2023년 11월 8일 수요일 11:13

CM19 231109Th on CTFT; convolution and modulation properties

Announcements

1. Messages from Instructor
   - Goal of Life: Self-Fulfillment and Contribution to Collective (자아실현, 공동체 기여)
     - Attitude: Exceed the expectations! (기대를 넘어서라!)
   - "아는 것이 힘이다. 배우고야 무슨 일이든 한다." -심훈, 상록수-
   - 학습 (배우고 익히기)
     - Theory Learning (이론 배우기)
     - Problem Solving (문제 풀기)
     - 배우기와 익히기의 균형
   - How to Learn
     - Stop and Think; 복기가 chess master 를 만든다; Find commonality, similarity, and difference
   - How to Practice

     Purposeful practice with 3F: Time, Focused Action (무엇을 향상시키려는 훈련인가?), Experience, Lesson, Immediate Feedback (코치의 중요성), Fix, Growth
   - How to solve? How to understand? ((S))implify and ((V))isualize:
     - essence, simplify; generalize; consider extremes; mental representation, representational examples;
     - Venn diagram, necessity and sufficiency, existence and uniqueness/plurality
       - An imperfect classification rule is better than no classification rule as long as you recognize the imperfection and update your rule.
     - block diagram, G(V,E); graph of a function, algebraic vs. geometric interpretations, mathematical vs. physical/engineering interpretations/implications

2. 지난 강의 수정/보완 사항
   - A Gaussian pulse can be used to approximate the Dirac delta function. Actually, any time-localized function with unit area can be used.
   - Notation <->

CTFS CTFT

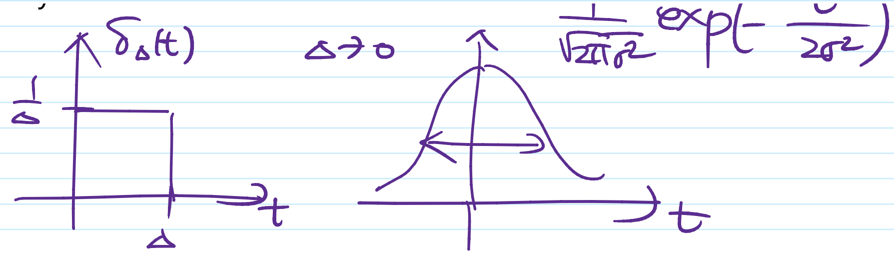
$\delta_\Delta(t)$ $\Delta \to 0$ $\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{t^2}{2\sigma^2}\right)$

function with unit area can be used.
- Notation <->
- gaussian <-> gaussian
- Impulse train <-> impulse train

CTFS CTFT

DTFS DTFT

$X(j\omega) \xleftrightarrow{\mathcal{F}} x(t)$



3. HW, Quiz, and Exam
   - HW10 will be assigned by this Friday. Due will be next Tuesday.
   - Quiz 2 is coming. Maybe on Tue. 11/20.

4. 강의 자료
   - Handouts and annotated pdfs are downloadable from the non-PLMS class homepage at http://cisl.postech.ac.kr/class/eece233/schedule.htm.
   - Recorded CMs are available from cisl.postech YouTube channel. Please place clickable time stamps.

5. 출석
   - 출결상태에 대해 전자출결시스템(https://wpsc.postech.ac.kr/eams/login)을 확인하세요. 정정사유가 있는 학생은 증빙를 제출해 주세요.

6. Overview (The numbers are the chapter numbers of the textbook)
   1. Signals and Systems
   2. Linear Time-Invariant Systems
   3. Fourier Series Representation of Periodic Signals
   4. Continuous-time Fourier Transform
   5. Discrete-time Fourier Transform
   6. Time and Frequency Characterization of Signals and Systems
   9. Laplace Transform
   10. Z-transform
   7. Sampling
   8. Communication Systems

