Derivation of ZF − DFE

①

- signal model

MF front end

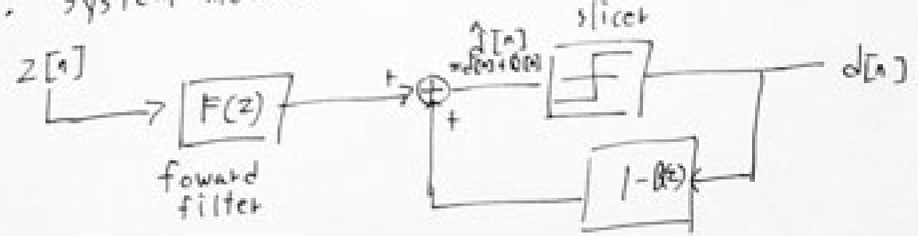
$$Z[n] = \sqrt{2P} \sum_{m=-\infty}^{\infty} d[m]\, g[n-m] + N[n]$$

$d[n]$ is uncorrelated, unit power

$g[n] = g[-n]^*$

$N[n]$ is gaussian r.p mean 0, PSD $2N_0 Q(e^{j2\pi f})$

- System model



- objective

minimize $E[(e[n])^2]$ over $F(z), B(z)$ $\quad \left(\begin{aligned} e[n] &\triangleq \hat{d}[n] - d[n] \\ &= \hat{N}[n] \end{aligned} \right.$

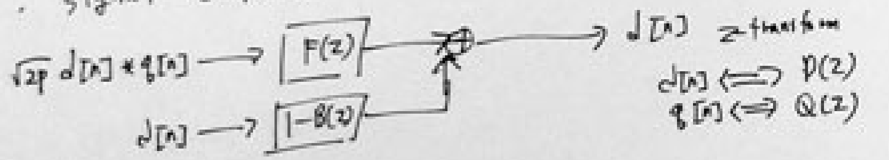
subject to ① No ISI at the slicer input $(\hat{d}[n] = d[n] + \hat{N}[n]$

② monic and causal $B(z)$

◎ Derivation

We first assume that a monic and causal $B(z)$ is given and fixed, find $F(z)$ that makes $\hat{d}[n] = d[n] + \hat{N}[n]$

- signal component



⇓

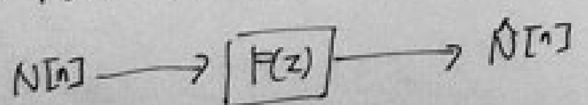
$$\sqrt{2P}\, D(z)Q(z)F(z) + D(z)(1-B(z)) = D(z), \quad \forall z$$

$$\Leftrightarrow \sqrt{2P}\, Q(z)F(z) = B(z), \quad \forall z$$

$$\therefore F(z) = \frac{B(z)}{\sqrt{2P}\, Q(z)}, \quad \forall z$$

$F(z)$ is uniquely determined in terms of $B(z)$ and $Q(z)$

- Noise component = estimation error



$$S_{\hat{N}\hat{N}}(z) = F(z)F\left(\tfrac{1}{z^*}\right)^* S_{NN}(z)$$

$$\Leftrightarrow S_{\hat{N}\hat{N}}(z) = 2N_0 Q(z) F(z) F\left(\tfrac{1}{z^*}\right)^*$$

$$= 2N_0 Q(z) \left(\frac{B(z)}{\sqrt{2P}\, Q(z)}\right)\left(\frac{B\left(\frac{1}{z^*}\right)^*}{\sqrt{2P}\, Q(z)}\right)$$

$$= \frac{N_0}{P} \times \frac{B(z)B\left(\frac{1}{z^*}\right)^*}{Q(z)}, \quad \forall z$$

$\left(\begin{aligned} &g[n] = g[-n]^* \\ &\Leftrightarrow Q(z) = Q\left(\tfrac{1}{z^*}\right)^* \end{aligned} \right.$ ②

(SNR of slicer input is $\dfrac{1}{E[\hat{N}[n]^2]} \triangleq \dfrac{1}{\sigma^2} = \dfrac{1}{\int_{0}^{1} S_{\hat{N}\hat{N}}(e^{j2\pi f})\,df}$

we already know about $\hat{N}[n-1]$ if $\hat{d}[n-1] - d[n-1]$

If $\hat{N}[n]$ is colored noise then $\hat{N}[n]$ and $\hat{N}[n-1]$ are correlated. $\quad (E[\hat{N}[n]\hat{N}[n-1]] = \rho\sigma^2.)$

Consider linear combination of $\hat{N}[n]$ and $\hat{N}[n-1]$.

$$\tilde{N} = \hat{N}[n] + b\hat{N}[n-1]$$

$$SNR(\tilde{N}) = \frac{1}{E[(\hat{N}[n] + b\hat{N}[n-1])^2]} = \frac{1}{\sigma^2(b^2 + 2b\rho + 1)}$$

This value is maximized when $b = -\rho$ and maximum value is $\dfrac{1}{\sigma^2(1-\rho^2)}$

This gives the inequality $\dfrac{1}{\sigma^2} \le \dfrac{1}{\sigma^2(1-\rho^2)}$ because $-1 \le \rho \le 1$

Thus SNR (slicer input) $\le SNR(\tilde{N})$

This implies that we can always increase the SNR by observing other correlated noise /

Thus, to minimize $E[(e[n])^2]$, $\hat{N}[n]$ must be white noise

$$S_{\hat{N}\hat{N}}(z) = \frac{N_0}{P} \times \frac{B(z)B\left(\frac{1}{z^*}\right)^*}{Q(z)} = \text{constant} \;\; \forall z \;\; = C$$

$$\Leftrightarrow B(z)B\left(\tfrac{1}{z^*}\right)^* = C \times \frac{P}{N_0} \times Q(z), \quad \forall z$$

suppose that $Q(z)$ can be well approximated by rational function, then $Q(z)$ can be expressed by $R(z)R\left(\frac{1}{z^*}\right)^*$

$$Q(z) = R(z)R\left(\tfrac{1}{z^*}\right)^*$$

$$\Leftrightarrow B(z)B\left(\tfrac{1}{z^*}\right)^* = C \times \frac{P}{N_0} \times R(z)R\left(\tfrac{1}{z^*}\right)^*$$

There are non unique $B(z)$ that satisfies this condition

$B(z) = k R(z) \quad \forall z, k \quad$ k is constant

However, $B(z)$ must be monic and causal, (3)

$R(z)$ must contain all poles of $Q(z)$ inside the unit circle and $B(z) = \frac{R(z)}{F[0]}$ for monic $B(z)$

then $S_{DD}(z) = \frac{P}{N_0} \times \frac{1}{|F[0]|^2}$, $\forall z$

To minimize $S_{DD}(z)$, $R(z)$ must be minimum phase.

($\because$ minimum phase sequence has the maximum energy containment around $n=0$

$R(z)$ is minimum phase, then $R(z)$ must contain all poles and zeros of $Q(z)$ inside the unit circle.

$\therefore$ $B(z)$ is uniquely defined

$B(z) = \frac{R(z)}{F[0]}$ ($R(z)$ contains all poles and zeros of $Q(z)$ inside unit circle

Thus, $F(z) = \frac{B(z)}{\sqrt{2p}\, Q(z)} = \frac{R(z)}{\sqrt{2p}\, Q(z)} \times \frac{1}{F[0]} = \frac{1}{\sqrt{4p}\, F[0]\, R(\frac{1}{z^*})^*}$

$\therefore$ $z[n] = \sqrt{2p} \sum_{m=-\infty}^{\infty} d[m] q[n-m] + N[n]$