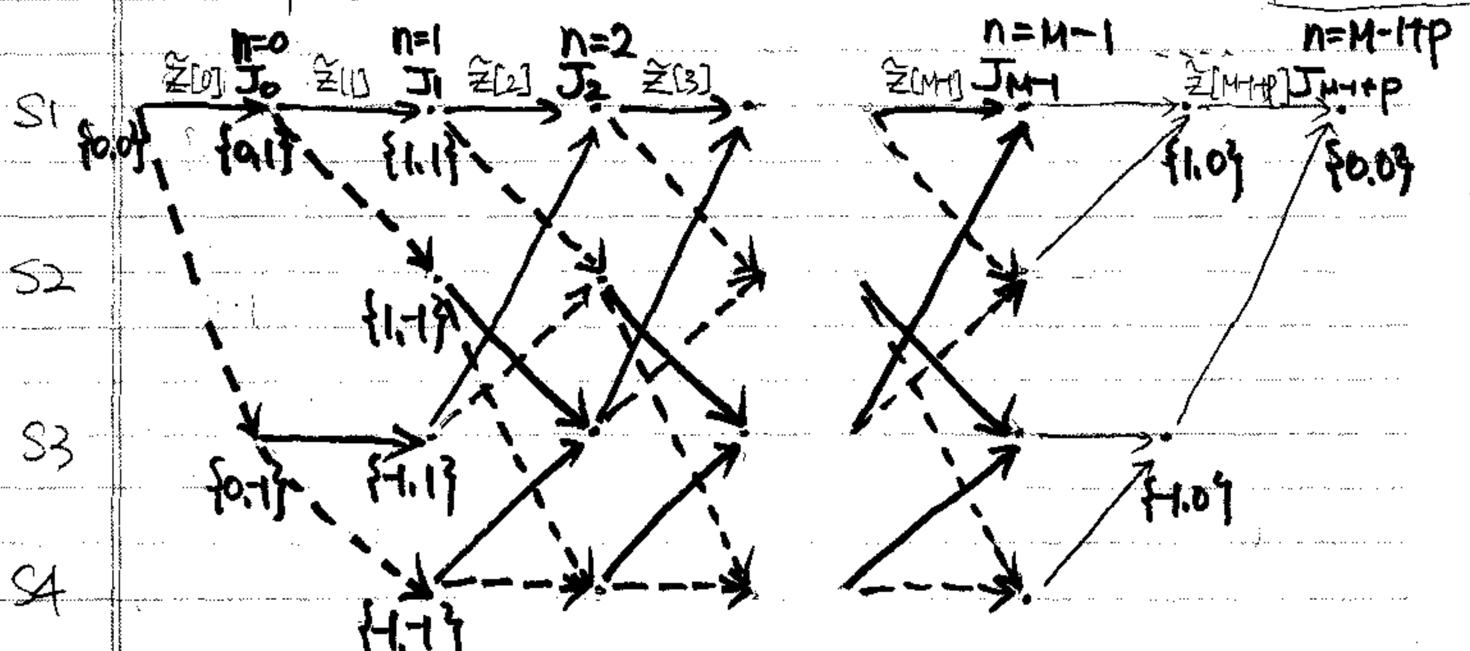
Since the correction term depends only on (i) the current observation, (ii) the past p data symbols, and (iii) the current data symbol, **we just need to keep $J_{n'}$ and $d[n'], \cdots, d[n'-p+1]$ for the next update.**

지금까지의 거리에 해당

So, we take $(d[n'], \cdots, d[n'-p+1])$ as a state.

$\therefore$ The MLSD can be performed on the trellis w/ $|A|^p$ states.

