CM18 231107T on CTFT of important time functions and CTFT properties

2023년 11월 7일 화요일 12:28

CM18 231107T

Announcements

1. Messages from Instructor
   - Goal of Life: Self-Fulfillment and Contribution to Collective (자아실현, 공동체 기여)
     - Attitude: Exceed the expectations! (기대를 넘어서라!)
   - "아는 것이 힘이다. 배우고야 무슨 일이든 한다." -심훈, 상록수-
   - 학습 (배우고 익히기)
     - Theory Learning (이론 배우기)
     - Problem Solving (문제 풀기)
     - 배우기와 익히기의 균형
   - How to Learn
     - Stop and Think; 복기가 chess master 를 만든다; Find commonality, similarity, and difference
   - How to Practice

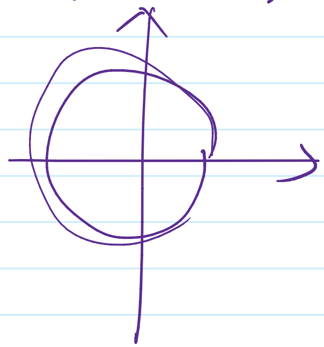
     Purposeful practice with 3F: Time, Focused Action, Experience, Lesson, Immediate Feedback, Fix, Growth
   - How to solve? How to understand? ((S))implify and ((V))isualize:
     - essence, simplify; generalize; consider extremes; mental representation, representational examples;
     - Venn diagram, necessity and sufficienciency, existence and uniqueness/plurality
       - An imperfect classification rule is better than no classification rule as long as you recognize the imperfection and update your rule.
     - block diagram, G(V,E); graph of a function, algebraic vs. geometric interpretations, mathematical vs. physical/engineering interpretations/implications

2. 지난 강의 수정/보완 사항
   - x[n] = e^i₩omega n is NOT a periodic signal in general. However, ₩omega is still

2. 지난 강의 수정/보완 사항
    - x[n] = e^j₩omega n is NOT a periodic signal in general. However, ₩omega is still called a (radian) frequency.

$x[n] = e^{j\omega_0 n}$, $\forall n$ $\quad \omega_0 = 2\pi \frac{n}{m}$

$x(t) = e^{j\omega_0 t}$, $\forall t$ $= x(t + \frac{2\pi k}{\omega_0})$



3. HW, Quiz, and Exam
    - HW10 will be assigned by this Friday. Due will be next Tuesday.

4. 강의 자료
    - Handouts and annotated pdfs are downloadable from the non-PLMS class homepage at http://cisl.postech.ac.kr/class/eece233/schedule.htm.
    - Recorded CMs are available from cisl.postech YouTube channel. Please place clickable time stamps.

5. 출석
    - 출결상태에 대해 전자출결시스템(https://wpsc.postech.ac.kr/eams/login)을 확인하세요. 정정사유가 있는 학생은 증빙를 제출해 주세요.

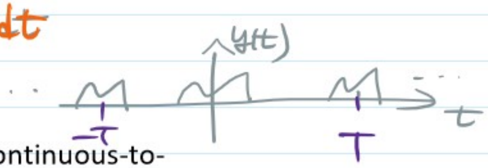
6. Overview (The numbers are the chapter numbers of the textbook)
    1. Signals and Systems
    2. Linear Time-Invariant Systems
    3. Fourier Series Representation of Periodic Signals
    4. Continuous-time Fourier Transform
    5. Discrete-time Fourier Transform
    6. Time and Frequency Characterization of Signals and Systems
    9. Laplace Transform
    10. Z-transform
    7. Sampling
    8. Communication Systems

7. Review
    - Q . Frequency response of an LTI system, commutativity of convolution, and Frequency Domain representation of the input?
    - Fourier's derivation of

- Q . Frequency response of an LTI system, commutativity of convolution, and Frequency Domain representation of the input?
- Fourier's derivation of
  - Synthesis equation $\quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$
  - Analysis equation $\quad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$

  aperiodic

  from CTFS by using periodic extension $\quad y(t) = \sum_{n=-\infty}^{\infty} x(t-nT)$



