2023년 11월 14일 화요일 12:25

CM20 231114T on DTFT and DTFT Properties

Announcements

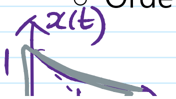
1. Messages from Instructor
   - Goal of Life: Self-Fulfillment and Contribution to Collective (자아실현, 공동체 기여)
     - Attitude: Exceed the expectations! (기대를 넘어서라!)
   - "아는 것이 힘이다. 배우고야 무슨 일이든 한다." -심훈, 상록수-
   - 학습 (배우고 익히기)
     - Theory Learning (이론 배우기)
     - Problem Solving (문제 풀기)
     - 배우기와 익히기의 균형
   - How to Learn
     - Stop and Think; 복기가 chess master 를 만든다; Find commonality, similarity, and difference
   - How to Practice

     Purposeful practice with 3F: Time, Focused Action (무엇을 향상시키려는 훈련인가?), Experience, Lesson, Immediate Feedback (코치의 중요성), Fix, Growth
   - How to solve? How to understand? ((S))implify and ((V))isualize:
     - essence, simplify; generalize; consider extremes; mental representation, representational examples;
     - Venn diagram, necessity and sufficiencency, existence and uniqueness/plurality
       - An imperfect classification rule is better than no classification rule as long as you recognize the imperfection and update your rule.
     - block diagram, G(V,E); graph of a function, algebraic vs. geometric interpretations, mathematical vs. physical/engineering interpretations/implications

$a_{m,n} \triangleq \frac{n}{m+n}$

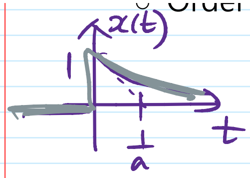
2. 지난 강의 수정/보완 사항
   - Order of limits Does matter.

$\lim_{m\to\infty}\left(\lim_{n\to\infty} \frac{n}{m+n}\right) = 1, \quad \lim_{n\to\infty}\left(\lim_{m\to\infty} \frac{n}{m+n}\right) = 0,$



$x(t) = e^{-at}u(t) \xrightarrow{F} X(j\omega) = \frac{1}{a+j\omega}$

○ Order of limits Does matter.



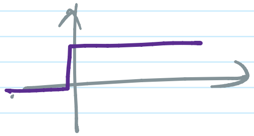
$x(t)=e^{at}u(t) \xleftrightarrow{FT} X(j\omega)=\frac{1}{a+j\omega}$

$x(t)=u(t) \xleftrightarrow{FT} X(j\omega)=\frac{1}{j\omega}+\pi\delta(\omega)$

$\lim_{a\to 0}\frac{1}{a+j\omega}=\frac{1}{j\omega}$

↓ as a→0

$=(u(t)-\frac{1}{2})+\frac{1}{2}$



$\frac{1}{2}e^{j0t} \leftrightarrow \pi\delta(\omega)$

$x(t)=ce^{j\omega_0 t}$

○ Time compression and dilation

$x(at) \xleftrightarrow{FT} \frac{1}{|a|}X(j\frac{\omega}{a})$

$\Rightarrow \begin{cases} \delta(t) \xleftrightarrow{FT} 1, \forall\omega \\ \delta(at) \leftrightarrow \frac{1}{|a|} \end{cases} \Rightarrow \begin{cases} |a|\delta(at)=\delta(t) \\ \frac{1}{|a|}\delta(\frac{\omega}{a})=\delta(\omega) \end{cases}$

○ Gaussian pulse

$\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{t^2}{2\sigma^2}) \xrightarrow[\sigma\to 0]{as} \delta(t)$

•

○ An LTI whose I/O relation described by an LCCDE is not necessarily causal.

Q. When

$x(t) \to \boxed{y'(t)+ay(t)=x(t)} \to y(t)$

$y'(t)+ay(t)=x(t)$, find $h(t)$.

