2023년 10월 17일 화요일 13:08

CM13 231019Th

Announcements

1. Messages from Instructor
   - An imperfect classification rule is better than no classification rule as long as you recognize the imperfection and update your rule.
   - ((S))implify and ((V))isualize:
     - essence, generalize, consider extremes,
     - Venn diagram, necessity and sufficiencienc, existence and uniqueness/plurality
     - block diagram, G(V,E); graph of a function, algebraic vs. geometric interpretations, mathematical vs. physical/engineering interpretations/implications
   - Purposeful practice with 3F: Time, Focused Action, Experience, Lesson, Immediate Feedback, Fix, Growth

     Purposeful practice is designed to improve performance, and is characterized by staying focused and positive, setting clear and specific goals, allocating time during your week, taking on tasks that go beyond your existing level of ability, and seeking feedback.

     출처: <https://www.google.com/search?q=purposeful+practice&rlz=1C1CHBD_koKR890KR890&oq=Purposeful+practice&gs_lcrp=EgZjaHJvbWUqBwgAEAAYgAQyBwgAEAAYgAQyBwgBEC4YgAQyBwgCEAAYgAQyBwgDEAAYgAQyBwgEEAAYgAQyCAgFEAAYFhgeMggIBhAAGBYYHjIICAcQABgWGB4yCAgIEAAYFhgeMggICRAAGBYYHtIBCDMzNTVqMGo3qAIAsAIA&sourceid=chrome&ie=UTF-8>

   - Stop and Think; similarity to and difference;
   - Self-Fulfillment and Contribution to Collective
   - Attitude: Exceed expectation.

2. 지난 강의 수정/보완 사항

3. HW, Quiz, and Exam
   - HW07 will be assigned by this Friday. Due will be next Tuesday.
   - Midterm Exam is approaching.
     - It will be taken on Thursday 10/26 at 19:00-23:00 in LG 102.
     - Open everything
       - Greek letters

- No communication with human being
  - You cannot carry any message-exchanging device/stationery when you go to the restroom.

4. 강의 자료
   - Handouts and annotated pdfs are downloadable from the non-PLMS class homepage at http://cisl.postech.ac.kr/class/eece233/schedule.htm.
   - Recorded CMs are available from cisl.postech YouTube channel. Please place clickable time stamps.

5. 출석
   - 출결상태에 대해 전자출결시스템(https://wpsc.postech.ac.kr/eams/login)을 확인하세요. 정정사유가 있는 학생은 증빙를 제출해 주세요.

6. Overview (The numbers are the chapter numbers of the textbook)
   1. Signals and Systems
   2. Linear Time-Invariant Systems DT/CT
   3. Fourier Series Representation of Periodic Signals
   4. Continuous-time Fourier Transform
   5. Discrete-time Fourier Transform
   6. Time and Frequency Characterization of Signals and Systems
   9. Laplace Transform
   10. Z-transform
   7. Sampling
   8. Communication Systems

$A \in \mathbb{R}^{N \times N}$

7. Review $\quad y = Ax \quad \lambda_i v_i = A v_i \quad (\lambda_i, v_i)$
   - Linear Algebra for same input-output vector lengths
     - ~~Defs. an~~ eigenvector of a square matrix, and its corresponding eigenvalue
       - □ non-zero vector?
       - □ zero eigenvalue?
     - Lemma. The eigenvalues of a square matrix can be found by solving the characteristic equation $\det(A-\lambda I)=0$ Nth poly-
       - □ det(A-lambda I) = 0 and the number of (distinct) eigenvalues
       - □ algebraic multiplicity of an eigenvalue
     - Corollary. Existence and Uniqueness of (lambda_i, v_i)
       - ◆ linearly independent eigenvectors
       - □ eigen-value
         - ◆ geometric multiplicity of an eigenvalue

$x \in \mathcal{D} \longrightarrow$ Linear System $\longrightarrow y \in \mathcal{R}$