/ex/ $p=2$. $A = \{+1, -1\}$



$\longrightarrow$: $d[n'] = 1$

$- - \rightarrow$: $d[n'] = -1$

$\longrightarrow$: $d[n'] = 0$

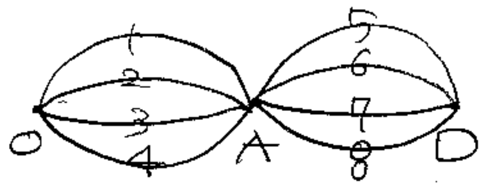
$\{\cup, \bigcirc\}$

↑ past ↑ current

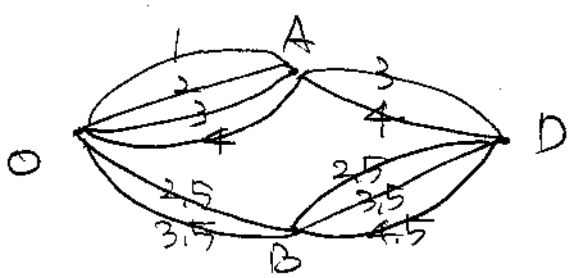
$p=2$

- Examples

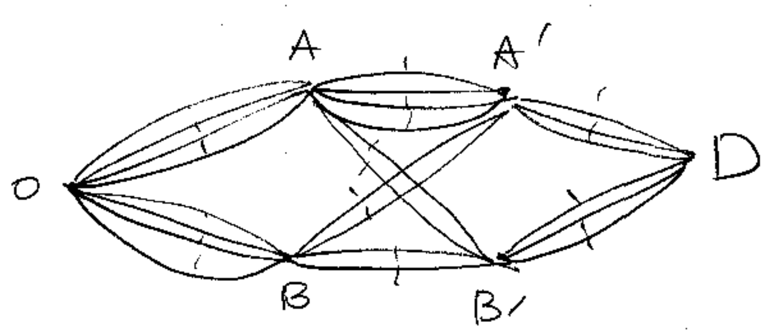
Q1. What is the minimum distance from O to D.



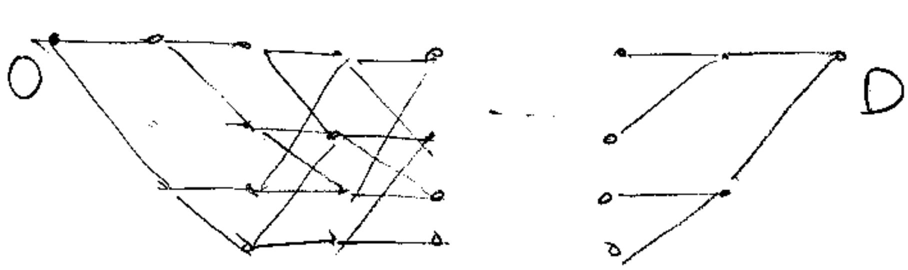
Q2. What is the minimum distance from O to D.



Q3. What is the minimum distance from O to D



Q4. What is the minimum distance from O to D



- Note that
  if the optimal path in the trellis

  that minimizes $\sum_{n=0}^{M-1+P-1} \left| \hat{z}[n] - \sqrt{2P} \sum_{m=0}^{M-1} d[m]\, r[n-m] \right|^2$

  passes through the kth state at time instant l

  then its partial path from $n=0$ till $n=l$ is the path that minimizes the partial distance

  $\sum_{n=0}^{l} \left| \hat{z}[n] - \sqrt{2P} \sum_{m=0}^{l} d[m]\, r[n-m] \right|^2.$

  So, at partial time instant 2, we need to track those 4 partial paths which arrive at the states S1, ..., S4, respectively, with the minimum partial distances.

  At $n=3$, by comparing 2 partial paths per state, we need to track those 4 partial paths which arrive at the states with the minimum partial distances.

  In this way, we keep only 4 $|A|^P$ paths at each time instant and eventually at $n = M-1+P-1$! select one complete path that has the minimum distance from $\{\hat{z}[n]\}_{n=0}^{M-1+P-1}$

  The complexity of this Viterbi algorithm is just proportional to $4 \times (M-1+P-1)$. In general, the complexity grows in $O(M)$ given P.

which is much smaller than the brute-force search whose complexity grows in $\underbrace{O(|A|^M)}_{\text{exponential complexity in } M} \; (\gg \underbrace{O(M)}_{\text{linear in } M})$

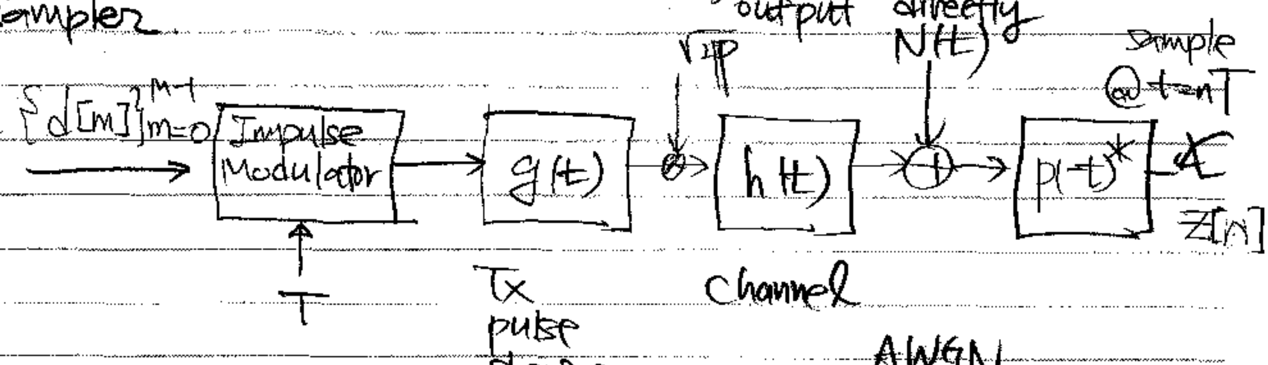
- This Forney's approach may not work if $\frac{1}{R(\frac{1}{z^*})^*}$ does not exist. < rational approximation X / whitening filter not stable.