- Def. of CTFT
  - Continuous-parameter to continuous-parameter Fourier Transform = continuous-to-continuous FT
    - Simply, we call it the CTFT.
  - existence and L^1($\boxed{R}$)

    $$|X(j\omega)| \le \int_{-\infty}^{\infty} |x(t)e^{-j\omega t}|dt = \int_{-\infty}^{\infty}|x(t)|dt < \infty.$$

  - sufficient condition
- Def. of CT inverse Fourier Transform
  - continuous-to-continuous inverse FT
  - existence
  - sufficient condition
- Def. Fourier-Plancherel Transform
  - Def. L^2(R) $\quad$ For $x(t) \in L^2(\mathbb{R})$, $X(j\omega) \triangleq \lim_{\Delta\to\infty}\int_{-\Delta}^{\Delta} x(t)e^{-j\omega t}dt$
- Examples
  - Dirac delta, delayed Dirac delta

□ Analysis, Synthesis와 무관하게 정의하기를

$$a \rightarrow \boxed{\text{C2C FT}} \rightarrow b \qquad b(y) = \int_{-\infty}^{\infty} a(x)e^{-jxy}dx$$

$$c \rightarrow \boxed{\text{C2C iFT}} \rightarrow d \qquad d(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} c(y)e^{jyz}dy$$

For symmetry, often defined as

$$b(y) = \int_{-\infty}^{\infty} a(x)e^{-j2\pi xy}dx$$
$$d(z) = \int_{-\infty}^{\infty} c(y)e^{j2\pi yz}dy$$

8. Preview
  - right-sided exponential function
  - rectangular pulse
  - sinc pulse
  - Gaussian pulse
  - CTFT and CTFS = CTFT of periodic signals
- Properties of CTFT

Example #1

X(f) = X(j2\pi f)로 정의하면,

\delta(t) <-> 1
e^{j2\pi f_0t} <-> \delta(f-f_0)

((S))

(a) $\quad x(t) = \delta(t)$

$\uparrow H(j\omega)$

Example #1

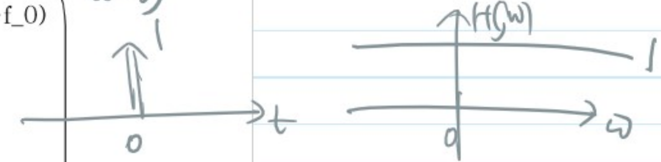
\delta(t) <-> 1
e^{j2\pi f_0t} <-> \delta(f-f_0)

이다.

((S))

(a) $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1, \ \forall \omega \in \mathbb{R}$$



$$\Downarrow$$

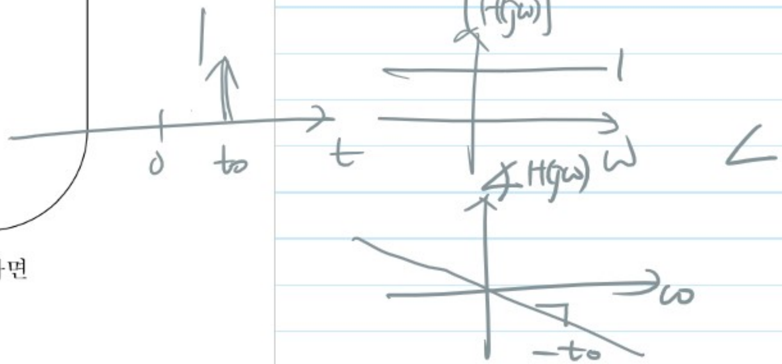
이것이 Fundamental assumption in Fourier transform to handle a delta ft이다.

