2023년 11월 14일 화요일 12:25

CM21 231114T on Properties of DTFT

Announcements

1. This video is supplementary to CM20.
   - Of course, the contents will be covered by the Quiz 2.
2. Overview (The numbers are the chapter numbers of the textbook)
   1. Signals and Systems
   2. Linear Time-Invariant Systems
   3. Fourier Series Representation of Periodic Signals
   4. Continuous-time Fourier Transform
   5. Discrete-time Fourier Transform
   6. Time and Frequency Characterization of Signals and Systems
   9. Laplace Transform
   10. Z-transform
   7. Sampling
   8. Communication Systems
3. Review
   - DTFT
     - Movitation of DTFT
       - Periodic extension fo an aperiodic signal
       - For sufficiently large period
       - synthesis and analysis equations
     - Math. vs. EE
       - Def. discrete-parameter to continuous-parameter Fourier Transformation
         - In EECE233, DTFT.
         - Existence and l^1(Z) sequences, i.e., absolutely summable sequences on Z
       - Def. continuous-parameter to discrete-parameter inverse Fourier Transformation
         - In EECE233, inverse DTFT.
         - Existence and L^1([0,2₩pi]) functions, i.e., absolutely integrable functions on [0,2₩pi]
       - Def. Fourier-Plancherel FT for l^2(Z) sequences

○ Properties of DTFT

Still More Properties

(MR) $x[n] = ce^{j\omega_0 n}$

5) Time Reversal $x[-n] \longleftrightarrow X(e^{-j\omega})$

$x[-n] = ce^{j\omega_0(-n)} = ce^{j(-\omega_0)n}$

$y[n] = x[-n]$

$Y(e^{j\omega}) = \sum y[n]e^{-j\omega n}$

$= \sum x[-n]e^{-j\omega n}$

$= \sum x[n]e^{j\omega n}$

$= \sum x[n]e^{-j(-\omega)n}$

$= X(e^{-j\omega})$

6) Conjugate Symmetry

Both are the same as CTFT.

$x[n]$ real $\Rightarrow X(e^{-j\omega}) = X^*(e^{j\omega})$ conjugate symmetrized

$\mathcal{F}$ $X(e^{j\omega})$

$\Downarrow$

$|X(e^{j\omega})|$ and $\Re e\{X(e^{j\omega})\}$ are even functions

$\angle X(e^{j\omega})$ and $\Im m\{X(e^{j\omega})\}$ are odd functions

and

$x[n]$ real and even $\Leftrightarrow X(e^{j\omega})$ real and even

$x[n]$ real and odd $\Leftrightarrow X(e^{j\omega})$ purely imaginary and odd

$x[n] = x[-n], \forall n \Rightarrow X(e^{j\omega}) = X(e^{-j\omega})$

$x^*[n] = x[n], \forall n \Rightarrow X^*(e^{j\omega}) = X(e^{-j\omega})$

$X(e^{j\omega})$ real

$X(e^{-j\omega}) = X(e^{j\omega})$

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Is There No End to These Properties?

7) Differentiation in Frequency

$$\frac{d}{d\omega}X(e^{j\omega}) = \frac{d}{d\omega}\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \rightarrow \sum_{n=-\infty}^{\infty}\frac{d}{d\omega}\left(x[n]e^{-j\omega n}\right)$$

$$\frac{d}{d\omega}X(e^{j\omega}) = -j\sum_{n=-\infty}^{+\infty} nx[n]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x[n](-jn)e^{-j\omega n} = (-j)\sum_{n=-\infty}^{\infty}(nx[n])e^{-j\omega n}$$

$\Downarrow$ $\mathcal{F}$

$nx[n] \longleftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$ Differentiation

$\left(\frac{1}{-j}\right)\frac{d}{d\omega}X(e^{j\omega}) \overset{\mathcal{F}^{-1}}{\longrightarrow} nx[n]$

$$nx[n] \xleftrightarrow{\mathcal{F}} j\frac{d}{d\omega}X(e^{j\omega})$$ Differentiation in frequency

$$\left(\frac{1}{-j}\right)\frac{d}{d\omega}X(j\omega) \xleftrightarrow{\mathcal{F}} nx[n]$$