$\mathcal{D} \subset \mathcal{U}$

$\mathcal{R} \subset \mathcal{U}$

$H v_k = \lambda_k v_k$

$v \rightarrow \lambda v \quad v \neq 0$

$x \rightarrow \boxed{H} \rightarrow y$

- linearly independent eigenvectors
- eigen-value
  - geometric multiplicity of an eigenvalue
  - dimension of an eigenspace
- Q. Why is the eigenstructure of a square matrix important?
  - Q. What about its application to a CT/DT system? Is it useful?
  - A.
  - Q. What about its application to a CT/DT linear system? Is it useful?
  - A.
- Defs. an eigen-function and the corresponding engen-value of a DT/CT linear system
  - the Fredholm integral equation and the Kernel
- Defs. an eigen-function and the corresponding engen-value of a DT/CT LTI system
- Lemmas. complex exponential signal
  - system function
  - Laplace transform, z-transform

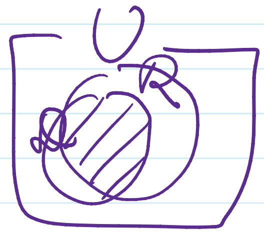
- Importance of signals consisting of complex exponential signals
  - ((S)) Signals consisting of complex sinusoidal signals
    - ((S)) Signals consisting of harmonically-related complex sinusoidal signals
      - The set of all CT/DT periodic signals
        - The set of all DT periodic signals

$x \to \boxed{H} \to y$

$v \to \lambda v \quad v \neq 0$

an eigenvector / function

$\mathbb{R}$



$x = \sum_{k=1}^{N} a_k v_k$

$L \Rightarrow ZIZO$

$y = Hx$

$= H\left(\sum a_k v_k\right) \quad (\lambda_1, v_1)$

$= \sum_k \lambda_k (a_k v_k)$

$e^{st} \to \boxed{h(t)} \to H(s)e^{st}$

$H(s) \triangleq \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$

H(s) is the LTF of h(t).

cf) circulant square matrix H $y = Hx$

$\simeq$ DT LTI system

$x[n] \to \boxed{h[n]} \to y[n]$

$z^n \to \boxed{h[n]} \to H(z)z^n$

$H(z) = \sum_{m=-\infty}^{\infty} h[m] z^{-m}$

H(z) is the z-TF of h[n].

8. Preview
   - DT Fourier series
     - existence
     - uniqueness
     - properties, lemmas, propositions, theorems

Def. $e^{j\omega t}$ $\quad e^{j2\pi f t}$

$\cos(\omega t)$ $\quad \cos(2\pi f t)$ → radian frequency [rad/s]

→ $\omega = 2\pi f \quad f = \frac{\omega}{2\pi} \quad [1/s] = \text{Hz}$

Q. Given a DT periodic signal

$\cos(\omega t)$ $\cos(2\pi f t)$ $\to$ $\omega = 2\pi f$ $f = \frac{\omega}{2\pi}$ $[1/s] = Hz$

Q. Given a DT periodic signal

$$x[n] = x[n+N], \ \forall n \quad N \in \mathbb{N}$$

find $(a_k, \omega_k)_k$ such that

$$x[n] = \sum_k a_k (e^{j\omega_k})^n, \ \forall n.$$

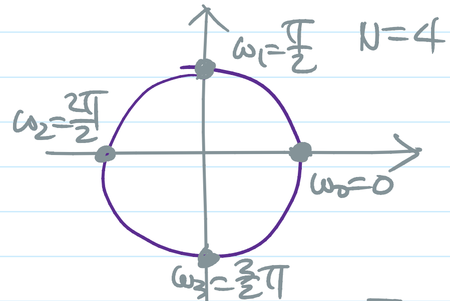
z^{n} ← exponent; z → base.

