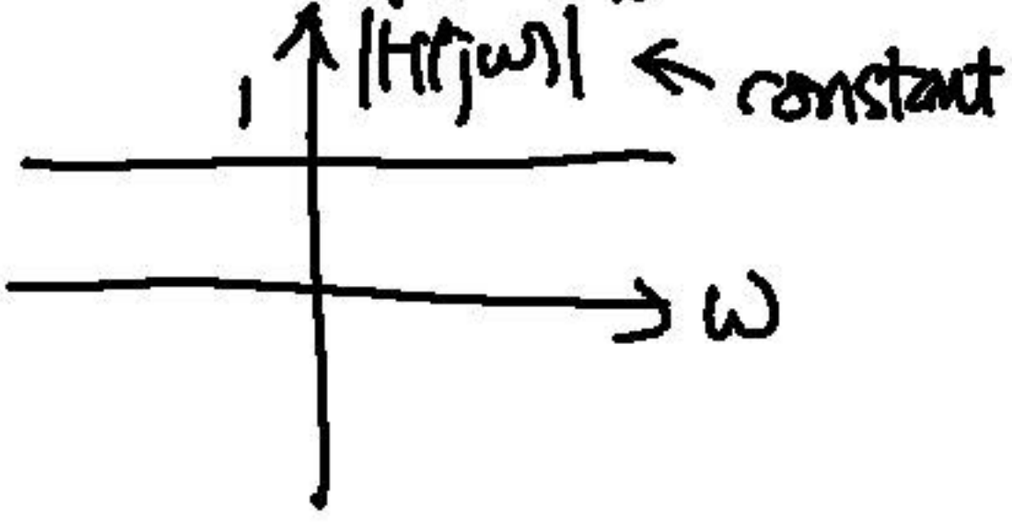
Exam #2

4. (a) $H(j\omega) = \dfrac{a+j\omega b}{a-j\omega b}$ — LPF, BPF, HPF

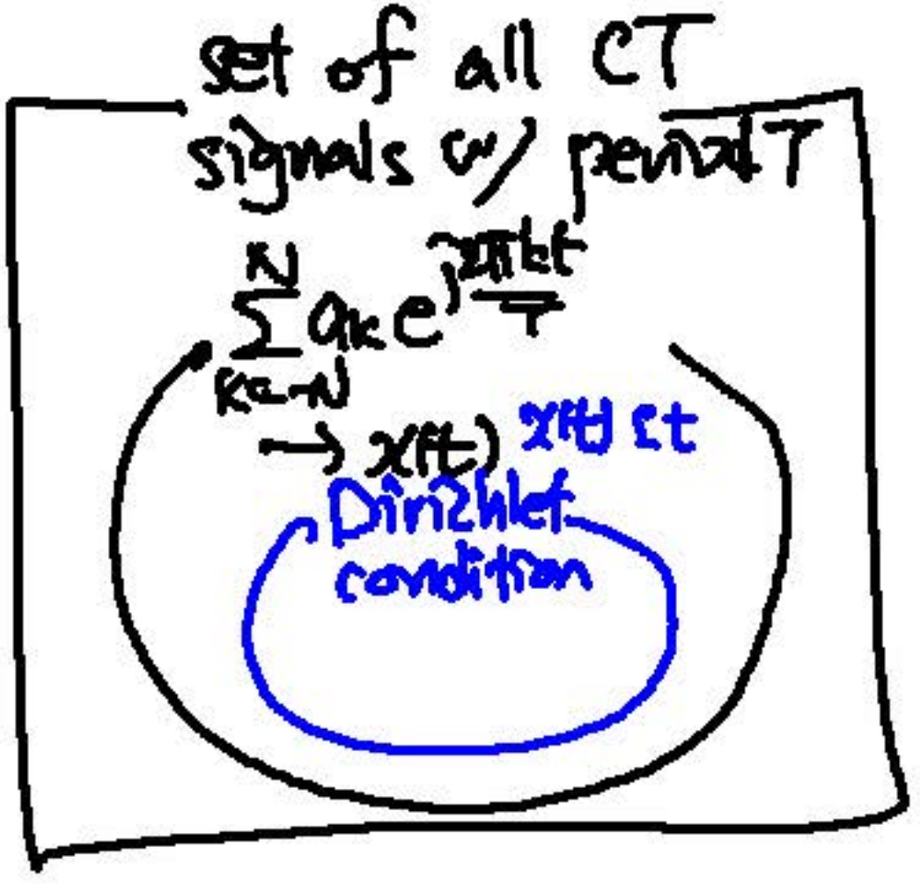
$|H(j\omega)| = \dfrac{\sqrt{a^2+\omega^2 b}}{\sqrt{a^2+\omega^2 b}} = 1, \forall \omega.$ = all-pass = frequency-flat



frequency-selective

4(b) the Dirichlet conditions $\int_{<T>} \left| x(t) - \sum_{k=-N}^{N} a_k e^{j\frac{2\pi kt}{T}} \right|^2 dt \to 0$