  It may not have low complexity if p is large so that the # of states is too large.

- Of course, other than Viterbi algorithm can be used for suboptimal but low-complexity sequence detection.

◯ Ungerböck's approach

- If the WMF observation model is not applicable what do we do?
  A more general approach would be that uses the symbol waveform matched filters and the baud-rate sampler.



where $p(t) = g(t) * h(t)$.

- The problem is that in general $z[n]$ has ISI terms and the noise sequence $N[n]$ is not white.
The DT symbol waveform is not causal but conjugate symmetrical.
If we define $\underline{z}$ that contains all the relevant samples from $\{z[n]\}_n$, then

$\underline{z} \sim CN(\underline{s}(\underline{d}), 2N_0 R)$ where $R_{ij} = \beta((i-j)T)$.

Thus, the direct application of the ML criterion leads to

$$\hat{\underline{d}} = \underset{\underline{d} \in A^M}{\arg\min} \ (\underline{z} - \underline{s}(\underline{d}))^H R^{-1} (\underline{z} - \underline{s}(\underline{d})) \quad \cdots (*)$$

which requires $|A|^M$ computations of the quadratic form.

- Ungerböck showed that the ML detection $(*)$ actually can be performed using a Viterbi algorithm that has **linear complexity in M**

How?

- From the detection theory, we know that the ML rule in CT AWGN is given by

$$\hat{\underline{d}} = \underset{\underline{d} \in A^M}{\arg\min} \int_{-\infty}^{\infty} \left| z(t) - \sqrt{2P} \sum_{m=0}^{M-1} d[m]\, p(t - mT) \right|^2 dt$$

$$= \underset{\underline{d}\in A^M}{\arg\max}\left[\mathrm{Re}\left\{\int_{-\infty}^{\infty} z(t)\sqrt{2P}\sum_{m=0}^{M-1} d[m]^* p(t-mT)^* dt\right\} \underbrace{- \frac{1}{2}\left\|\sqrt{2P}\sum_{m=0}^{M-1} d[m]\, p(t-mT)\right\|^2}_{J(d[0],\cdots,d[M-1])}\right] \quad \text{---}(**)$$

- The first term in (**) can be rewritten as

$$\mathrm{Re}\left\{ \sqrt{2P}\sum_{m=0}^{M-1} d[m]^* \underbrace{\int_{-\infty}^{\infty} z(t)\, p(t-mT)^* dt}_{z[m]} \right\}$$

which just processes the MF output $z[m]$.

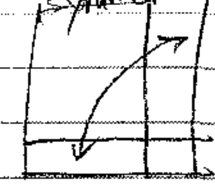
- The second term in (**) can be rewritten as

$$-P\sum_{i=0}^{M-1}\sum_{j=0}^{M-1} d[i]^* \tilde{p}((i-j)T)\, d[j]$$

- Thus, if we assume that

$$\tilde{p}(mT) = 0 \quad \text{for} \quad |m| \geq P$$

Hermitian symmetrical



then

$$J(d[0],\cdots,d[n-1]) = J(d[0],\cdots,d[n-2])$$
$$+\mathrm{Re}\left\{ \sqrt{2P}\, d[n-1]^* z[n-1] - P|d[n-1]|^2 \tilde{p}(0) - 2P\sum_{j=2}^{P_1} d[n-1]^* \tilde{p}((j-2)T)\, d[n-j] \right\}$$

So, at each recursive update, # of possible states is $|A|^p$ because the current $d[n-1]$ and the past $(p-1)$ data symbols $d[n-2], d[n-3], \ldots, d[n-p]$ determine the amount of the correction/update term.

- This again can be implemented using a Viterbi algorithm. The complexity grows in $O(M)$ given p. However, if p large, then the # of states is too large to be efficiently implemented.

Preview

Instead of MLSD, we consider **linear receivers** that mitigate (somehow) the effect of ISI at the input to the decision device (usually a threshold device or a slicer).

Of course, there are nonlinear receivers but we do not study in this course.

These receivers are called **equalizers**. In a wide-sense, the MLSD is a nonlinear equalizer.