Matched filter bound

○ Bound on what?

- When we talk about a bound, we need to specify on what the bound is a bound.

  For example, $\frac{1}{2}e^{-x^2}$ is an upper bound on $Q(x)$ for $x \geq 0$, i.e.,

  $$Q(x) \leq \frac{1}{2}e^{-x^2}, \quad \forall x \geq 0$$

- The matched filter bound is the term used for a few different situations. We study these situations in this note.

○ The matched filter bound on SNR.

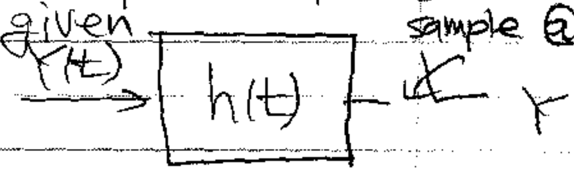
- Consider the following observation model

  $$Y(t) = S(t) + N(t)$$

  where $S(t)$ is a known deterministic signal, and $N(t)$ is a proper-complex additive white noise with two-sided PSD σ^2.

- Q1. What is the signal-to-noise ratio (SNR) at the output sample of a linear filter?

  IOW, given



  where

  $$Y = S(t) * h(t)\Big|_{t=T_0} + N(t) * h(t)\Big|_{t=T_0}$$

What is $\text{SNR} \triangleq \dfrac{\left| s(t) * h(t) \big|_{t=T_0} \right|^2}{E\left\{ \left| N(t) * h(t) \big|_{t=T_0} \right|^2 \right\}}$?

- A1. It depends on $h(t)$ and T_0.

Q2. Then, what is an upper bound on the SNR?

- A2. Since

$$\text{SNR} = \frac{\left| \int_{-\infty}^{\infty} s(t)\, h(T_0 - t)\, dt \right|^2}{\sigma^2 \int_{-\infty}^{\infty} |h(t)|^2 dt}$$

$$\leq \frac{\left| \sqrt{\int_{-\infty}^{\infty} |s(t)|^2 dt} \sqrt{\int_{-\infty}^{\infty} |h(T_0 - t)|^2 dt} \right|^2}{\sigma^2 \int_{-\infty}^{\infty} |h(t)|^2 dt}$$

$$= \frac{\int_{-\infty}^{\infty} |s(t)|^2 dt}{\sigma^2}$$

where the equality holds if and only if

$$s(t) = k\, h(T_0 - t)^* \quad \text{for some } k \neq 0$$

$$\left(\Leftrightarrow h(t) = k' s(T_0 - t)^* \quad \text{for some } k' \neq 0. \right)$$

$\dfrac{\int_{-\infty}^{\infty} |s(t)|^2 dt}{\sigma^2}$ is an upper bound on the SNR.

This upper bound is called the matched filter bound on the SNR.

○ The matched filter bound on BER.

• Consider the following observation model.

$$Y(t) = A \sum_{m=M}^{N} d[m]\, p(t-mT) + N(t)$$

where $Pr(d[m]=1) = Pr(d[m]=-1) = 1/2$ and $\{d[m]\}_m$ is an i.i.d sequence, and $N(t)$ is a proper complex AWGN w/ two-sided PSD N_0.

• Q1. What is the BER (bit error rate) of the optimum receiver?

• A1. It depends on the optimality criterion and $p(t)$.

Q2. What is the BER of the bit-by-bit MAP detector?

• A2. ~~It is difficult to find.~~ However we can find a lower bound. The bit-by-bit MAP detector views the observation $Y(t)$ as

$$Y(t) = \underbrace{A\, d[n]\, p(t-nT)}_{\text{desired signal}} + \Big(\underbrace{A \sum_{m \neq n} d[m]\, p(t-mT)}_{\text{interference}} + \underbrace{N(t)}_{\text{noise}} \Big) \quad (*)$$

The matched filter output $\underline{Y} = p(-t)^* * Y(t)\Big|_{t=MT}^{NT}$ is sufficient. Note that we can put $(*)$ as

$$\underline{Y} = d[n]\, \underline{s} + \underline{I} + \underline{N}$$

Since $\underline{I}$ is independent of $\underline{N}$ and $d[n]$, the BER of the optimum detector is lower bounded by the BER of that for