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega t} d\omega \quad \text{— Synthesis equation for } \delta(t)$$

(b) $x(t) = \delta(t - t_0)$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t - t_0) e^{-j\omega t} dt$$

$$= e^{-j\omega t_0} \quad \text{— Linear phase shift in } \omega$$



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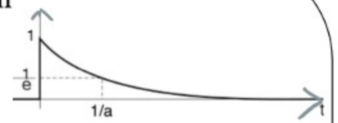
주의. e^{j=omega} 의 어떤 harmonic 들을 모두 countable sum하면 impulse train이 나오고

여기서는 e^{j=omega}를 모든 주파수에서 uncountable sum (=integrate)하면 impulse가 나온다는 말씀.

Example #2: Exponential function

right-sided exponential function은 circuit theory, control에서 아주 중요한 신호이다. 이의 CTFT를 반드시 기억하자.

$x(t) = e^{-at}u(t)$, $a > 0$
$-a<0$

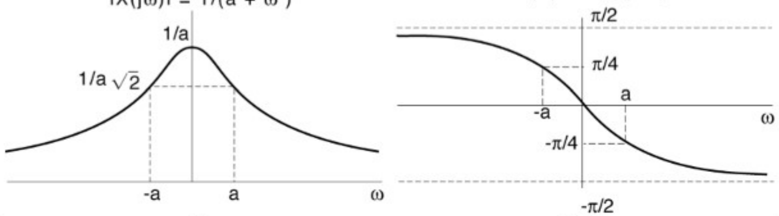


$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt = \int_0^{+\infty} \underbrace{e^{-at} e^{-j\omega t}}_{e^{-(a+j\omega)t}} dt$$

$$= -\left(\frac{1}{a + j\omega}\right) e^{-(a+j\omega)t} \Big|_0^{\infty} = \frac{1}{a + j\omega}$$

$|X(j\omega)| = 1/(a^2 + \omega^2)^{1/2}$

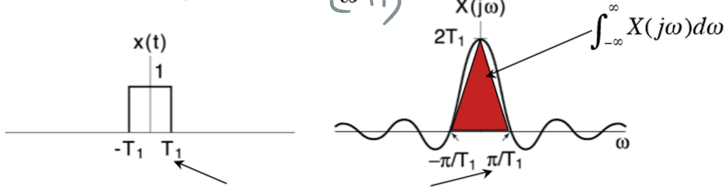
$\angle X(j\omega) = -\tan^{-1}(\omega/a)$



Even *Odd*

Example #3: A square pulse in the time-domain

$$X(j\omega)=\int_{-T_1}^{T_1} e^{-j\omega t}\,dt = \frac{T_1\, 2\sin(\omega T_1)}{(\omega T_1)} = \text{sinc}(\ )$$



Note the inverse relation between the two widths ⇒ Uncertainty principle

Useful facts about CTFT's

$X(0)=\int_{-\infty}^{+\infty} x(t)\,dt$ Example above: $\int_{-\infty}^{\infty} x(t)\,dt = 2T_1 = X(0)$

$x(0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,d\omega$ Ex. above: $x(0)=1=\frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} X(j\omega)\,d\omega}_{2\pi}=\frac{1}{2\pi}\cdot\frac{1}{2}\left(2T_1\times\frac{2\pi}{T_1}\right)$

$=\frac{1}{2\pi}\times$ (Area of the triangle)

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X(0) is not the DC component. 2X(0)Delta_f is the DC component around |f| \leq Delta_f.

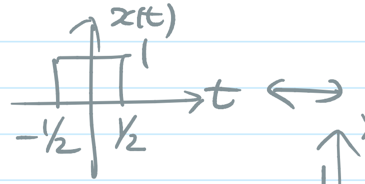
We have a duality. To obtain the value of x(t) at t=0, integrate the CTFT and divide it by 2\pi .

strictly time−limited signal은 무한대까지 주파수 성분을 갖는다.

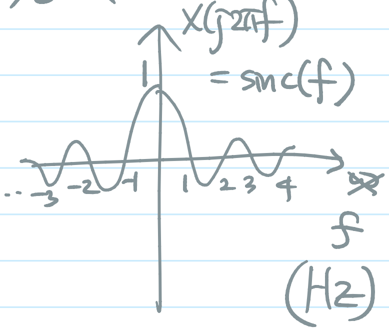
strictly band−limited signal은 무한대까지 time 성분을 갖는다.