7. Related Courses offered in 24-1
   - EECE302 전자수학 A (확률 및 랜덤프로세스); 월수 2시; ~~신호및시스템~~

7. Related Courses offered in 24-1

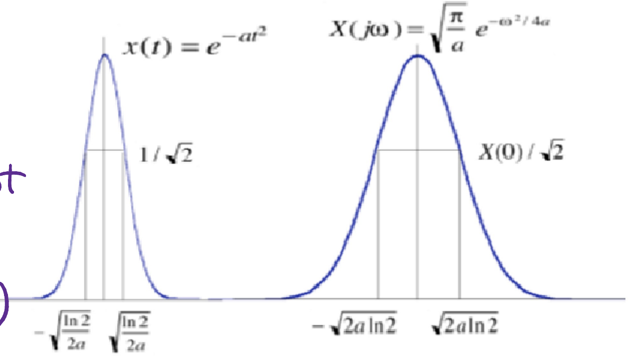
- EECE302 전자수학 A (확률 및 랜덤프로세스); 월수 2시; ~~신호및시스템~~
- EECE361/DISU361 전자파응용; 화목 11시; 전자기학 개론
- EECE441 디지털통신개론; 월수 11시; 신호및 시스템, 전자수학
- EECE442/NGCN301 통신및네트워크개론; 화목 2시
- EECE443/NGCN302 통신및네트워크실험; 월5시-10시
- EECE451 디지털신호처리개론; 화목 3시반; 신호및시스템

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

8. Review
- 수학적 operations로서
  - Continuous-Time Fourier Tranform(ation) = CTFT
    - continuous-parameter to continuous parameter Fourier Transformation에서 time과 frequency로.
    - Def.
      - Existence and L^1(R) functions, i.e., absolutely integrable functions on R.
      - Fourier-Plancherel TF and L^2(R) functions, i.e., square integrable functions on R.
  - inverse CTFT
    - continuous-parameter to continuous parameter Fourier Transformation에서 frequency와 time으로.
- 신호 및 시스템에서
  - Synthesis equation
  - Analysis equation
  - Time <-> Frequency; The problem is that L^1 in TD is not necessarilly L^1 in FD.
- Examples
  - Dirac delta <-> constant
  - delayed Dirac delta <-> linear phase shift
  - right-sided exponential function <-> a 1st-order rational function of j\omega
  - rectangular pulse <->sinc pulse
  - Gaussian pulse <-> Gaussian pulse
  - Dirac delta in the FD <-> complex sinusoidal in TD
    - periodic signal <-> equally spaced Dirac deltas
      - CTFS of periodic signal
      - CTFT of periodic signal
    - impulse train <-> impulse train
    - CTFT and CTFS = CTFT of periodic signal

$$x(t) = \mathrm{rect}\left(\frac{t}{2T_1}\right) \longleftrightarrow X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = \frac{2\sin \omega T_1}{\omega}$$



$$\delta(\omega-\omega_0) \overset{FT}{\longleftrightarrow} \frac{1}{2\pi} e^{j\omega_0 t}$$

$$e^{j\omega_0 t} \overset{FT}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0)$$

$$x(t) = \sum_k a_k e^{jk\omega_0 t}$$

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT) \longleftrightarrow X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T}\delta\left(\omega - \frac{k2\pi}{T}\right)$$

- CTFT of periodic signal
  - impulse train <-> impulse train
  - CTFT and CTFS = CTFT of periodic signal
- Properties of CTFT
  - linearlity in TD <-> linearity in FD
  - shift in TD <-> linear-phase shift in FD
  - real in TD <-> conjugate-symmetrical in FD
    - real-even mag; real-odd phase; real-even real-part; pure-imagainary odd j times imaginary part

$x(t) = \sum_k a_k e^{jk\omega_0 t}$

$\updownarrow$

$X(j\omega) = \sum_k a_k 2\pi\delta(\omega - \omega_0 k)$

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \longleftrightarrow X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T}\delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_o})$$

Properties of the CT Fourier Transform

1) Linearity $\quad ax(t) + by(t) \longleftrightarrow aX(j\omega) + bY(j\omega)$

Given $x(t) \Leftrightarrow X(j\omega)$,

And $Y(j\omega)$ when $y(t) = x(t - t_0)$.

2) Time Shifting $\quad x(t - t_0) \longleftrightarrow e^{-j\omega t_o} X(j\omega)$

$$\text{Proof:} \quad \int_{-\infty}^{\infty} x(\underbrace{t - t_o}_{t'}) e^{-j\omega t} dt = e^{-j\omega t_o} \underbrace{\int_{-\infty}^{\infty} x(t') e^{-j\omega t'} dt'}_{X(j\omega)}$$

FT magnitude unchanged

$$\left| e^{-j\omega t_0} X(j\omega) \right| = \left| X(j\omega) \right|$$

Linear change in *FT* phase

x(t)를 time shifting을 하며 X(j\omega)에 linear phase shift가 추가된다.

$$\angle(e^{-j\omega_0 t} X(j\omega)) = \angle X(j\omega) - \omega t_0$$

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CTFT Properties (continued)