A. ①

$(j\omega+a)Y(j\omega)=X(j\omega)$

$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{1}{j\omega+a}$

$x(t)=\delta(t)$

$\frac{d}{dt}(e^{-at}u(t))$

$=-ae^{-at}u(t)+e^{-at}\delta(t)$

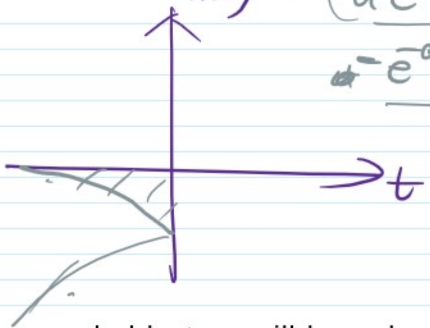
$\frac{d}{dt}(-e^{-at}u(-t))$ $\quad \delta(t)$

$h(t)=ae^{-at}u(t)$

$X(j\omega) = \frac{1}{j\omega + a}$

$x(t) = \delta(t)$

$y(t) = \begin{cases} e^{-at} u(t) & \text{causal} \\ -e^{-at} u(-t) & \text{anti-causal} \end{cases}$



$h(t) = a e^{-at} u(t)$

$- e^{-at} (-\delta(t))$

$= \delta(t)$

3. HW, Quiz, and Exam
   - There will be a supplementary CM. No attendance required. The recorded lecture will be uploaded.
   - HW11 will be assigned by this Friday. Due will be next Tuesday.
   - Quiz 2 will be taken on Tue. 11/21.

4. 강의 자료
   - Handouts and annotated pdfs are downloadable from the non-PLMS class homepage at http://cisl.postech.ac.kr/class/eece233/schedule.htm.
   - Recorded CMs are available from cisl.postech YouTube channel. Please place clickable time stamps.

5. 출석
   - 출결상태에 대해 전자출결시스템(https://wpsc.postech.ac.kr/eams/login)을 확인하세요. 정정사유가 있는 학생은 증빙를 제출해 주세요.

6. Overview (The numbers are the chapter numbers of the textbook)
   1. Signals and Systems
   2. Linear Time-Invariant Systems
   3. Fourier Series Representation of Periodic Signals
   4. Continuous-time Fourier Transform
   5. Discrete-time Fourier Transform
   6. Time and Frequency Characterization of Signals and Systems
   9. Laplace Transform
   10. Z-transform
   7. Sampling
   8. Communication Systems

7. Related Courses offered in 24-1
   - EECE302 전자수학 A (확률 및 랜덤프로세스); 월수 2시; 신호및시스템
   - EECE361/DISU361 전자파응용; 화목 11시; 전자기학 개론
   - EECE441 디지털통신개론; 월수 11시; 신호및 시스템, 전자수학
   - EECE442/NGCN301 통신및네트워크개론; 화목 2시
   - EECE443/NGCN302 통신및네트워크실험; 월5시-10시
   - EECE451 디지털신호처리개론; 화목 3시반; 신호및시스템

8. Review
   - Properties of CTFT
     - Convolution Property; TD convolution <-> FD multiplication
       - Frequency reponse revisited
       - Examples
         - phasor analysis for an LTI system with real-valued impulse response and a real-valued sinusoidal input signal.
           - a differentiator in initial rest
           - an integrator in initial rest
           - the notion of impedance
         - an ideal LPF
           - impulse response and causality
           - step response and sine integral, Gibbs' phenomenon
         - cascading filters
           - integration of product of sinc pulses
           - convolution of gaussian pulses
         - convolution of two right-sided exponential functions
         - A system whose I/O relation is described by an LCCDE with initial rest condition satisfied
           - linear constant-coefficient differential equation (LCCDE)
     - Parseval's relation
       - Energy Spectral Density (ESD)
     - Multiplication Property; TD multiplication <-> FD convolution
       - Duality 쌍대성
       - Example
         - amplitude modulation (AM) in communications

$$x(t) = A\cos(\omega_0 t + \theta) = \mathrm{Re}\{A e^{j(\omega_0 t+\theta)}\}$$