$$\underline{\tilde{r}} = d[n]\,\underline{s} + \underline{N}$$

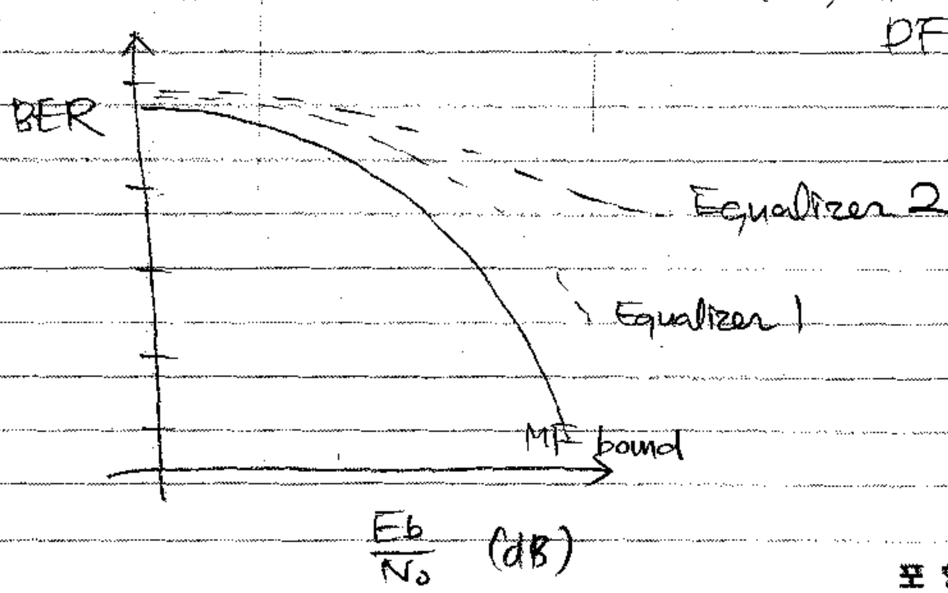
which is $$Q\left(\sqrt{\frac{A^2\int_{-\infty}^{\infty}|p(t)|^2dt}{\frac{N_0}{2}}}\right)$$

This is called the MF bound on BER for linearly modulated signal w/ BPSK symbols.

- The BERs of the ZF-LE and the MMSE-LE are both lower bounded by this matched filter bound

Actually, this MF bound is the lower bound on the BER of any linear & non-linear receiver that demodulates an uncoded BPSK data symbols.

(ex) MLSD
DFE



BW of a real-valued 2nd-order random process (Peebles, Sklar)

○ PSD

Def. ~~Let~~ $X(t)$ ~~be~~ a real-valued 2nd-order random process. Then, its power spectral density $S_{XX}(f)$ is defined as

$$S_{XX}(f) = \mathcal{F}\left[A\{ E[X(t)X(t+\tau)] \} \right]$$

where $\mathcal{F}$ denotes the Fourier transform, A denotes the time average, and E denotes the expectation

- $A\{a(t)\} \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} a(t)\,dt$

  $\mathcal{F}\{b(\tau)\} \triangleq \int_{-\infty}^{\infty} b(\tau)\, e^{-j2\pi f\tau}\,d\tau$

- Alternately,

$$S_{XX}(f) = \int_{-\infty}^{\infty} \left(\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E[X(t)X(t+\tau)]\,dt \right) e^{-j2\pi f\tau}\,d\tau$$

○ Property

- $S_{XX}(f)$ is real, even, and non-negative for all f

Q1. Show the above property is true for a real-valued WSS random process.

~~A1~~. (Outline)

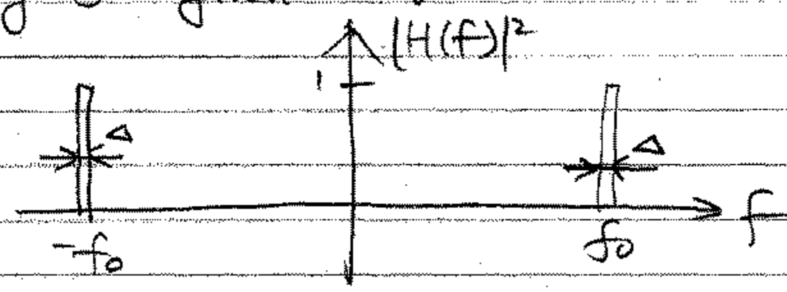
(i) We show that $R_{XX}(\tau) \triangleq E\{X(t)X(t+\tau)\}$ is real and even.

$\Rightarrow S_{XX}(f)$ is real and even.

(ii) Using the fact that

$$S_{YY}(f) = S_{XX}(f)\,|H(f)|^2,$$

we filter $X(t)$ using an LTI system w/ the impulse response $h(t)$ whose energy spectral density is given by



Then, $\int_{-\infty}^{\infty} S_{YY}(f)\,df \cong S_{XX}(f_0)\,2\Delta$

(iii) We show that $\int_{-\infty}^{\infty} S_{YY}(f)\,df = E\{Y(t)^2\} \geq 0$

$\therefore S_{XX}(f)$ is real, even, and non-negative.

Q.2. Show the above property is true for a general real-valued 2nd-order random process.

A.2. It would be great if we can show

(i) $\tilde{R}_{XX}(\tau) \triangleq A[E\{X(t)X(t+\tau)\}]$ is real & even &

(ii)' $\tilde{R}_{YY}(\tau) = \tilde{R}_{XX}(\tau) * h(\tau) * h(-\tau)^*$ (Why?)