$\sum_{k=-N}^{N} a_k e^{j\frac{2\pi kt}{T}} \longrightarrow x(t)$ in the mean-square sense.



where $a_k = \frac{1}{T}\int_{<T>} x(t) e^{-j\frac{2\pi kt}{T}} dt$

4.(d) $\mathcal{L}\left\{\left|\sum_{k=-\infty}^{\infty} a_k\right|^2\right\} = \mathcal{L}\left\{\sum_{k=-\infty}^{\infty} |a_k|^2\right\}$

↑ caligraphic letter

$\mathcal{L}\{a_m a_n^*\} = 0, \forall m \neq n.$

RHS $= \mathcal{L}\left\{\sum_{l=-\infty}^{\infty} |a_l|^2\right\}$

$$\text{LHS} = \mathcal{L}\left\{\left(\sum_{k=-\infty}^{\infty} a_k\right)\left(\sum_{k'=-\infty}^{\infty} a_{k'}\right)^*\right\}$$

$$= \mathcal{L}\left\{\sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty} a_k a_{k'}^*\right\}$$

$$= \sum_{k}\sum_{k'} \mathcal{L}\{a_k a_{k'}^*\} = \sum_{l=-\infty}^{\infty} \mathcal{L}\{|a_l|^2\}$$

5. $x[n] = x[n+N]$, $\forall n$

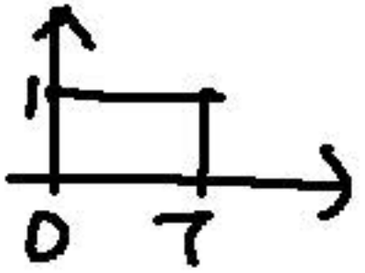
$p(t) \xleftrightarrow{\mathcal{F}} P(j\omega)$ $\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} |P(j\omega)|^2 d\omega = E$.

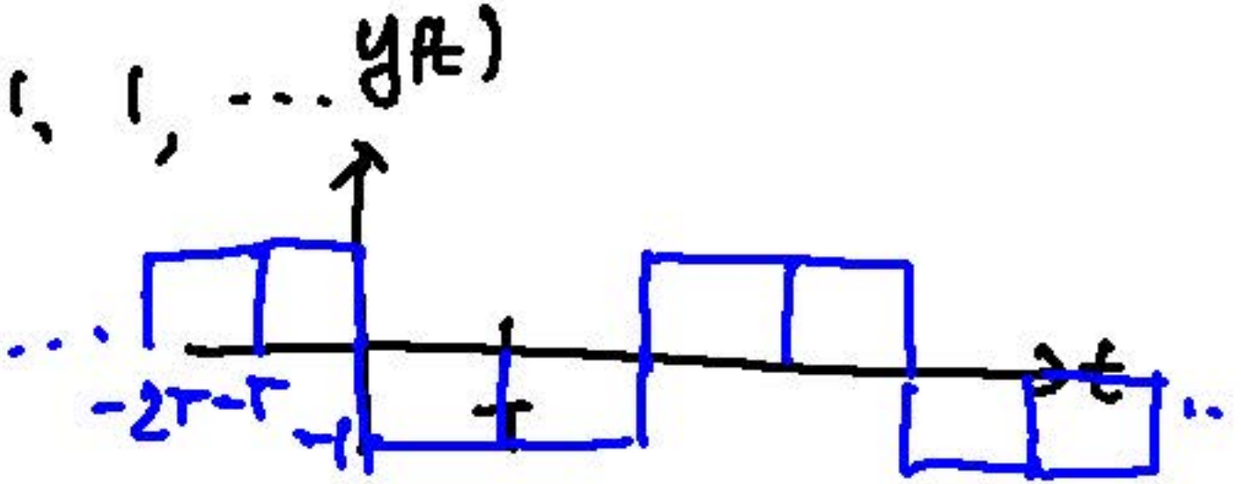
$$y(t) = \sum_{n=-\infty}^{\infty} x[n]\, p(t-nT)$$

(e) $N = 2$ $\quad \cdots x[0]\ x[1] \cdots$

$x[n] = \cdots, -1, -1, 1, 1, \cdots$

$p(t) =$





(f) $y(t+NT) = y(t), \forall t$

$$LHS = \sum_{n=-\infty}^{\infty} x[n]\, p(t+NT-nT)$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, p(t-(n-N)T)$$

$$n-N = k$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{x[k+N]}_{= x[k]}\, p(t-kT)$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, p(t-kT) = y(t).$$

