Quiz #2

2015년 1월 26일 월요일

오후 4:06

1. (7 points)
Suppose that $\underline{X}$ is a real-valued random vector with characteristic function $\Phi_{\underline{X}}(\underline{\omega})$. Answer the following questions.

(a) (2 points)
What is the definition of $\Phi_{\underline{X}}(\underline{\omega})$? State it using the expectation operator $E\{\cdot\}$ and using the pdf $f_{\underline{X}}(\underline{x})$ of $\underline{X}$ assuming that it exists.

(b) (2 points)
Using the operator $E\{\cdot\}$, find the characteristic function of $\underline{Y} \triangleq A\underline{X}+\underline{b}$. Also find it in terms of $\Phi_{\underline{X}}(\underline{\omega})$.

(c) (1 point) When $\underline{X}$ is a real-valued Gaussian random vector w/ $E\{\underline{X}\}=\underline{\mu}$ and $Cov\{\underline{X}\}=C$, what is $\Phi_{\underline{X}}(\underline{\omega})$ in terms of $\underline{\mu}$ & C?

(d) (2 points) In (c), find the characteristic function of $\underline{Y}=A\underline{X}+\underline{b}$ and show that $\underline{Y}$ is a real-valued Gaussian r.v. w/
$E\{\underline{Y}\}=A\underline{\mu}+\underline{b}$ and $Cov\{\underline{Y}\}=ACA^T$.

2. (13 points) When $\underline{X} \sim N(\underline{\mu}, C)$, answer the following questions, where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}} dy.$$

(a) (2 points) Sketch the graphs of $\frac{d}{dx}Q(x)$ and $Q(x)$ as functions of x.

(b) (2 points) When $\underline{\mu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ & $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, sketch the regions $\{\underline{x} \in \mathbb{R}^2 : x_1 \geq x\}$ and $\{\underline{x} \in \mathbb{R}^2 : x_1 \geq 0 \text{ \& } \|\underline{x}\| \geq x\}$ when $x \geq 0$.

(c) (2 points) Using the results in (b), show that

$$Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}} \quad \text{for } x \geq 0.$$

(d) (3 points) Using the results in (b), show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta \text{ for } x \geq 0.$$

(e) (2 points) Find $Pr(\underline{a}^T \underline{X} > b)$ in terms of $Q(\cdot)$, $\underline{a}$, b, $\underline{\mu}$, and C.

(f) (2 points) Find $Pr(|\underline{a}^T \underline{X}| > b)$ in terms of $Q(\cdot)$, $\underline{a}$, b, $\underline{\mu}$, and C, when $b > 0$.

3. (8 points) When $\underline{X}_1 \sim N(\underline{\mu}_1, \sigma^2 I_{2m})$ and $\underline{X}_2 \sim N(\underline{\mu}_2, \sigma^2 I_{2m})$ with natural number m, answer the following questions.

(a) (2 points) When $m=1$, $\underline{\mu}_1 = \underline{0}$, find $Pr(\|\underline{X}_1\| > x)$ in terms of an exponential function.

(b) (2 points) When $m=1$, $\|\underline{\mu}_1\| = \|\underline{\mu}_2\| \neq 0$, show that

$$Pr(\|\underline{X}_1\| > x) = Pr(\|\underline{X}_2\| > x).$$

(Do not use Marcum's Q-function.)

(c) (1 point) In (b), what is $Pr(\|\underline{X}_1\| > x)$ in terms of Marcum's Q-function $Q(\cdot, \cdot)$?

(d) (2 points) Find $Pr(\|\underline{X}_1\| > x)$ in terms of generalized Marcum's Q-function $Q_m(\cdot, \cdot)$.

(e) (1 point) What is the relation b/w generalized Marcum's Q-function $Q_m(\cdot, \cdot)$ and the chi-square cdf $F_R(r)$ of $2m$ degrees of freedom.

4. (5 points) Suppose that a real-valued WSS bandpass Gaussian random process $X(t)$ written as

$$X(t) = X_c(t)\cos 2\pi f_c t - X_s(t)\sin 2\pi f_c t$$

has the PSD $S_{XX}(f)$. Answer the following questions.

(a) (1 point) Show that $E\{X_c(t)\} = E\{X_s(t)\} = 0, \forall t$.

(b) (4 points) Show that

$$R_{X_cX_c}(t, t+\tau) = R_{X_cX_c}(0,\tau), \quad \forall t, \forall \tau,$$
$$R_{X_sX_s}(t, t+\tau) = R_{X_sX_s}(0,\tau), \quad \forall t, \forall \tau,$$
$$R_{X_cX_c}(0,\tau) = R_{X_sX_s}(0,\tau), \quad \forall \tau,$$
$$R_{X_cX_s}(t, t+\tau) = R_{X_cX_s}(0,\tau), \quad \forall t, \forall \tau$$
$$R_{X_sX_c}(t, t+\tau) = R_{X_sX_c}(0,\tau), \quad \forall t, \forall \tau$$

and $R_{X_cX_s}(0,\tau) = R_{X_sX_c}(0,-\tau) = -R_{X_cX_s}(0,-\tau), \forall \tau$

5. (5 points) In Problem 4, let

$$R_{X_cX_c}(0,\tau) \xleftrightarrow{\mathcal{F}} S_{X_cX_c}(f),$$
$$R_{X_sX_s}(0,\tau) \xleftrightarrow{\mathcal{F}} S_{X_sX_s}(f),$$
$$R_{X_cX_s}(0,\tau) \xleftrightarrow{\mathcal{F}} S_{X_cX_s}(f), \text{ and}$$
$$R_{X_sX_c}(0,\tau) \xleftrightarrow{\mathcal{F}} S_{X_sX_c}(f), \text{ and}$$

answer the following questions.

(a) (2 points) Show that $S_{X_cX_c}(f) = S_{X_sX_s}(f) = S_{X_cX_c}(-f)$ and $S_{X_cX_s}(f) = -S_{X_sX_c}(f) = -S_{X_cX_s}(-f)$.

(b) (3 points) Derive the relation among $S_{XX}(f)$, $S_{X_cX_c}(f)$, and $S_{X_cX_s}(f)$.