This outline should give you a lesson: If I don't know about the derivation of a simpler result, then it is impossible for me to generalize the result.

Showing (i)' requires the belief that

$$\lim_{T\to\infty} \int_{-T}^{T} a(t)\,dt = \lim_{T\to\infty} \int_{-T+\alpha}^{T+\alpha} a(t)\,dt \quad \forall \alpha.$$

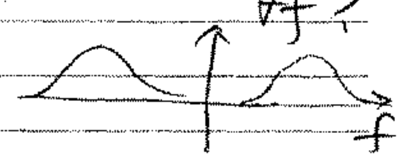
Actually, I have no idea whether (ii)' is true or not. You try it and let me know the result. Thanks in advance!

○ BW

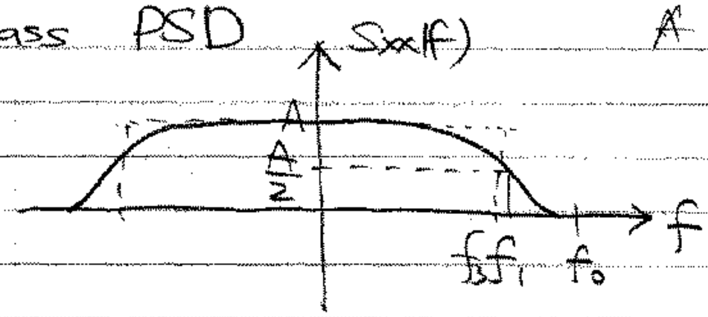
- We consider two cases
  (i) A lowpass PSD
  (ii) A bandpass PSD

Q. $S_{XX}(f) \le S_{XX}(0)$, $\forall f$?

A.



- A lowpass PSD



a. Strict-sense bandwidth : f_0 (Existence not guaranteed)
b. 3-dB bandwidth : f_1

c. Root mean square (RMS) bandwidth :

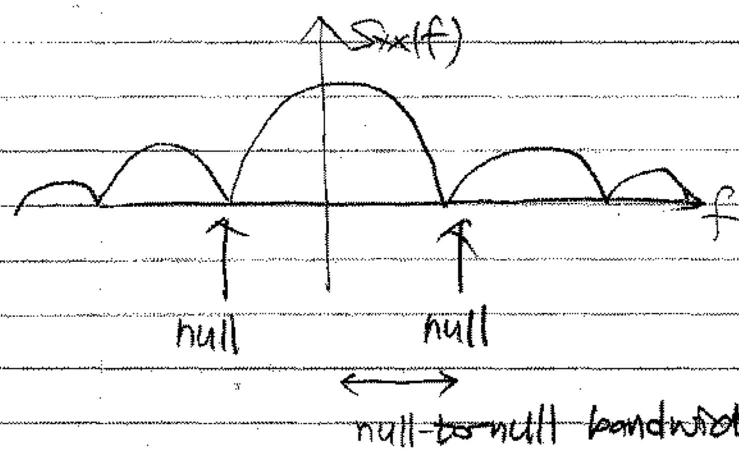
$$W_{rms} = \sqrt{\int_{-\infty}^{\infty} f^2 \frac{S_{xx}(f)}{\int_{-\infty}^{\infty} S_{xx}(f')df'} df}$$

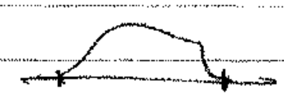
d. Equivalent rectangular bandwidth : f_3

$$S_{xx}(0) f_3 = \int_0^{\infty} S_{xx}(f)\, df$$

("Noise equivalent bandwidth" if $x(t)$ is noise.)

e. Null-to-null bandwidth





$f_4 = f_0$ mostly case

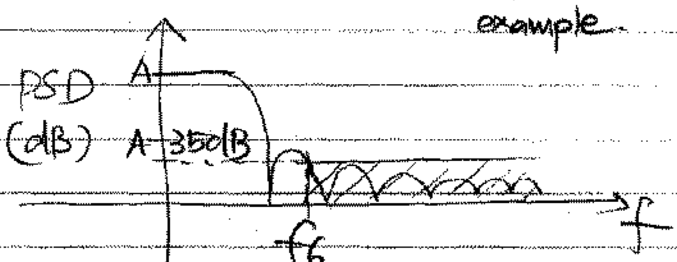
f. fractional power containment bandwidth

ex) 99%-power bandwidth : f_5

$$\int_0^{f_5} S_{xx}(f)df = 0.99 \int_0^{\infty} S_{xx}(f)df$$

h. Bounded PSD at 35 dB : f_6 (same idea as 3-dB BW)

example



∘ A bandpass PSD : DIY.

○ Remarks

information-theoretically

- The strict-sense bandwidth is the only meaningful definition of BW.
- These various definitions have their own uses.
- FCC does not regulated in terms of one of these definitions. It imposes a "spectral mask" looking like