$$= \mathrm{Re}\{A e^{j\theta} e^{j\omega_0 t}\}$$

$$= A\,\frac{e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}}{2}$$

$$X(j\omega) = A\{e^{j\theta}\delta(\omega-\omega_0) + e^{-j\theta}\delta(\omega+\omega_0)\}$$

$$H(j\omega) = H^*(-j\omega) \quad \text{conjugate symmetrical}$$

real coefficients

$$Y(j\omega) = \pi e^{j\theta} H(j\omega_0)\delta(\omega-\omega_0) + \pi e^{-j\theta} \underbrace{H(-j\omega_0)}_{H^*(j\omega_0)}\delta(\omega+\omega_0)$$

$$y(t) = \frac{A}{2}\{H(j\omega_0) e^{j(\omega_0 t+\theta)} + H^*(j\omega_0) e^{-j(\omega_0 t+\theta)}\}$$

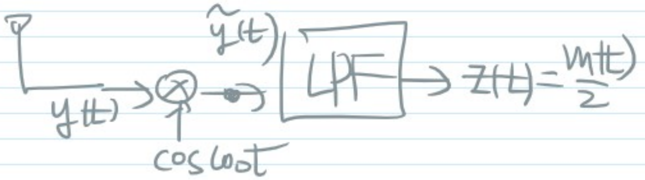
- Duality 쌍대성
- Example
  - amplitude modulation (AM) in communications
    - Recall that h(t)*delta(t-t_0) = h(t-t_0)
    - Similarly, H(jw)*delta(w-w_0) = H(j(w-w_0))

$y(t) = \frac{A}{2}\{H(j\omega_0)e^{j(\omega_0 t+\theta)} + H(-j\omega_0)e^{-j(\omega_0 t+\theta)}\}$

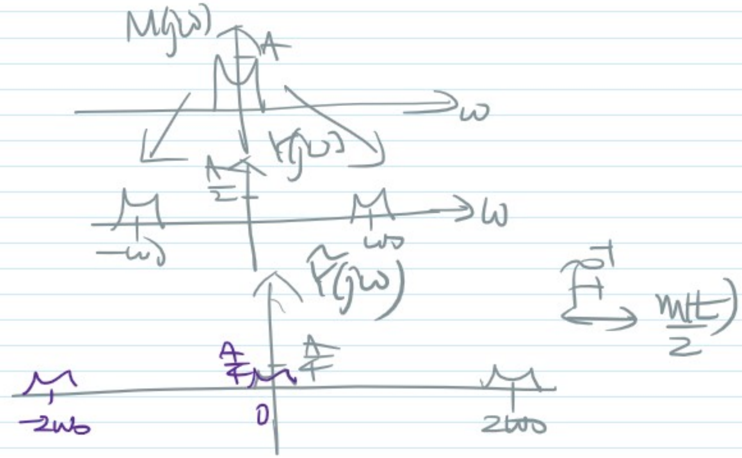
$\mathrm{Re}\{Ae^{j\theta} H(j\omega_0) e^{j\omega_0 t}\}$

$r(t)=s(t)p(t) \Leftrightarrow R(j\omega)=\frac{1}{2\pi} S(j\omega) * P(j\omega)$

$y(t) = m(t)\cos\omega_c t$



$Y(j\omega)=\frac{1}{2}M(j\omega) * \{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\}$



9. Preview
   - DTFT
     - Movitation of DTFT
       - Periodic extension fo an aperiodic signal
       - For sufficiently large period
       - synthesis and analysis equations
     - Math. vs. EE
       - Def. discrete-parameter to continuous-parameter Fourier Transformation
         - In EECE233, DTFT.
         - Existence and l^1(Z) sequences, i.e., absolutely summable sequences on Z
       - Def. continuous-parameter to discrete-parameter inverse Fourier Transformation
         - In EECE233, inverse DTFT.
         - Existence and L^1([0,2₩pi]) functions, i.e., absolutely integrable functions on [0,2₩pi)
       - Def. Fourier-Plancherel FT for l^2(Z) sequences

aperiodic

자 이제 DT signal을
complex sinusoids의 linear combination으로 나타내는 것을 생각하자.

