Derivation of MMSE – DFE

- signal model

MF front end

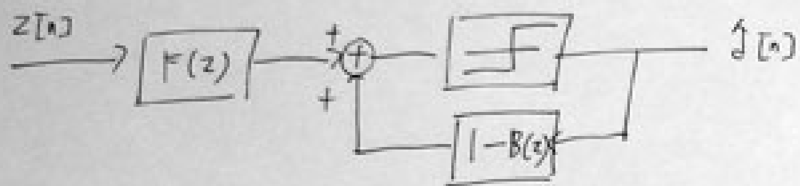
$$z[n] = \sqrt{2p}\sum_{m=-\infty}^{\infty} d[m]\, g[n-m] + N[n]$$

$d[n]$ is uncorrelated, unit power

$g[n] = g[-n]^*$

$N[n]$ is gaussian R.P mean 0, PSD $2N_0 Q(e^{j\omega T})$

- System model



- objective

minimize $E[\, \|e[n]\|^2 \,]$ $\quad$ $e[n] \triangleq$ estimation error

$F(z), B(z)$

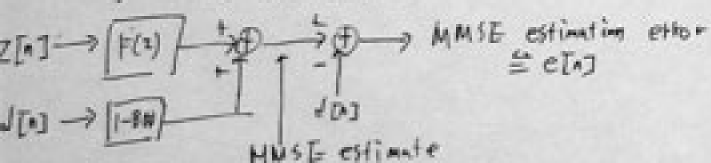
subject to monic and causal $B(z)$

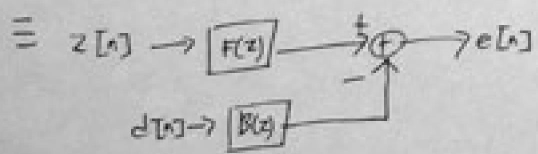
◎ Derivation

- Inner optimization

We first assume that a monic and causal $B(z)$ is given and fixed.

By the orthogonality principle the MMSE estimation error must be orthogonal to observation.





orthogonality principle leads to

$$E[\, z[n]^* e[m+n] \,] = 0$$

$$= E\left[z[n]^* \left(z[n'] * f[n'] - d[n'] * b[n'] \right)\Big|_{n'=m+n} \right]$$

$\Updownarrow$ z-transform

$$\Phi_{ze}(z) = \Phi_{zz}(z)F(z) - \Phi_{zd}(z)B(z) = 0$$

$\Phi_{zz}(z)$ is z-transform of $E[\, z[n]^* z[m+n] \,]$

$$= E\left[\left(\sqrt{2p}\, d[n]^* * g[n]^* + N[n]^*\right)\left(\sqrt{2p}\, d[n] * g[n] + N[n]\right)\Big|_{n=m+n} \right]$$

$$\Phi_{zz}(z) = 2p\,(Q(z))^2 + 2N_0 Q(z)$$

$\Phi_{zd}(z)$ is z-transform of $E[\, z[n]^* d[m+n] \,]$

$$= E\left[\left(\sqrt{2p}\, d[n]^* * g[n]^* + N[n]^*\right) d[m+n] \right]$$

$$\Phi_{zd}(z) = \sqrt{2p}\, Q(z)$$

$$\Leftrightarrow \quad \Phi_{zz}(z)F(z) = \Phi_{zd}(z)B(z)$$

$$\therefore \quad F(z) = \frac{\Phi_{zd}(z)}{\Phi_{zz}(z)} B(z) = \frac{\sqrt{2p}\, Q(z)}{2p(Q(z))^2 + 2N_0 Q(z)} B(z)$$

$$= \frac{\sqrt{2p}\, B(z)}{2p\, Q(z) + 2N_0}$$

$$= \frac{B(z)}{\left(\sqrt{2p}\, Q(z) + \frac{2N_0}{\sqrt{2p}}\right)} \triangleq \frac{B(z)}{Q'(z)}$$

feed forward filter $F(z)$ is uniquely determined by the orthogonality principle, in terms of $B(z)$ and $Q'(z)$