3) Conjugate Symmetry

$$x(t) \text{ real} \longleftrightarrow X(-j\omega) = X^*(j\omega)$$

$$\Downarrow$$

$$|X(-j\omega)| = |X(j\omega)| \qquad \textit{Even}$$

$$|X(-j\omega)| = |X(j\omega)| \qquad \textit{Even}$$

$$\angle X(-j\omega) = -\angle X(j\omega) \qquad \textit{Odd}$$

Or

$$\mathrm{Re}\{X(-j\omega)\} = \mathrm{Re}\{X(j\omega)\} \qquad \textit{Even}$$

$$\mathrm{Im}\{X(-j\omega)\} = -\mathrm{Im}\{X(j\omega)\} \qquad \textit{Odd}$$

When $x(t)$ is real (all the physically measurable signals are *real*), the negative frequency components do *not* carry any additional information from the positive frequency components. $\omega \geq 0$ will be sufficient.

- time compression vs. dilation
- Given a Fourier Transform pair x and X, find the CTFTs of ...

The Properties Keep on Coming ...

Q. audio signal을 빠르게 play하면 고음화 되는데 어떻게 하면 높이는 일정하게 하면서 빠르게 play할 수 있을까?

-> MATLAB project에서 구현?!

~~(RM)~~

C(MR)

$x(t) = ce^{j\omega_0 t}$

4) Time-Scaling $\quad x(at) \longleftrightarrow \dfrac{1}{|a|} X\left(j\dfrac{\omega}{a}\right)$

E.g. $a > 1 \rightarrow at > t$ compressed in time $\leftrightarrow$ stretched in frequency

$\Downarrow a = -1$

5) Time−Reverse $\quad x(-t) \longleftrightarrow X(-j\omega)$

$\Downarrow$

a) $x(t)$ real and even $\quad x(t) = x(-t) = x^*(t)$

$$\Rightarrow X(j\omega) = X(-j\omega) = X^*(j\omega) \text{ — Real \& even}$$

b) $x(t)$ real and odd $\quad x(t) = -x(-t) = x^*(t)$

$$\Rightarrow X(j\omega) = -X(-j\omega) = -X^*(j\omega) \text{ — Purely imaginary \& odd}$$

일반적으로 X(j \omega)는 complex valued function of \omega이다.

c) $X(j\omega) = \mathrm{Re}\{X(j\omega)\} + j\,\mathrm{Im}\{X(j\omega)\}$

$\qquad\qquad\qquad\qquad\uparrow \qquad\qquad\qquad \uparrow$

For real $\quad x(t) = Ev\{x(t)\} + Od\{x(t)\}$

이 관계 c)는 communication system에서 real bandpass signal을 이해하는데 있어 매우 중요하다.

For real $x(t) = Ev\{x(t)\} + Od\{x(t)\}$ bandpass signal을 이해하는데 있어 매우 중요하다.

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반드시 기억할 것

x(t) <-> X(j\omaga)
x(-t) <-> X(-j\omega)
x(t)^* <-> X(-j\omega)^*
x(-t)^* <-> X(j\omega)^*

9. Preview
   - Properties of CTFT
     - Convolution Property; TD convolution <-> FD multiplication
       - Frequency reponse revisited
       - Examples
         - phasor analysis for an LTI system with real-valued impulse response and a real-valued sinusoidal input signal.
           - a differentiator in initial rest
           - an integrator in initial rest
           - the notion of impedence
         - an ideal LPF
           - impulse response and causality
           - step response and sine integral, Gibbs' phenomenon
         - cascading filters
           - integration of product of sinc pulses
           - convolution of gaussian pulses
         - convolution of two right-sided exponential functions
         - A system whose I/O relation is described by an LCCDE with initial rest condition satisfied
           - linear constant-coefficient differential equation (LCCDE)
     - Parseval's relation
       - Energy Spectral Density (ESD)
     - Multiplication Property; TD multiplication <-> FD convolution
       - Duality 쌍대성
       - Example

◆ amplitude modulation (AM) in communications

EECE233 Signals and Systems

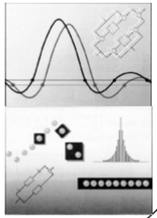
앞서 Lecture Note #8에서 CT signal에 대해 FT과 inverse FT을 배웠다. 여기서는 이들의 properties에 대해 배운다.