자 이제 DT signal을
complex sinusoids의 linear combination으로 나타내는 것을 생각하자.

앞서 CTFF에서 complex sinusoids 의 linear combination을 표현하기 위해
integral을 사용하였다:
\int_{-\infty}^{\finty}a(\omega)e^{j\omega t}d\omega.

이제 DTFT에서 complex sinusoids의 linear combination을 표현하기 위해 여전히
\int_{-\infty}^{\finty}a(\omega)e^{j\omega n}d\omega
를 써야 할까?

-\infty에서 +\infty까지 필요없이 2\pi 주기 만큼만 필요하다. !!!!!

Q. 우리는 x[n] = \int_{0}^{2\pi} a(\omega)e^{j\omega n}d\omega
로 나타낼 수 있는 x[n]들의 집합을 알고 싶다.
A. 아무도 이 집합을 찾지 못했다. 단지 부분 집합들을 찾았을 뿐.

DTFT & inverse DTFT exist!

∵ Fourier-Plancherel TF으로 relax.

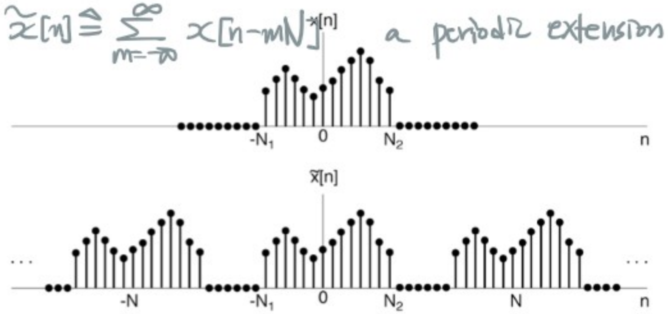
The Discrete-time Fourier Transform

Derivation: (Analogous to CTFT except $e^{j\omega n} = e^{j(\omega+2\pi)n}$)

Recall $e^{j\omega n}$ is Not periodic in general.

- $x[n]$ - aperiodic and (for simplicity) of finite duration
- N is large enough so that $x[n] = 0$ if $|n| \geq N/2$
- $\tilde{x}[n] = x[n]$ for $|n| \leq N/2$ and periodic with period N

Def. $\tilde{x}[n] \triangleq \sum_{m=-\infty}^{\infty} x[n-mN]$ a periodic extension of $x[n]$.



$\tilde{x}[n] = x[n]$ for *any* n as $N \longrightarrow \infty$

DTFT Derivation (Continued)

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \quad , \quad \omega_0 = \frac{2\pi}{N}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}$$

Define $\quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ $\quad \boxed{\text{— periodic in } \omega \text{ with period } 2\pi}$

$$\Downarrow$$

$$a_k = \frac{1}{N} X(e^{jk\omega_0})$$

DTFT Derivation (Home Stretch)

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \underbrace{\frac{1}{N} X(e^{jk\omega_0})}_{a_k} e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0 \quad (*)$$

As $N \to \infty$: $\quad \tilde{x}[n] \to x[n]$ for every n

$$\omega_o \to 0, \quad \sum \omega_o \to \int d\omega$$

The sum in $(*)$ $\to$ an integral

$\boxed{X(e^{j(\omega+2\pi)}) = X(e^{j\omega}), \forall\omega}$

$\Downarrow$ The DTFT Pair

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega}) e^{j\omega n} d\omega \qquad \text{Synthesis equation}$$

← Continuous-parameter to discrete — "

$$x[n] = \frac{1}{2\pi}\int_{2\pi} \boxed{X(e^{j\omega})} e^{j\omega n} d\omega$$ **Synthesis equation**