Q. True or False: A CT signal is either time−limited or band−limited.
A. No.
There are CT signals neither time−limited nor band−limited. Ex. Gaussian pulse in the next slide.

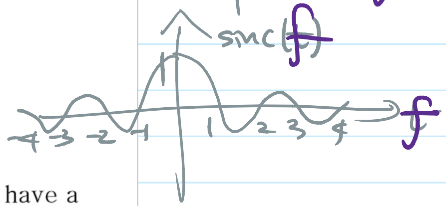
$\boxed{\sin\theta \simeq \theta \text{ for s. small } \theta.}$



$\longleftrightarrow$



$$\text{sinc}(f) \triangleq \begin{cases} \frac{\sin \pi f}{\pi f}, & f \neq 0 \\ 1, & f = 0 \end{cases}$$



$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad \forall x$$

Bell Curve

Example #4: $x(t) = e^{-at^2}$ — A Gaussian, important in probability, optics, etc.

$a > 0$

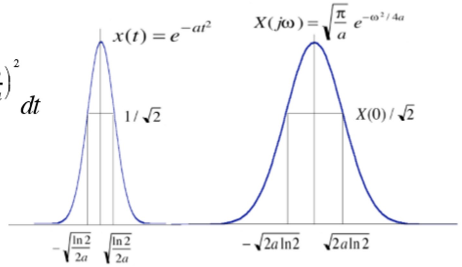
$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-a\left[t^2 + j\frac{\omega}{a}t + \left(\frac{j\omega}{2a}\right)^2\right] + a\left(\frac{j\omega}{2a}\right)^2} dt$$

$$= \underbrace{\left[\int_{-\infty}^{\infty} e^{-a\left(t+\frac{j\omega}{2a}\right)^2} dt\right]}_{\sqrt{\pi/a}} \cdot e^{-\frac{\omega^2}{4a}}$$

$$= \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}$$

Also a Gaussian!



(Pulse width in t)•(Pulse width in ω)
$\Rightarrow \Delta t \cdot \Delta\omega \sim (1/a^{1/2}) \cdot (a^{1/2}) = 1$

Uncertainty Principle! Cannot make *both* Δt and $\Delta\omega$ arbitrarily small.

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반드시 기억할 것:

standard Gaussian random variable N(0,1)의 characteristica function은 여전히 bell curve이다.
In other words,
the CTFT of a bell curve is a bell curve.

If you multiply the standard deviations of the two bell curves of a CTFT

In other words,
the CTFT of a bell curve is a bell curve.

If you multiply the standard deviations of the two bell curves of a CTFT pair, then the product is a constant.

As a rectangular pulse, a Gaussian pulse can be used to approximate the Dirac delta function when the width decreases and height increase with the product fixed.

이제 CTFS와 CTFT를 연결해 보자.

$X(f)=\delta(f-f_0) \overset{\mathcal{F}^{-1}}{\Longleftrightarrow} x(t)=e^{j2\pi f_0 t}$

CT Fourier Transforms of Periodic Signals

Suppose $\quad X(j\omega)=\delta(\omega-\omega_0)$

$\Downarrow$

$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\delta(\omega-\omega_0)e^{j\omega t}d\omega=\frac{1}{2\pi}e^{j\omega_0 t}$ — periodic in t with frequency ω_o

$x(t)$ has no component other than $\omega=\omega_0$.
$\Rightarrow x(t) \propto e^{j\omega_0 t}$?

That is $\quad \boxed{e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0)}$ — All the energy is concentrated in one frequency — ω_o

cf) $X(j2\pi f)=\delta(f-f_0)$
$\Rightarrow x(t)=e^{j2\pi f_0 t}$

e^{j2\pi f_0 t} <-> \delta(f-f_0) 이다.