(g) $y(t) = \sum_{n=-\infty}^{\infty} x[n]\, p(t-nT) = y(t+NT), \forall t$

$= \sum_{k=-\infty}^{\infty} b_k e^{j\frac{2\pi kt}{NT}}$

where $b_k = \frac{1}{NT} \int_{\langle NT \rangle} y(t) e^{-j\frac{2\pi kt}{NT}} dt$

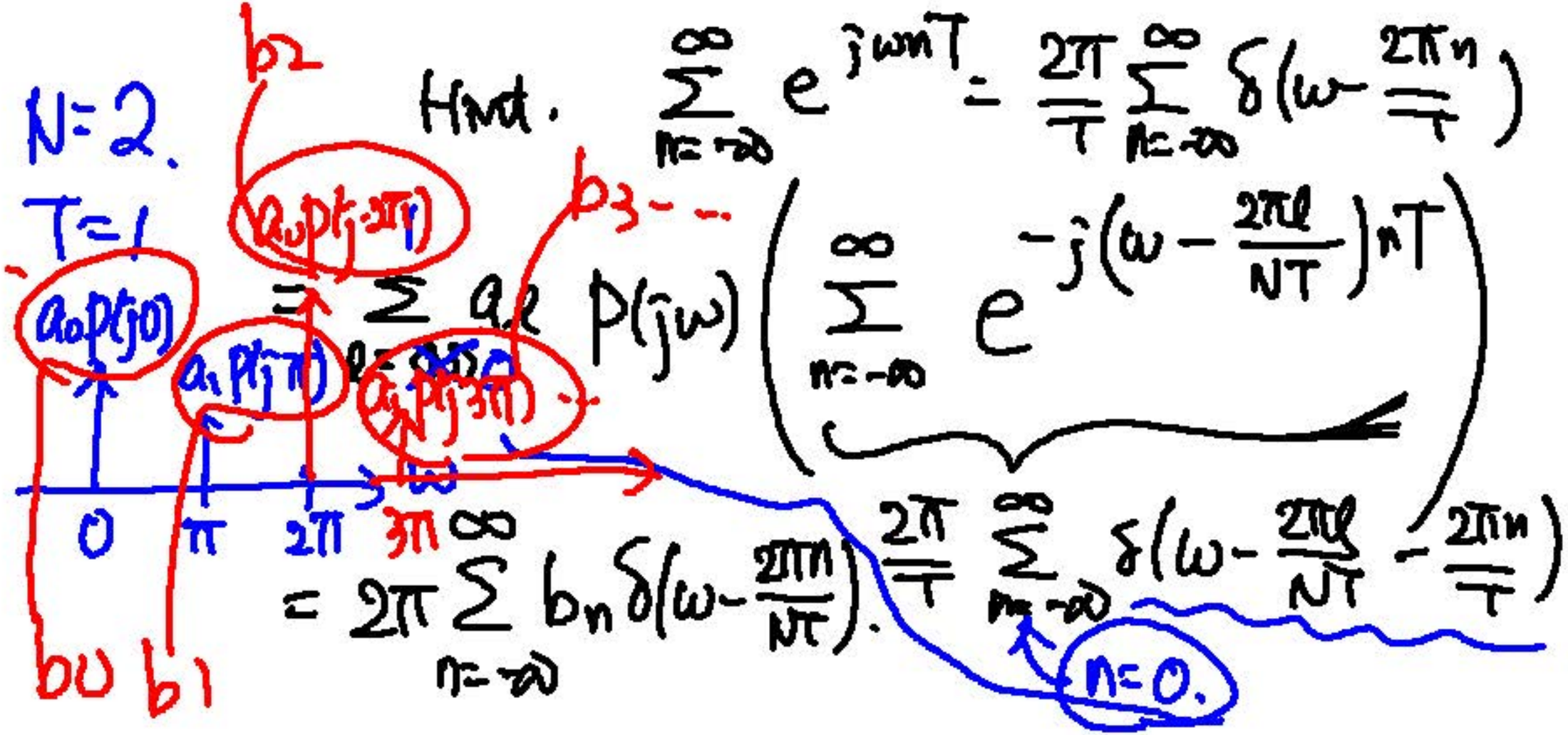
$Y(j\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega nT} P(j\omega)$

$x[n] = x[n+N], \forall n$

$= \sum_{l=\langle N \rangle} a_l e^{j\frac{2\pi ln}{N}}$

where $a_l = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j\frac{2\pi ln}{N}}$

$$Y(j\omega) = \sum_{n=-\infty}^{\infty} \left(\sum_{l=\langle N \rangle} a_l e^{j\frac{2\pi l n}{N}} \right) e^{-j\omega nT} P(j\omega)$$