A. We can ALWAYS find N pairs of (a_k, w_k)

s.t. $$x[n] = \sum_{k=0}^{N-1} a_k \underbrace{(e^{j\omega_k})}_{\text{a complex \# on the unit circle.}}{}^n, \ \forall n$$

$n = 0, \cdots, N-1$

$$\omega_k = \frac{2\pi}{N} k$$



$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ e^{j\omega_0} & e^{j\omega_1} & - & & e^{j\omega_{N-1}} \\ e^{j2\omega_0} & & & & \\ e & & & & \\ \vdots & & & & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix}$$

Q1. When x[n] is a periodic signal with period N, is it always possible to represent x[n] in the form of

x[n] = \sum_{k=<N>} a_k exp(j 2\pi kn/N) ?

A1. Yes.

Q2. How?

A2.
[
x[0]
x[1]
x[2]
...
x[N−1]
] = [exp(j*2*\pi (i−1)(j−1)/N)]
[
a_0
a_1
...
a_{N−1}
]

$$x[n] = x[n+N], \forall n = \sum_{k=0}^{N-1} a_k (e^{j\frac{2\pi}{N}k})^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

$$= \sum_{k=0}^{N-1} a_k H\left(e^{j\frac{2\pi}{N}k}\right) (e^{j\frac{2\pi}{N}k})^n$$

a_1
...
a_{N−1}
]

Then, we have
x = U \sqrt{N} a
where U is unitary.
Thus, we obtain a = inv(U)x/sqrt{N}.

DT Fourier Series Representation

If x[n] is a DT signal with period N, then...

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

This summation is a finite sum, which is always well defined.

$\sum_{k=\langle N\rangle}$ = Sum over *any* N consecutive values of k

— This is a *finite* series

$\{a_k\}$ - Fourier (series) coefficients

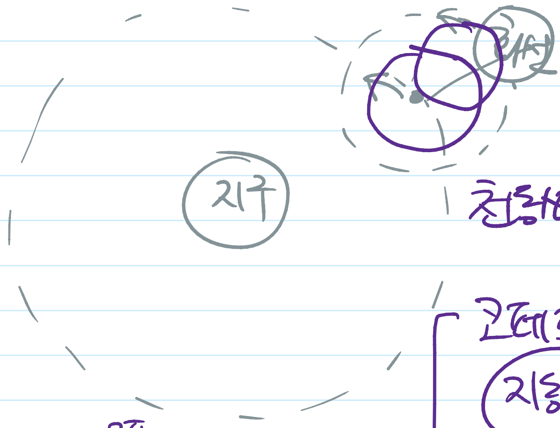
Questions:

1) What DT periodic signals have such a representation?

A. All

표기법

$k = \langle N \rangle$



천동설.

코페르니쿠스

지동설

$a_k(6)$ 주기 "

Questions:

1) What DT periodic signals have such a representation?

2) How do we find a_k?

A. All.
Draw a Venn diagram.

A. Both of the questions are related to what you learned in linear algebra.

Convert the questions to equivalent linear algebra problems.

Answer to Question #1:

Any DT periodic signal has a Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$\Downarrow$$

$$x[0] = \sum_{k=<N>} a_k$$

$$x[1] = \sum_{k=<N>} a_k e^{jk\omega_0}$$

$$x[2] = \sum_{k=<N>} a_k e^{j2k\omega_0}$$

$$\vdots$$

$$x[N-1] = \sum_{k=<N>} a_k e^{j(N-1)k\omega_0}$$

Put these equations in an equation in terms of a matrix and vectors.