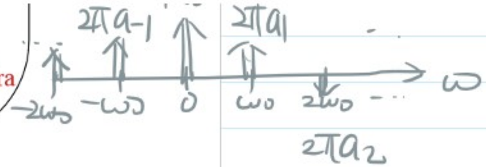
More generally, if $x(t)=x(t+T)$, then

$x(t)=\sum_{k=-\infty}^{+\infty}a_k e^{jk\omega_0 t} \longleftrightarrow X(j\omega)=\sum_{k=-\infty}^{+\infty}2\pi a_k\delta(\omega-k\omega_0)$ Discrete spectra

More generally, if $x(t) = x(t+T)$, then

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \longleftrightarrow X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$ Discrete spectra

CTFS



Example #4:

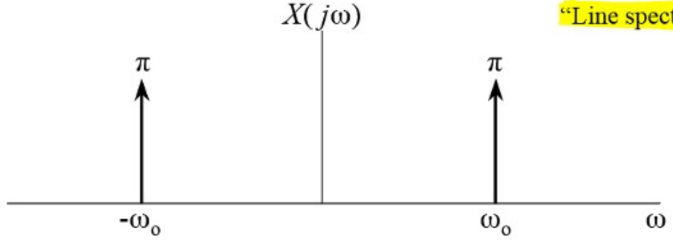
$$x(t) = \cos\omega_0 t = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

f를 쓸 경우도 생각해 보자.

$$\Updownarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$\pi = \frac{1}{2} \cdot 2\pi$



"Line spectra"

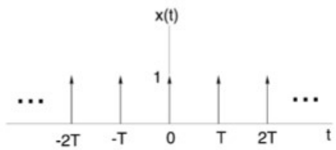
(i) $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \sum \delta(t-nT)e^{-j\omega t}dt = \sum_n e^{-j\omega nT} = X(j(\omega + \frac{2\pi k}{T})), \forall k$

(ii) $x(t) = \sum_k a_k e^{jk\omega_0 t} \longleftrightarrow X(j\omega) = \sum_k 2\pi a_k \delta(\omega - k\omega_0)$ ← periodic ft.

Example #5:

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$ — Old friend, sampling function

$$x(t) \longleftrightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_o t}dt = \frac{1}{T}$$



$$\Downarrow\ x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T} e^{jk\omega_o t}$$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T}\delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_o})$$

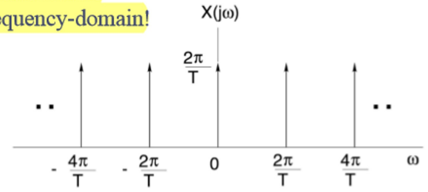
\omega대신에 f를 쓰면 X(f)에서는 2\pi가 없다.

Same function in the frequency-domain!

X(jω)

\omega대신에 f를 쓰면 X(f)에서는 2\pi가 없다.

Same function in the frequency-domain!



Note in this case, periodic in both time domain (with a period T) and frequency domain (with a period $2\pi/T$)

Properties of the CT Fourier Transform

1) Linearity $\quad ax(t) + by(t) \longleftrightarrow aX(j\omega) + bY(j\omega)$

2) Time Shifting (delay) $\quad x(t - t_0) \longleftrightarrow e^{-j\omega t_o} X(j\omega)$

Proof: $\int_{-\infty}^{\infty} x(\underbrace{t - t_o}_{t'}) e^{-j\omega t} dt = e^{-j\omega t_o} \underbrace{\int_{-\infty}^{\infty} x(t') e^{-j\omega t'} dt'}_{X(j\omega)}$

FT magnitude unchanged

$$\left| e^{-j\omega t_0} X(j\omega) \right| = \left| X(j\omega) \right|$$

Linear change in *FT* phase shift

x(t)를 time shifting을 하며 X(j\omega)에 linear phase shift가 추가된다.

$$\angle(e^{-j\omega_0 t} X(j\omega)) = \angle X(j\omega) - \omega t_0$$

CTFT Properties (continued)