↑ Any 2π interval in ω

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$$ **Analysis equation**

← Continuous-parameter to discrete- " inverse FT

← Discrete-parameter to continuous- " FT

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EECE233 Signals and Systems

앞서 DTFT의 heuristic derivation을 보았고 이 lecture note에서는 examples와 properties를 배운다

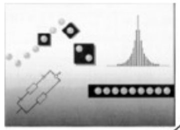
Lecture Note #10

Prof. Joon Ho Cho

(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

CTFT일때와 유사점, 차이점에 주목하면서 일사천리로 진행한다. 휘릭~

- Covers O & W pp. 361-384
- Examples of the DT Fourier Transform
- Properties of the DT Fourier Transform
- The Convolution Property and its Implications and Uses



* These lecture notes are based on the lecture notes used in 6.003 taught by Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.

1

DT Fourier Transform Pair

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$x[n] \longleftrightarrow X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$ — Analysis equation

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$ — Synthesis equation

Note for DTFT the *FT* and inverse *FT* are not symmetric. One is integration over a finite interval (2π), and the other is summation over infinite terms.

Convergence Issues

Synthesis Equation: None, since integrating over a finite interval

Analysis Equation: Need conditions analogous to CTFT, e.g.

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$ — Finite energy

$\ell^2(\mathbb{Z})$ square ~~integrable~~ summable sequence

or

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$ — Absolutely summable

Thm. DTFT exists if if x[n] is an l^1 sequence.

$\ell^1(\mathbb{Z})$ sequence = absolutely summable

$\exists X(e^{j\omega})$

Examples

Parallel with the CT examples in Lecture #8

1) $x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n] e^{-j\omega n} = 1, \ \forall \omega$$

$$= X(e^{j(\omega+2\pi)}).$$

2) $x[n] = \delta[n-n_o]$ — shifted unit sample

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n-n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

$\uparrow \in \mathbb{Z}$

— Same amplitude (=1) as above, but with a *linear* phase $-\omega n_o$.

shift by

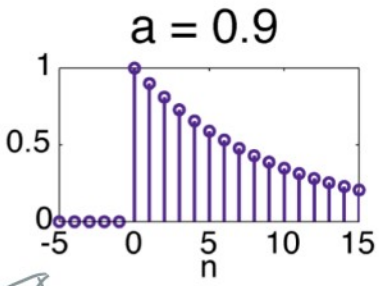
cf) $x(t) = e^{-at}\boxed{u(t)}$ $\boxed{e^{st}}$

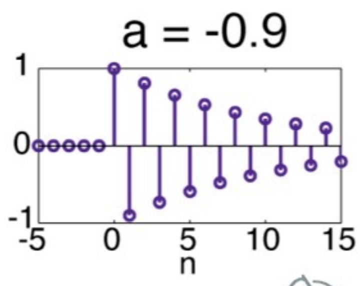
$\boxed{z^n.}$

More Examples

right-sided

3) $x[n] = a^n \boxed{u[n]}, \ |a| < 1$ — Exponentially decaying function



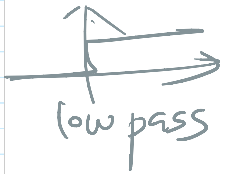


Smaller

Smaller

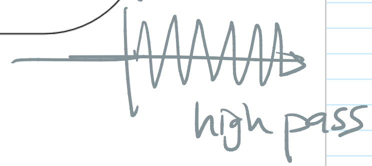
주의.

Similar to



주의.
exponential function z^n 은 모든 DT LTI의 eigenfunction의 후보 이지만 right xor left-sided real exponential은 아니다.

Similar to



~~$x[n] = -a^n u[-n-1]$~~

More Examples (cont.) $x[n] = a^n u[n]$, $|a| < 1$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \underbrace{\left(ae^{-j\omega}\right)^n}_{|ae^{-j\omega}|<1}$$

$$= \frac{1}{1-ae^{-j\omega}} = \frac{1}{(1-a\cos\omega)+ja\sin\omega}$$

Let $z = e^{j\omega}$. Then

$$\frac{1}{1-az^{-1}} = \frac{z}{z-a} \rightarrow z \neq a.$$

Recall $x(t) = e^{-at}u(t)$ $\overset{FT}{\Updownarrow}$ $X(j\omega) = \frac{1}{a+j\omega}$

Q. CT에서 j\omega의 유리함수였던 것과 유사하게 DT right-sided real exponential도 e^{j\omega} 의 rational function으로 나온다. 그렇다면 주파수가 서로 다른 두 right-sided real exponetial의 convolution은 ???