Lecture Note #9

Prof. Joon Ho Cho

(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

- Covers O & W pp. 306-333, 358-361
- The convolution Property of the CTFT
- Frequency Response and LTI Systems Revisited
- Multiplication Property and Parseval's Relation
- The DT Fourier Transform



* These lecture notes are based on the lecture notes used in 6.003 taught by Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.

The CT Fourier Transform Pair

$$x(t) \longleftrightarrow X(j\omega)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \text{— x(t)의 } FT$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{— } X(j\omega)\text{의 Inverse FT}$$

Last lecture: some properties

Today: further exploration

Math: Convolution의 FT은 FT들의 Product이다.

EE: LTI system output의 CTFT는 input CTFT와 frequency response의 곱이다.

Convolution Property

$$y(t) = h(t) * \boxed{x(t)} \longleftrightarrow Y(j\omega) = H(j\omega) X(j\omega)$$

where $h(t) \longleftrightarrow H(j\omega)$

Basically a consequence of the eigenfunction property

$$a\, e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega) a\, e^{j\omega t}$$

$$x(t) = \int_{-\infty}^{+\infty} \underbrace{\left(\frac{1}{2\pi} X(j\omega) d\omega \right)}_{\text{coefficient}} e^{j\omega t}$$

Σ_k $e^{j w_k t}$

$\Downarrow$ superposition

$$y(t) = \int_{-\infty}^{+\infty} \underbrace{\left(H(j\omega) \cdot \frac{1}{2\pi} X(j\omega) d\omega \right)}_{\text{New coefficient}} e^{j\omega t}$$

이 성질은, two indepedent random variables의 합의 pdf는 두 pdf의 convolution이고, 따라서 two indepedent random variables의 합의 characteristic function은 두 characteristic functions의 곱이라는 성질을 연상시킨다.

이라는 성질을 얻 상시킨다.

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{\boxed{H(j\omega)X(j\omega)}}_{= Y(j\omega)} e^{j\omega t}\, d\omega$$

New coefficient

$y(t)$ $H(j\omega)e^{j\omega t}$

Now, we have a crude tool to design the impulse resp. of a filter.
1. Design the frequency response of the desired filter.
2. Find the inverse CTFT. Done!

The Frequency Response Revisited

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * x(t)$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$\Downarrow$$

The frequency response $H(j\omega)$ of a CT LTI system is simply the Fourier transform of its impulse response $h(t)$

Example #1: $x(t) = e^{j\omega_o t} \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$

Recall $e^{j\omega_o t} \longleftrightarrow 2\pi\delta(\omega - \omega_o)$

$$Y(j\omega) = H(j\omega)X(j\omega) = H(j\omega)2\pi\delta(\omega - \omega_0) = 2\pi H(j\omega_0)\delta(\omega - \omega_0)$$

$$\Downarrow \text{ inverse } FT$$

$$y(t) = H(j\omega_0)e^{j\omega_0 t}$$

Now it is clear to see frequency shaping effects of earlier demos.

이 결과로부터 phasor analysis 가 나온다. linear circuit의 입력이 sinusoid일때 출력을 계산하기 위해서 굳이 differential equation을 풀 필요가 없이 algebraic equation을 풀면 해결이 된다.

in initial rest. $\equiv$ $x(t) \rightarrow \boxed{u_1(t)} \rightarrow y(t) \equiv X(j\omega) \rightarrow \boxed{j\omega} \rightarrow Y(j\omega)$

Example #2: A differentiator

$x(t) \rightarrow \boxed{\frac{d}{dt}} \rightarrow y(t)$

$$y(t) = \frac{dx(t)}{dt} \quad \text{— an LTI system}$$

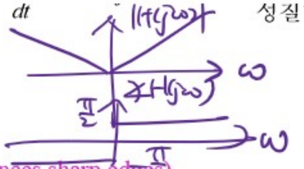
From differentiation property $\Rightarrow$ $\frac{d}{dt} \xleftrightarrow{FT} j\omega$

$|H(j\omega)|$ ω $\angle H(j\omega)$ $e^{j\frac{\pi}{2}}$

$\Downarrow$

phasor analysis에서 미분은 j\omega로 적분은 1/(j\omega)로 바꾼 것은 바로 Fourier transform의 성질때문이었다.

dt 성질때문에었다.