N equations for N unknowns, $a_0, a_1, \ldots a_{N-1}$

A More Direct Way to Solve for a_k

Finite geometric series

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N & , \alpha = 1 \\ \frac{1-\alpha^N}{1-\alpha} & , \alpha \neq 1 \end{cases}$$

$$\Downarrow \alpha = e^{jk\omega_0}$$

$$\sum_{k=<N>} a_k \left(e^{j\frac{2\pi}{N}k}\right)^n = \sum_{k=0}^{N-1} a_k \left(e^{j\frac{2\pi}{N}k}\right)^n$$

$$\left(e^{j\frac{2\pi}{N}k}\right)^n = \left(e^{j\frac{2\pi}{N}(k+N)}\right)^n \quad \forall n, \forall k$$

$$= e^{j\frac{2\pi}{N}kn} \, e^{j\frac{2\pi}{N}N \cdot n} = 1, \forall n$$

$$= \sum_{k \in \{0, N+1, 2N+2, -N+3, \ldots\}}$$

$$a_k = a_{k+N}, \forall k.$$

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$$\begin{bmatrix} a_0 \\ \vdots \\ a_{N-1} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} \frac{1}{\sqrt{N}} & \cdots \\ & \ddots \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$a_0 = \frac{1}{N} \sum_{k=0}^{N-1} x[k]$$

$$a_1 = \frac{1}{N} \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}k} x[k]$$

$$a_2 = \frac{1}{N} \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}\cdot 2k} x[k]$$

$$\Downarrow \ \alpha = e^{jk\omega_0}$$

$$\sum_{n=<N>} e^{jk\omega_0 n} = \sum_{n=0}^{N-1}(e^{jk\omega_0})^n = \sum_{n=0}^{N-1}(e^{jk2\pi/N})^n$$

$$= \begin{cases} N & , \ k=0,\pm N,\pm 2N \\ \dfrac{1-e^{jk\omega_0 N}}{1-e^{jk\omega_0}} = 0 & , \ \text{otherwise} \end{cases}$$

$$= N\delta[k-mN]$$

omega_0= 2\pi/N

$$a_2 = \frac{1}{N}\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}\cdot 2k}\, x[k]$$

So, from

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$\Downarrow \ \sum_{n=<N>} e^{-jm\omega_0 n} \ \textit{to both sides}$$

— Taking an “inner product” of $x[n]$ and $e^{jm\omega_0 n}$

$$\sum_{n=<N>} x[n]e^{-jm\omega_0 n} = \sum_{n=<N>}\left(\sum_{k=<N>} a_k e^{jk\omega_0 n}\right)e^{-jm\omega_0 n}$$

$$= \sum_{k=<N>} a_k \underbrace{\left(\sum_{n=<N>} e^{j(k-m)\omega_0 n}\right)}_{N\delta[k-m] \text{ — orthogonality}}$$

$$= Na_m$$

$$\Downarrow$$

DT Fourier Series Pair $\left(\omega_o = \dfrac{2\pi}{N}\right)$

DT Fourier Series Pair $\left(\omega_o = \frac{2\pi}{N}\right)$

Theorem. If x[n] is periodic with period N, then

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \quad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk\omega_0 n} \quad \text{(Analysis equation)}$$

(*)

This is a Fourier series representation of a DT periodic signal x[n].

This shows how to find the Fourier series coefficients.

Note: It is convenient to think a_k as being defined for *all* integers k. So:

1) $a_{k+N} = a_k$ — *Unique* for DT. Since $x[n] = x[n+N]$, there are only N pieces of information, whether in the time-domain or in the frequency-domain.

2) We only use N consecutive values of a_k in the synthesis equation

existence (o)

unique. (o)

일반화하면, k를 N으로 나누었을때 서로다른 나머지를 갖는 a_k들을 쓰면 언제나 성립한다. 굳이 연속적인 k들을 쓸 필요는 없다.

Q. Given x[n], find its Fourier series representation.

Example #1: Sum of a pair of sinusoids

$$x[n] = \cos(\pi n/8) + \cos(\pi n/4 + \pi/4)$$

— periodic with period $N = 16 \Rightarrow \omega_o = \pi/8 = \frac{2\pi}{16}$

$$x[n] = \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right] + \frac{1}{2}\left[e^{j\pi/4}e^{j2\omega_0 n} + e^{-j\pi/4}e^{-j2\omega_0 n}\right]$$

더 이상 sine cosine을 쓰지 말라!

$$\Downarrow$$

$a_0 = 0$

$a_1 = 1/2$

$a_{-1} = 1/2$

$a_2 = e^{j\pi/4}/2$

$a_{-2} = e^{-j\pi/4}/2$

$a_3 = 0$

$a_{-3} = 0$

$\vdots$

$a_{15} = a_{-1+16} = a_{-1} = 1/2$

$a_{66} = a_{2+4\times16} = a_2 = e^{j\pi/4}/2$

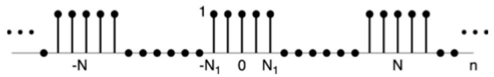
$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Q. Find the Fourier series representation of x[n].