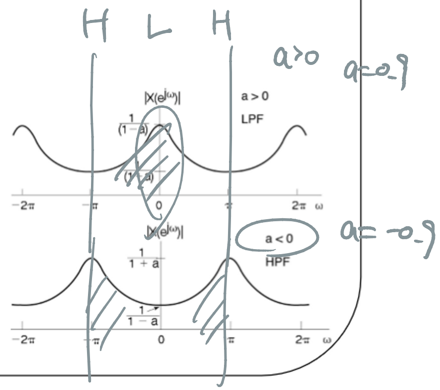
A. 두 right-sided real exponential의 weighted sum이 된다. (주파수가 같은 경우는 답이 조금 달라지므로 유의한다. 뒤에 다시 나온다.) 왜냐하면 곱이 되었다가 다시 partial fraction expansion하면 weight 만 달라진채 합으로 나오기 때문이다.

More Examples (cont.) $x[n] = a^n u[n]$, $|a| < 1$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

$$\omega = 0: \quad X(e^{j\omega}) = \frac{1}{\sqrt{1 - 2a + a^2}} = \frac{1}{1-a}$$

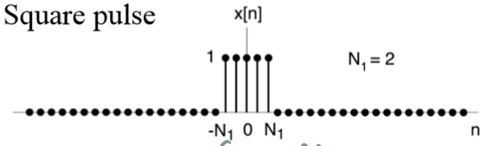
$$\omega = \pi: \quad X(e^{j\omega}) = \frac{1}{\sqrt{1 + 2a + a^2}} = \frac{1}{1+a}$$



Still More

Still More

4) DT Square pulse



FIR (Finite-Impulse-Response) LPF

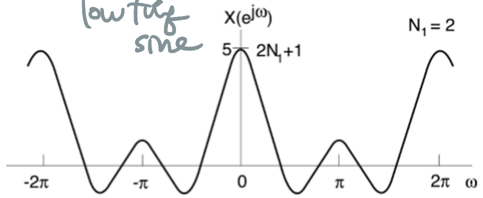
high freq sinc

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \sum_{n=-N_1}^{N_1} (e^{-j\omega})^n = \frac{\sin[\omega(N_1+1/2)]}{\sin[\omega/2]} = X(e^{j(\omega-2\pi)})$$

변수치환으로 유도 해봐라!

low freq sinc

Similar to a *sinc*, but periodic with a period 2π.



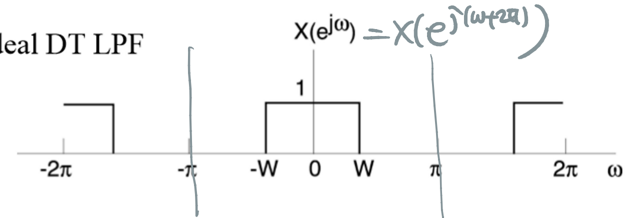
DT Dirichlet kernel이다.

여기서 N_1이 무한대로 가면
주기 2\pi인 comb signal이 얻어진다?!

이것이 DTFT의 key assumption이다.

Still More ...