3) Conjugate Symmetry

$x(t)$ real $\longleftrightarrow X(-j\omega) = X^*(j\omega)$ Conjugate Symmetrical

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$x(t)$ real $\longleftrightarrow X(-j\omega) = X^*(j\omega)$ Conjugate Symmetrical

$\Downarrow$

$|X(-j\omega)| = |X(j\omega)|$ real Even

$\angle X(-j\omega) = -\angle X(j\omega)$ " Odd

Or $\mathrm{Re}\{X(-j\omega)\} = \mathrm{Re}\{X(j\omega)\}$ " Even

cf) $z=a+jb$

$\mathrm{Re}\{z\}=a, \mathrm{Im}\{z\}=b$ $\mathrm{Im}\{X(-j\omega)\} = -\mathrm{Im}\{X(j\omega)\}$ " Odd

When $x(t)$ is real (all the physically measurable signals are *real*), the negative frequency components do *not* carry any additional information from the positive frequency components. $\omega \geq 0$ will be sufficient.

Lemma. x(t)가 real이면
X(j\omega)는 conjugate symmetrical 하고
its magnitude는 even, its phase는 odd, its real part는 even. its imaginary part
는 odd이다. 그림을 그려 보자.

Q. audio signal을 빠르게 play하면 고음화 되는데 어떻게 하면 높이는 일정하게 하면서 빠르게 play할 수 있을까?

-> MATLAB project에서 구현?!

The Properties Keep on Coming ...

4) Time-Scaling $x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$ E.g. $a > 1 \to at > t$ compressed in time $\leftrightarrow$ stretched in frequency

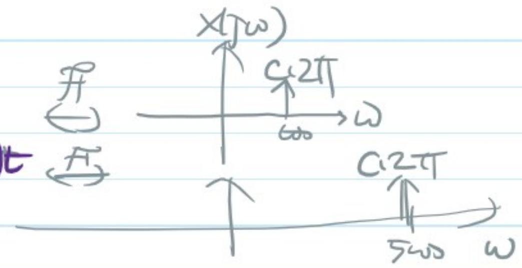
dilation

(MR) $x(t)=ce^{j\omega_0 t}$

$y(t)=x(5t)=ce^{j(5\omega_0)t}$

$\Downarrow a = -1$

5) Time-Reverse $x(-t) \longleftrightarrow X(-j\omega)$ (MR) $x(t)=ce^{j\omega_0 t}$, $y(t)=x(-t)=ce^{j\omega_0 t}$



cf) $\omega_0=0$ $\frac{1}{|a|}\delta\left(\frac{\omega}{a}\right) = \delta(\omega)$ Why?

$\Downarrow$

a) $x(t)$ real and even $x(t) = x(-t) = x^*(t)$

$\Rightarrow X(j\omega) = X(-j\omega) = X^*(j\omega)$ — Real & even

b) $x(t)$ real and odd $x(t) = -x(-t) = x^*(t)$

$\Rightarrow X(j\omega) = -X(-j\omega) = -X^*(j\omega)$ — Purely imaginary & odd

일반적으로 X(j\omega)는 complex valued function of \omega이다.

c) $X(j\omega) = \mathrm{Re}\{X(j\omega)\} + j\mathrm{Im}\{X(j\omega)\}$

For real $x(t) = Ev\{x(t)\} + Od\{x(t)\}$

이 관계 c)는 communication system에서 real bandpass signal을 이해하는데 있어 매우 중요하다.

반드시 기억할 것 (MR) $x(t)=Ce^{j\omega_0 t} \Longleftrightarrow X(j\omega)=2\pi\delta(\omega-\omega_0)$

x(t) <-> X(j\omaga)

$y(t)=$ x(-t) <-> X(-j\omega) $\to Y(j\omega)=X(-j\omega)$

$y(t)=$ x(t)^* <-> X(-j\omega)^* $\to Y(j\omega)=X^*(-j\omega)$

$y(t)=$ x(-t)^* <-> X(j\omega)^* $\to Y(j\omega)=X^*(j\omega)$