$e^{j\frac{\pi}{2}}$

$H(j\omega) = j\omega$

1) Amplifies high frequencies (enhances sharp edges)

2) $+\pi/2$ phase shift ($j = e^{j\pi/2}$)

Larger at high ω_o phase shift

$$\frac{d}{dt}\sin\omega_o t = \omega_o\cos\omega_o t = \omega_o\sin(\omega_o t + \pi/2)$$

$$\frac{d}{dt}\cos\omega_o t = -\omega_o\sin\omega_o t = \omega_o\cos(\omega_o t + \pi/2)$$

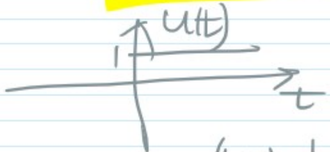
Example #2: Edge enhancement using DT differentiator

Example #3: Impulse Response of an *Ideal* Lowpass Filter

cf) integrator under initial rest

$\equiv x(t) \rightarrow \boxed{u_{-1}(t)} \rightarrow y(t) \equiv$

$X(j\omega) \rightarrow \boxed{\frac{1}{j\omega} + \pi\delta(\omega)} \rightarrow Y(j\omega)$



$= (u(t) - \frac{1}{2}) + \frac{1}{2}$

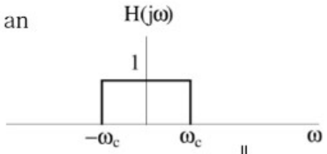
$\rightarrow \frac{2\pi}{2}\delta(\omega)$

$\int_{-\Delta}^{\Delta}(\ )\cdot e^{-j\omega t}dt$

존재 as $\Delta \rightarrow \infty$

Example #3: Impulse Response of an *Ideal* Lowpass Filter

Q1. Is the square pulse an L^1 function?
A1. Yes.
Q2. Is the CTFT of the square pulse an L^1 function?
A2. No.



Q3. Is the CTFT of the square pulse an L^2 function?
A3. Yes.
따라서 iFT은 존재하지 않지만 iFPT가 존재하고 MS sense 로 x(t)가 된다.

$\Downarrow$

$$h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t} d\omega = \frac{\sin\omega_c t}{\pi t} = \frac{\omega_c}{\pi}\mathrm{sinc}\left(\frac{\omega_c t}{\pi}\right)$$

$$\mathrm{sin}\,c(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

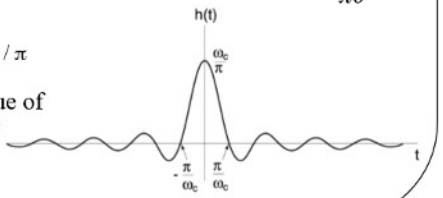
Questions:

1) Is this a causal system? No.
2) What is $h(0)$?

$$h(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)d\omega = \omega_c/\pi$$

3) What is the steady-state value of the step response, i. e. $s(\infty)$?

$$s(\infty) = \int_{-\infty}^{\infty} h(t)dt = H(j0) = 1$$



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Def. The sine integral is defined as
Si(x) = int_{0}^{x} sin(t)/t dt.

이것을 이용하여 CTFT에서의 Gibbs' phenomenon을 증명할 수 있다.

$h(t) \leftrightarrow H(j\omega)$

TD

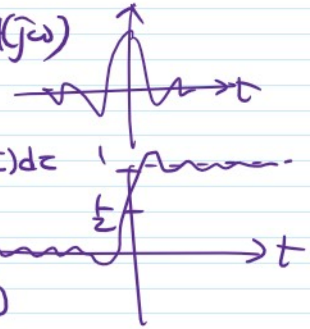
impulse response

step

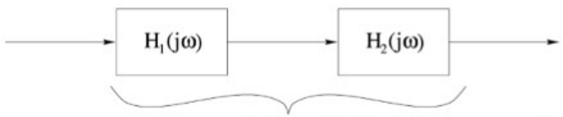
$s(t) \triangleq \int_{-\infty}^{t} h(\tau)d\tau$

"

$= Si(x)$



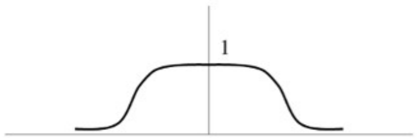
Example #4: Cascading filtering operations



$H(j\omega) = H_1(j\omega)\,H_2(j\omega)$

convolution이 CTFT들의 곱으로 나타났으므로 당연한 결과이다

e.g. $H_1(j\omega) = H_2(j\omega)$



$\Longrightarrow$ $H(j\omega) = H_1^2(j\omega)$ has a sharper frequency selectivity

두 filter가 동일한 non-ideal LPF이면 resultant filter는 더 narrow한 LPF이다.