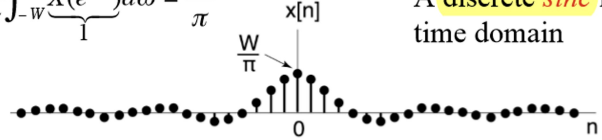
5) Ideal DT LPF



1 W sin Wn

$$x[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n} d\omega = \frac{\sin Wn}{\pi n}$$

$$x[0] = \frac{1}{2\pi}\int_{-W}^{W} \underbrace{X(e^{j\omega})}_{1} d\omega = \frac{W}{\pi}$$

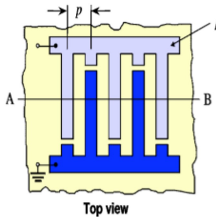


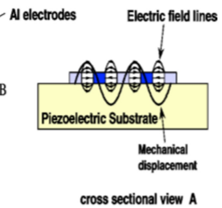
A discrete *sinc* in time domain

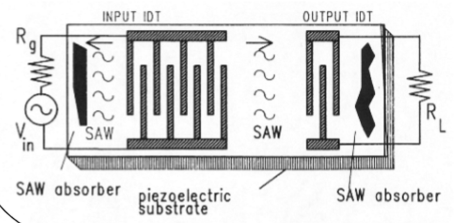
즉, continuous sinc이든 discrete sinc 이든 이를 impulse response로 갖는 LTI system 은 ideal LPF이다.

Eg. Compact BPF in your cell phone and TV

SAW (Surface-Acoustic-Wave) filters







Operation principle:

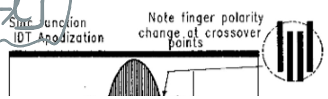
- Convert electrical signals into sound (acoustic waves).
- At output, pick up the propagating sound signals after a delay $t = x/c_s$, where x is the distance traveled and c_s is the speed of sound.
- This is a good example that time = space with a scaling factor.

An echo microphone implemented by delay line. ---- ?

SAW Bandpass Filters (cont.)

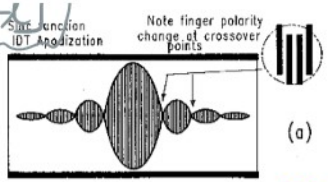
- Since $h[n]$ is proportional to the overlap length of the

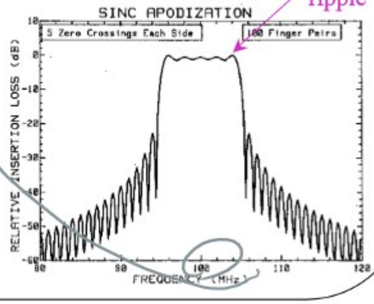
$h[n] = \sum h[k]\delta[n-k]$



- Since $h[n]$ is proportional to the overlap length of the electrode pair, we can design and fabricate any $h[n]$ desired. Here we use a modulated *sinc* (multiplied with $\cos\omega_o n$) which results in a bandpass filter.
- Note the passband ripples due to the Gibbs' phenomenon.

$h[n] = \sum h[l]\delta(n-l)$





CT에서와 마찬가지로 DT에서도 DTFS와 DTFT와의 관계를 알아보자.

DTFT's of Sums of Complex Exponentials

Recall CT result: $x(t) = e^{j\omega_0 t} \longleftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$

What about DT: $x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = ?$ $X(e^{j(\omega+2\pi)})$

a) We expect an impulse (of area 2π) at $\omega = \omega_o$

b) But $X(e^{j\omega})$ must be periodic with period 2π, In fact

$$X(e^{j\omega}) = 2\pi \sum_{m=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi m)$$

Note: The integration in the synthesis equation is over 2π period, only need $X(e^{j\omega})$ in *one* 2π period. Clearly,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{2\pi \sum_{m=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi m)}_{X(e^{j\omega})} e^{j\omega n} d\omega = e^{j\omega_o n}$$

DTFT of Periodic Signals

중요:
주기 DT signal의
DTFS coefficients와
DTFT 의 관계에 대해
확인하자.

$$x[n] = x[n+N]$$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \omega_0 = \frac{2\pi}{N}$$

Finite sum

From last page:

$$e^{jk\omega_0 n} \longleftrightarrow 2\pi \sum_{m=-\infty}^{+\infty} \delta(\omega - k\omega_0 - 2\pi m)$$

DIY:
Do it
yourself!