Hint. $\sum_{n=-\infty}^{\infty} e^{j\omega nT} = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T}\right)$

$$= \sum_{l=\langle N \rangle} a_l P(j\omega) \left(\sum_{n=-\infty}^{\infty} e^{-j\left(\omega - \frac{2\pi l}{NT}\right)nT} \right)$$

$$= 2\pi \sum_{n=-\infty}^{\infty} b_n \delta\left(\omega - \frac{2\pi n}{NT}\right) \cdot \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi l}{NT} - \frac{2\pi n}{T}\right)$$

$n=0$.

$$\therefore\ b_l = \frac{c}{?}\, a_{\mathrm{mod}(l,2)}\, P\left(j\,\frac{2\pi l}{NT}\right)$$

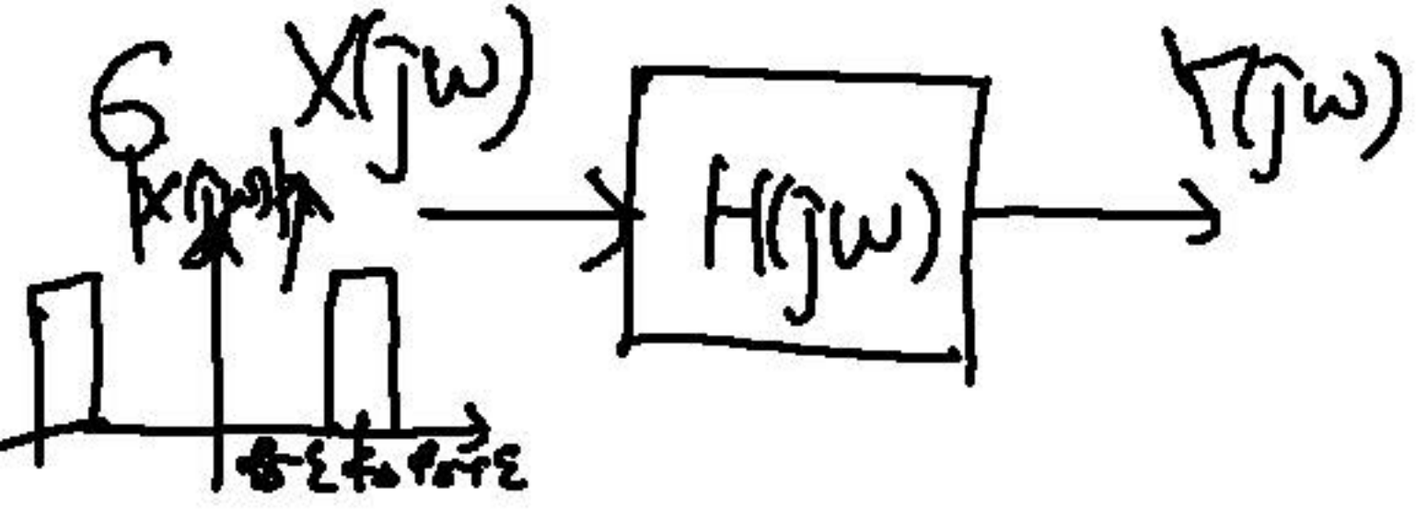
/ex/ mod (0,2) = 0

Mod (1,2) = 1

mod (2,2) = 0

(g)

(h) ← Final exam



$$\theta(\omega) \triangleq \angle H(j\omega_0) + \left.\frac{d}{d\omega}\angle H(j\omega)\right|_{\omega=\omega_0} (\omega-\omega_0)$$

$H(j\omega)$

$= |H(j\omega)|\, e^{j\angle H(j\omega)}$

"very small"

"very large"

"very close"

$$X(j\omega)H(j\omega) \simeq |H(j\omega_0)|\, e^{j\{\angle H(j\omega_0) + \left.\frac{d\angle H(j\omega)}{d\omega}\right|_{\omega=\omega_0}(\omega-\omega_0)\}}$$

$$= X(j\omega)\, |H(j\omega_0)|\, e^{j\angle H(j\omega_0)}\, e^{-j\omega_0 \angle H'(\omega_0)}\, e^{j\angle H'(\omega_0)\omega}$$

$\angle H'(\omega_0)$

$\tau(\omega_0) \triangleq -\left.\frac{d\angle H(j\omega)}{d\omega}\right|_{\omega=\omega_0}$ "$C(j\omega_0)$"

$$Y(j\omega) \simeq C(j\omega_0)\, e^{-j\omega\tau(\omega_0)}\, X(j\omega)$$

$$y(t) = \frac{1}{2\pi} C(j\omega_0)\, x(t - \tau(\omega_0))$$

$$\tau(\omega_0) = -\left.\frac{d\angle H(j\omega)}{d\omega}\right|_{\omega=\omega_0}$$