중요: 시스템이 LTI이면 input에 없던 frequency 성분이 output에서 생겨나지 않는다.

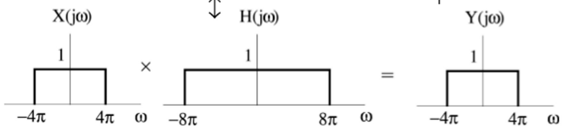
Q. int sinc(x)sinc(3x)dx = pi. int 1 3 5 = pi. Then, int 1 3 5 ... 15 dx?
A. < pi. Why?

Example #5:

$$\underbrace{\frac{\sin 4\pi t}{\pi t}}_{x(t)} * \underbrace{\frac{\sin 8\pi t}{\pi t}}_{h(t)} = ?$$

$Y(j\omega) = X(j\omega)$

$\Rightarrow y(t) = x(t)$



Example #6:

$$e^{-at^2} * e^{-bt^2} = ?$$

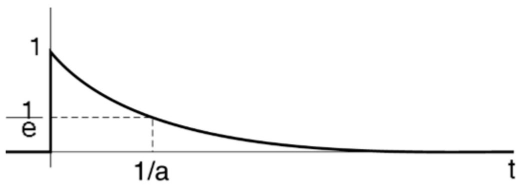
$$\sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}} \times \sqrt{\frac{\pi}{b}} e^{-\frac{\omega^2}{4b}} = \frac{\pi}{\sqrt{ab}} e^{-\frac{\omega^2}{4}\left(\frac{1}{a}+\frac{1}{b}\right)}$$

Gaussian × Gaussian = Gaussian, Gaussian ∗ Gaussian = Gaussian

전자수학A에서, "두 independent Gaussian random variables의 합은 Gaussian random variable 임을 보여라."는 문제를 풀때 characteristic function을 이용해서 푼다.

이 과목에서 가장 중요한 교훈(lesson):
time domain에서 잘 안되면 frequency domain에서 생각하라!!!!! vice versa

Review from the last lecture, right-sided exponential



$$x(t) = e^{-at} u(t)\ , \quad a > 0$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt = \int_{0}^{+\infty} e^{-at} e^{-j\omega t} dt$$

여기서 반드시 기억 할 것은, right-sided real

$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} \frac{\sin 3\pi x}{3\pi x} dx = 1$$

$$\int_{-\infty}^{\infty} 1 \quad 3 \quad 5 \quad dx = 1$$

$$\int \quad dx = 1$$

$$\int \quad \frac{1}{15} dx < 1$$

부분분수분해 if $a \neq b$

$$\frac{1}{x+a} \frac{1}{x+b} = \frac{1}{b-a}\left(\frac{1}{x+a} - \frac{1}{x+b}\right)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt = \int_0^{+\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)t}} dt$$

$$= -\left(\frac{1}{a+j\omega}\right)e^{-(a+j\omega)t}\Big|_0^{\infty} = \boxed{\frac{1}{a+j\omega}}$$

$j\omega \to s$

여기서 반드시 기억할 것은, right−sided real exponential의 CTFT는 rational function of j omega 라는 것이다.

$\frac{1}{s+a}$ $\quad s=-a$

s

Q. 고등학교 시험문제에 이 문제가 나왔다면?

A. time−domain에서 convolution하는 것은 고등학교 적분실력으로 할 수 있을 것이다. 시도해 보자.

그러나 이제는, ??? frequency−domain에서 할 수 있다!!!! 있어야 한다!!!

Example #7: $h(t) = e^{-t}u(t)$, $x(t) = e^{-2t}u(t)$

$$y(t) = h(t) * x(t) = ?$$

$$\Downarrow$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{(1+j\omega)} \cdot \frac{1}{(2+j\omega)}$$

— a *rational* function of $j\omega$, ratio of polynomials of $j\omega$

Q. 자 이제 inverse FT 을 해야 하는데 어떻게 하나?

A. 부분분수 분해를 하여 다시 right−sided exponential function들의 FT이 1/(a+j \omega)임을 이용한다.

$\Downarrow$ Partial fraction expansion

$$Y(j\omega) = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

$\Downarrow$ inverse FT

$$y(t) = [e^{-t} - e^{-2t}]u(t)$$

Q. When

$x(t) \to \boxed{y'(t)+ay(t)=x(t)} \to y(t)$

$y'(t)+ay(t)=x(t)$, find $h(t)$.