$$X(e^{j\omega}) = \sum_{k=<N>} a_k \boxed{\left[2\pi \sum_{m=-\infty}^{+\infty} \delta(\omega - k\omega_0 - 2\pi m) \right]}$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right) \overset{a_k = a_{k+N}}{=} X(e^{j(\omega+2\pi)})$$

↑
Infinite sum

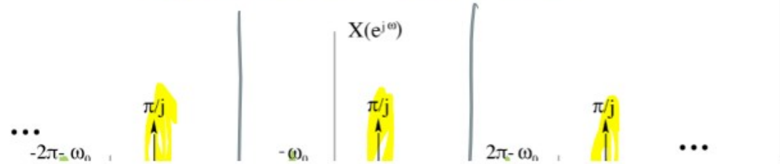
$a_k = a_{k+N}, \forall k$

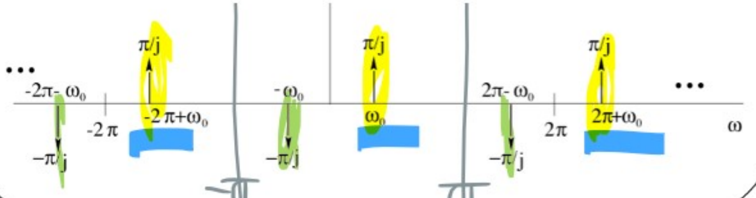
$$\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}$$

Example #1: DT sine function

$$x[n] = \sin \omega_0 n = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$$

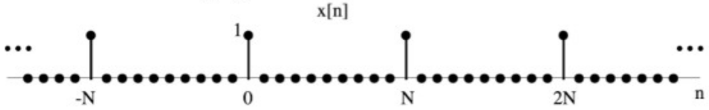
$$X(e^{j\omega}) = \frac{\pi}{j} \sum_{m=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi m) - \frac{\pi}{j} \sum_{m=-\infty}^{+\infty} \delta(\omega + \omega_0 - 2\pi m)$$





Example #2: DT periodic impulse train

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \qquad \omega_o = 2\pi / T$$

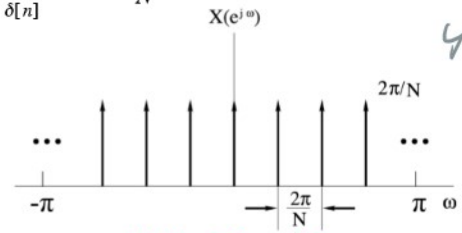


$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=0}^{N-1} \underbrace{x[n]}_{\delta[n]} e^{-jk\omega_0 n} = \frac{1}{N}$$

$$\Downarrow$$

$$X(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

— Sampling function again!



Properties of the DT Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n}$$ — Analysis equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad \text{— Analysis equation}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n} d\omega \qquad \text{— Synthesis equation}$$

1) Periodicity: $X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$ — Different from CTFT

2) Linearity: $ax_1[n] + bx_2[n] \longleftrightarrow aX_1\left(e^{j\omega}\right) + bX_2\left(e^{j\omega}\right)$

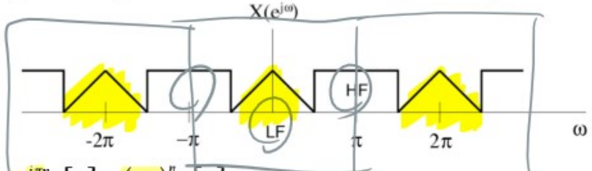
More Properties — EZ

3) Time Shifting $x[n - \boxed{n_0}] \longleftrightarrow e^{-j\omega n_0} X\left(e^{j\omega}\right)$

4) Frequency Shifting $e^{j\omega_0 n} x[n] \longleftrightarrow X\left(e^{j(\omega-\omega_0)}\right)$ special case of modulation property

— Important implications in DT because of periodicity

Example



$\omega_0 = \pi$, $y[n] = e^{j\pi n} x[n] = (-1)^n x[n]$

$Y(e^{j\omega}) = X(e^{j(\omega-\pi)})$