Example #2: DT Square wave



$$a_0 = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n] = \frac{(2N_1+1)}{N} = a_N = a_{-N} = a_{6N} = \cdots$$

For $k \neq$ multiple of N:

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} \overset{n=m-N_1}{=} \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\omega_0(m-N_1)}$$

$$= \frac{1}{N} e^{jk\omega_0 N_1}\sum_{m=0}^{2N_1}(e^{-jk\omega_0})^m = \frac{1}{N} e^{jk\omega_0 N_1}\frac{1-e^{-jk\omega_0(2N_1+1)}}{1-e^{jk\omega_0}}$$

Euler's Identity

$$= \frac{1}{N}\frac{\sin[k(N_1+1/2)\omega_0]}{\sin(k\omega_0/2)} = \frac{1}{N}\frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)}$$

$\Downarrow$

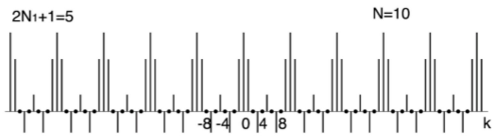
Q. Rewrite this answer using sinc functions, where
sinc(x) = 1 for x=0
sin(\pi x)/(\pi x) for x \ne 0

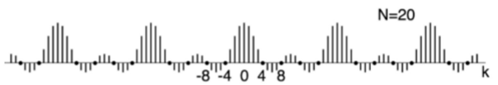
There are two conventions in defining a sinc function.
1. normalized sinc function
Sa(x) = sinc(x) = sin(pi x)/(pi x)
2. unnormalized sinc function
Sa(x) = sinc(x) = sin(x)/(x)

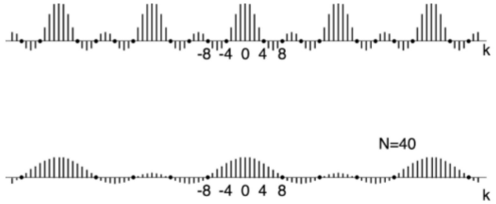
A related function: sine integral
Si(x) = int_{0}^{x} sin(y)/y dy

Example #2: DT Square wave (continued)

$$a_k = \frac{1}{N}\frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)}$$







Convergence Issues for DT Fourier Series:

Not an issue, since all series are finite sums.

Properties of DT Fourier Series: Lots, just as with CT Fourier Series

Example:

$$x[n] \longleftrightarrow a_k$$

x[n]이 N주기함수이고
DTFS coefficients가 a_k일때,

Q1. e^{j2\pi Mn/N}x[n]는 주기함수인가? 만약 그렇다면 주기는 얼마인가? 만약 그렇다면 이의 DTFS coefficients는?

$$e^{jM\omega_0 n}x[n] \longleftrightarrow b_k = ?$$

$$x[n]e^{jM\omega_0 n} = \sum_{r=<N>} a_r e^{jr\omega_0 n} e^{jM\omega_0 n}$$

$$\overset{k=r+M}{=} \sum_{k=<N>} a_{k-M} e^{jk\omega_0 n}$$

$$\Downarrow$$

$$b_k = a_{k-M}$$

Frequency shift
$jk\omega_o \to j(k-M)\omega_o$

Q2. 주기함수 x[n−n_0]의 DTFS coefficients는?

Q3. If x[n] is real and periodic, what is the property of the Fourier series coefficients?