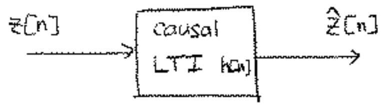
Derivation of causal linear MMSE equalizer

- **Observation model** : WMF output observation



$z[n] = \sqrt{2P}\, d[n] * r[n] + N[n]$

where $r[n]$ causal and $N[n]$ is a Gaussian r.p. with PSD $2N_0$.

- **Objective**

Estimate $d[n]$ based on $z[n], z[n-1], z[n-2], \cdots$ in MMSE sense.

- **Derivation**

By orthogonality principle,

$$E\left\{ (z[n] * h[n] - d[n])^* z[\ell] \right\} = 0, \quad \forall \ell \le n$$

$$\Leftrightarrow \quad E\left\{ \left(\sum_{m=0}^{\infty} h[m] z[n-m] - d[n] \right)^* z[\ell] \right\} = 0, \quad \forall \ell \le n$$

$$\Leftrightarrow \quad \sum_{m=0}^{\infty} h^*[m] E\left\{ z^*[n-m] z[\ell] \right\} - E\left\{ d^*[n] z[\ell] \right\} = 0, \quad \forall \ell \le n$$

$$\Leftrightarrow \quad \sum_{m=0}^{\infty} h^*[m] R_{zz}[\ell-n+m] = R^*_{zd}[n-\ell], \quad \forall \ell \le n$$

$n - \ell = k$

$$\Leftrightarrow \quad \sum_{m=0}^{\infty} h^*[m] R_{zz}[m-k] = R^*_{zd}[k], \quad \forall k \ge 0$$

$$\Leftrightarrow \quad h^*[k] * R_{zz}[k] = R^*_{zd}[k], \quad \forall k \ge 0$$

$$\Leftrightarrow \quad \left[H^*(z^*) R_{zz}(z) - R^*_{zd}(z^*) \right]_+ = 0 \qquad \cdots (*)$$

where $[X(z)]_+$ denotes the uni-lateral z-transform of $x[n]$ when $X(z)$ is the bi-lateral z-transform of $x[n]$.

We are going to spectrally factorize $R_{zz}(z)$ by combining the fact that

1. $R_{zz}[n]$ has conjugate symmetry, i.e., $R_{zz}[n] = R^*_{zz}[-n]$ and
2. $R_{zz}(z)$ can be approximated by a rational function.

$$\Rightarrow \quad R_{zz}(z) = \underbrace{\hat{R}_{zz}(z)}_{\text{causal}} \underbrace{\hat{R}^*_{zz}\left(\tfrac{1}{z^*}\right)}_{\text{anti-causal}}$$

Thus, $(*)$ can be rewritten as

$$\left[H^*(z^*) \hat{R}_{zz}(z) \hat{R}^*_{zz}\left(\tfrac{1}{z^*}\right) - R^*_{zd}(z^*) \right]_+ = 0$$

Let $G(z) = H^*(z^*) \hat{R}_{zz}(z)$, then $G(z)$ only consists of constant and negative powers of z.

$$\Leftrightarrow \quad \left[G(z) \hat{R}^*_{zz}\left(\tfrac{1}{z^*}\right) - R^*_{zd}(z^*) \right]_+ = 0$$

$$\Leftrightarrow \quad \left[\hat{R}^*_{zz}\left(\tfrac{1}{z^*}\right) \left(G(z) - \frac{R^*_{zd}(z^*)}{\hat{R}^*_{zz}\left(\frac{1}{z^*}\right)} \right) \right]_+ = 0$$

Here, $G(z) - \dfrac{R^*_{zd}(z^*)}{\hat{R}^*_{zz}\left(\frac{1}{z^*}\right)}$ must have only positive powers of z because $\hat{R}^*_{zz}\left(\frac{1}{z^*}\right)$ consists of constant and positive powers of z.

$$\Leftrightarrow \quad \left[G(z) - \frac{R^*_{zd}(z^*)}{\hat{R}^*_{zz}\left(\frac{1}{z^*}\right)} \right]_+ = 0$$

$$\Leftrightarrow \quad G(z) = \left[\frac{R^*_{zd}(z^*)}{\hat{R}^*_{zz}\left(\frac{1}{z^*}\right)} \right]_+$$

$$\Leftrightarrow \quad H^*(z^*) = \frac{1}{\hat{R}_{zz}(z)} \left[\frac{R^*_{zd}(z^*)}{\hat{R}^*_{zz}\left(\frac{1}{z^*}\right)} \right]_+$$

Thus, $H(z) = \dfrac{1}{\hat{R}^*_{zz}(z^*)} \left[\dfrac{R_{zd}(z)}{\hat{R}_{zz}(z^{-1})} \right]_+$. Here, $R_{zz}(z) = 2PQ(z) + 2N_0$ and $R_{zd}(z) = \sqrt{2P}\, R^*\left(\frac{1}{z^*}\right)$