$\Updownarrow$

$(j\omega+a)Y(j\omega)=X(j\omega)$

$H(j\omega)=\frac{Y(j\omega)}{\ldots} = \frac{1}{\ldots}$

RLC circuit에서 입출력관계는 언제나 LCCDE로 나타낼 수 있다.

Q. 적분기는 어디 있는가?

A. 치환하여 정부

Example #8: LTI Systems Described by LCCDE's
(Linear-constant-coefficient differential equations)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

LCCDE라고 하여 언제나 LTI시스템을 나타내는 것은 아니며, initial condition에 따라 LTI system 이 아닐 수 있다.

Using the Differentiation Property

frequency response

Q. 적분기는 어디 있는가?
A. 치환하여 적분기를 없애 풀고 나서 다시 적분하면 된다.

또는 적분을 넣어서 풀어도 된다.
Q. 적분기의 frequency response는 무엇인가?
A. 1/(j\omega)

=> phasor analysis가능...

$$\sum_{k=0} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0} b_k \frac{d^k x(t)}{dt^k}$$

Using the Differentiation Property

$$\frac{d^k x(t)}{dt^k} \longleftrightarrow (j\omega)^k X(j\omega)$$

$\Downarrow$ Transform both sides of the equation

$$\left(\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega)\right) = \left(\sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)\right)$$

$\Downarrow$

$$Y(j\omega) = \underbrace{\left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}\right]}_{H(j\omega)} X(j\omega)$$

1) *Rational*, can use PFE to get $h(t)$

2) If $X(j\omega)$ is rational
i.e. $x(t) = \sum \alpha_i e^{-\beta_i t} u(t)$,
then $Y(j\omega)$ is also rational

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LCCDE라고 하여 언제나 LTI시스템을 나타내는 것은 아니며, initial condition에 따라 LTI system 이 아닐 수 있다.

frequency response가 유리함수이므로 부분분수 분해를 하여 inverse Fourier transform을 할 때 causality를 적용하면 impulse response들이 right−sided exponential function들로 나온다.

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{j\omega + a}$$

$x(t) = \delta(t)$.

$$y(t) = \begin{cases} e^{-at}u(t) & \text{causal} \\ -e^{-at}u(-t) & \text{anti-causal} \end{cases}$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

중요!!!

이제 우리는 원하는 CT LTI system의 CTFT를 design함으로써 원하는 system을 얻는 법을 배웠다.
그러나 실제로 구현하려면 쉽지않다. 예를 들어 LPF가 되도록 무얼가지고 어떻게 만들면 되는가?

여기서 보듯, LCCDE으로 input output relation을 만들면 circuit theory에서 배운 실력으로 system을 실제로 구현할 수 있는데, component수를 잘 조절하여 원하는 주파수 응답과 아주(?) close한 주파수 응답을 가지는 LCCDE를 만들어 linear circuit으로 구현하면 되겠다는 결론을 얻는다!!!!!

Pade approximation!!!
See https://www.youtube.com/watch?v=szMaPkJEMrw
See https://www.youtube.com/watch?v=Uzjx8uOL6as

cf) Power series approximation

$$\int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt$$

Parseval's Relation

$$= \int_{-\infty}^{\infty} x(t) \left(\cdots \right)^* dt$$

$\int_{-\infty} x(t)\, x^*(t)\, dt$

$= \int_{-\infty}^{\infty} x(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \right)^* dt$

$= \cdots$

Parseval's Relation

$$E_{xx} = \underbrace{\int_{-\infty}^{+\infty} |x(t)|^2 dt}_{\text{TD}} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega}_{\text{FD}}$$

Total energy in the time-domain

Total energy in the frequency-domain

$\frac{1}{2\pi}|X(j\omega)|^2$ — energy spectral density (ESD)

$\frac{|X(j\omega)|^2}{2\pi}$

이를 energy spectral density라고 한다. 이유는??

적분해서 energy가 되니까.
그게 다일까?

어떤 주파수에서 very narrow bandpass filter를 만들면 그 출력의 에너지는 이 에너지스펙트럼밀도의 그 주파수 성분 곱하기 필터의 bandwidth가 된다.

Q. 유도해 보라.

Multiplication Property

Since *FT* is highly symmetric,

$$x(t) = \mathfrak{F}^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega, \quad X(j\omega) = \mathfrak{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

thus if $x(t) * y(t) \longleftrightarrow X(j\omega)\cdot Y(j\omega)$

time domain에서의 곱은 frequency domain에서의 convolution
그런데 radian frequency를 쓰면 2\pi때문에 헷갈린다.
그냥 Hz를 쓰면 간단하다.
2\pi 어쩌고하는 계수가 나타나지 않는다.