- outer minimization

$$e[n] = z[n] * f[n] - d[n] * b[n]$$

$$E(z) = Z(z)F(z) - D(z)B(z)$$

$$= Z(z)\frac{B(z)}{Q'(z)} - D(z)B(z)$$

$$= B(z)\left(\frac{Z(z)}{Q'(z)} - D(z) \right) \triangleq B(z)E'(z)$$

we have

$$\Phi_{e'e'}(z) = \frac{\Phi_{zz}(z)}{(Q'(z))^2} - 2\frac{\Phi_{zd}(z)}{(Q'(z))} + \Phi_{dd}(z)$$

$$= \frac{\sqrt{2p}\,Q(z)}{Q'(z)} - \frac{2\sqrt{2p}\,Q(z)}{Q'(z)} + 1$$

$$= \frac{Q'(z) - \sqrt{2p}\,Q(z)}{Q'(z)} = \frac{2N_0}{\sqrt{2p}} \times \frac{1}{Q'(z)}$$

Thus

$$\Phi_{ee}(z) = B(z)B\left(\tfrac{1}{z^*}\right)^* \Phi_{e'e'}(z)$$

$$= \frac{2N_0}{\sqrt{2p}} \times \frac{B(z)B\left(\frac{1}{z^*}\right)^*}{Q'(z)}$$

If estimation error sequence is colored, $E[\|e[n]\|^2] = \sigma^2$

$E[e[n]^* e[n-1]] = \rho\sigma^2$

we set $e[n-1]$, the estimation error for the previous symbols, as a new observation,

linear combination of $e[n]$ and $e[n-1]$

$\tilde{e} = e[n] + b\,e[n-1]$

$E[\|\tilde{e}\|^2] = \sigma^2(b^2 + 2b\rho + 1)$

This value is minimized when $b = -\rho$ and minimum value is $\sigma^2(1-\rho^2)$

This gives the inequality $\sigma^2 \geq \sigma^2(1-\rho^2)$ because $-1 \leq \rho \leq 1$

This implies that we can always increase the SNR by observing other correlated noise

Thus, to minimize $E[\|e[n]\|^2]$, estimation error sequence $e[n]$ must be white

$\Phi_{ee}(z) = C$, $\forall z$

$\Leftrightarrow\ B(z)B\left(\frac{1}{z^*}\right)^* \frac{2N_0}{\sqrt{2p}}\frac{1}{Q'(z)} = C$, $\forall z$

$B(z)B\left(\frac{1}{z^*}\right)^* = C \times \frac{\sqrt{2p}}{2N_0} Q'(z)$

suppose that $Q'(z)$ can be well approximated by rational function, then $Q(z)$ can be expressed by $G(z)G\left(\frac{1}{z^*}\right)^*$

$Q'(z) = G(z)G\left(\frac{1}{z^*}\right)^*$

$\Leftrightarrow\ B(z)B\left(\frac{1}{z^*}\right)^* = C \times \frac{\sqrt{2p}}{2N_0} \times G(z)G\left(\frac{1}{z^*}\right)^*$, $\forall z$

There are non unique $B(z)$ that satisfies this condition

$B(z) = kG(z)$ $\forall z$, k is constant

However $B(z)$ is must be monic and causal

$G(z)$ must contain all poles of $Q'(z)$ inside the unit circle and $B(z) = \frac{G(z)}{g[0]}$ for monic $B(z)$

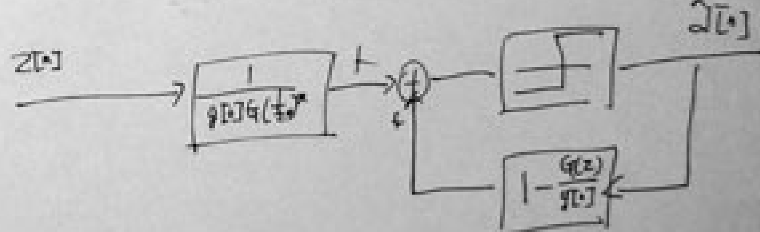
then $\Phi_{ee}(z) = \frac{2N_0}{\sqrt{2p}} \times \frac{1}{|g[0]|^2}$, $\forall z$

To minimize $\Phi_{ee}(z)$, we want $G(z)$ that has the maximum value of $|g[0]|^2$, therefore $G(z)$ must be the minimum phase

$B(z)$ is uniquely defined as

$\therefore B(z) = \frac{G(z)}{g[0]}$ ($G(z)$ must contain all poles and zeros of $Q(z)$ inside unit circle)

$$\therefore F_{opt}(z) = \frac{\frac{G(z)}{g[0]}}{Q'(z)} = \frac{1}{g[0]\,G\left(\frac{1}{z^*}\right)^*}$$



$$Q'(z) = \sqrt{2p}\,Q(z) + \frac{2N_0}{\sqrt{2p}} = G(z)G\left(\tfrac{1}{z^*}\right)^*$$