$z(t) =$ $\quad Z(j\omega) =$

then the other way around is also true

$$x(t)\cdot y(t) \longleftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta) Y(j(\omega-\theta)) d\theta$$

— Definition of convolution in ω

then the other way $x(t)\cdot y(t) \longleftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega)$

around is also true $= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$ — Definition of convolution in ω

— A consequence of *Duality*

Examples of the Multiplication Property

((S))

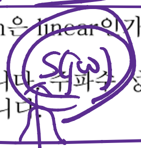
real baseband

voice

$$r(t) = s(t)\cdot p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

Q. s(t)가 input이고 r(t)가 output일때 이 system은 linear인가? 만약 linear라면 LTI인가?

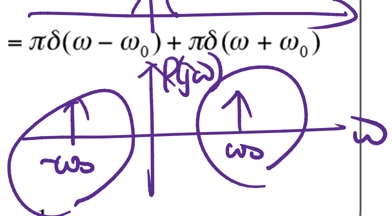
A. linear이나 nonzero \omega에 대해 LTI는 아니다. 주파수 성분을 보면, input에 없던 성분이 생겨난다. 고로 LTI가 아니다.



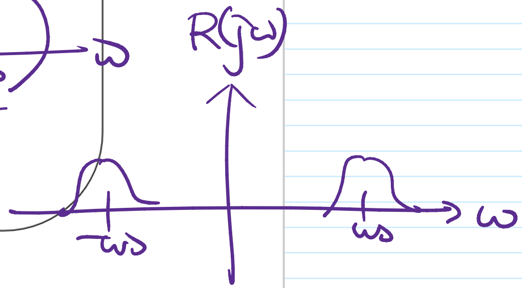
For $p(t) = \cos\omega_0 t \longleftrightarrow P(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$

carrier

통신에서 up−conversion! ⇓

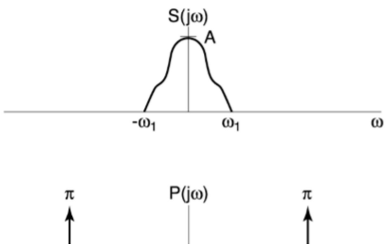


$$R(j\omega) = \frac{1}{2}S(j(\omega-\omega_0)) + \frac{1}{2}S(j(\omega+\omega_0))$$



Example (continued)

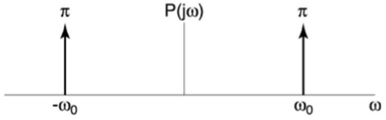
$r(t) = s(t)\cdot\cos(\omega_0 t)$
Amplitude modulation (*AM*)



Q. 여기서 s(t)가 음성신호라고 하고 cos(\omega_0 t)가 반송파 신호라고 하면 출력은?

A. AM signal

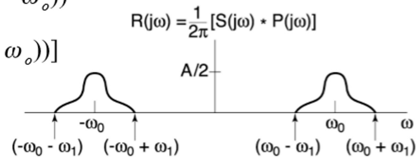
Amplitude modulation (*AM*)



파 신호라고 하면 출력은?

A. AM signal

$$R(j\omega) = \frac{1}{2}[S(j(\omega - \omega_o)) + S(j(\omega + \omega_o))]$$



Drawn assume
$\omega_o - \omega_1 > 0$
i.e. $\omega_o > \omega_1$

Q. AM signal의 Fourier transform은?
A. s(t)의 FT를 좌우로 frequency shifting한 것.
=> radio station들의 frequency division duplexing법으로 서로 다른 주파수를 써서 주파수 영역에서 신호들이 overlap이 되지 않도록 한다.

Q. Time domain에서 생각해도 AM은 쉽게 이해할 수 있다.
즉
송신단에서는 메시지 신호 s(t)에 carrier 신호 cos[\omega_0 t]를 곱하여 보내고
수신단에서는 수신신호
s(t-t_0)cos[\omega_0 (t-t_0)]
에 cos[\omega_0 (t-t_0)]를 곱하고 LFP하면 s(t)/2를 얻는다. Why?

그러나 cos[\omega_0 (t-t_0)]를 아는 것은 불가능하다. 따라서 그냥 cos[\omega_0 t+ \theta)]를 곱하면 어떤 일이 벌어질까? 이 문제를 어떻게 해결할까? -> 정보통신공학개론을 보라.