8) Parseval's Relation

$$E_{xx} = \sum_{n=-\infty}^{+\infty}|x[n]|^2 = \frac{1}{2\pi}\int_{2\pi}\left|X(e^{j\omega})\right|^2 d\omega$$

Total energy in time domain — Total energy in frequency domain

Def. CT에서와 마찬가지로 energy spectral density라 불린다.

$$\text{cf) } E_{xx} = \int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2 d\omega.$$

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$$RHS = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(e^{j\omega})X^*(e^{j\omega})\,d\omega$$

$$X(e^{j\omega}) = \sum x[n]e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(e^{j\omega})\sum_n x^*[n]e^{j\omega n}\,d\omega$$

$$= \sum_n x^*[n]\boxed{\frac{1}{2\pi}\int_{\langle 2\pi\rangle}X(e^{j\omega})e^{j\omega n}d\omega}$$

$x[n]$

$$= \sum_n |x[n]|^2 = E_{xx}.$$

The Convolution Property

$$x[n] \xrightarrow{e^{j\omega n}} \boxed{h[n]} \to y[n] = h[n] * x[n]$$

$H(e^{j\omega})e^{j\omega n}$

$$\boxed{Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})}$$

$$x[n] = \frac{1}{2\pi}\int_{\langle 2\pi\rangle}X(e^{j\omega})e^{j\omega n}d\omega \to H(e^{j\omega})e^{j\omega n}$$

Frequency response $H(e^{j\omega})$ = DTFT of the unit sample response

Example #1: $x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = 2\pi\sum_{k=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi k)$

$$Y(e^{j\omega}) = H(e^{j\omega})\cdot 2\pi\sum_{k=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi k)$$

$$= 2\pi\sum_{k=-\infty}^{+\infty}H\left(e^{j(\omega_0+2\pi k)}\right)\delta(\omega-\omega_0-2\pi k)$$

$$\overset{H\text{ periodic}}{=} H(e^{j\omega_0})\cdot 2\pi\sum_{k=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi k)$$

$$\Downarrow$$

$$\boxed{y[n] = H(e^{j\omega_0})e^{j\omega_0 n}}$$

$$y[n] = \frac{1}{2\pi}\int_{\langle 2\pi\rangle}\boxed{H(e^{j\omega})X(e^{j\omega})}e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{\langle 2\pi\rangle}\boxed{Y(e^{j\omega})}e^{j\omega n}d\omega$$

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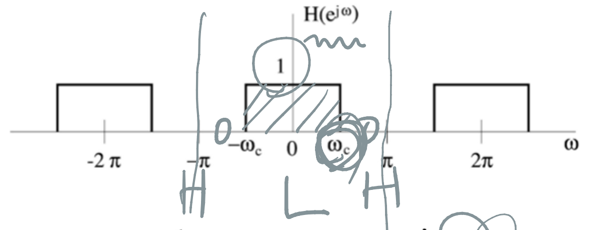
Now, we have a crude tool to design a digital filter.
1. Find the frequency response of the desired filter.
2. Apply Inverse DTFT to find h[n]. Done!!!
MATLAB에서는 h[n]을 저장하여 convolution하는 것으로 구현할 수 있는데, 이렇게 software적으로 하면 느리다. 빠른 system을 구현하려면 hardware적으로 구현해야한다. 어떻게?????

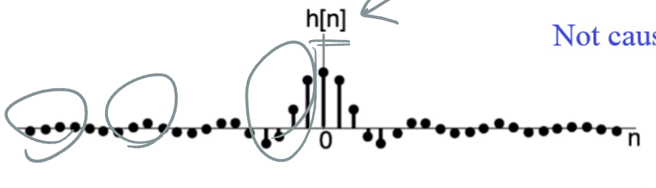
Reminder: Ideal Lowpass Filter

hardware적으로 구현해야한다. 어떻게?????

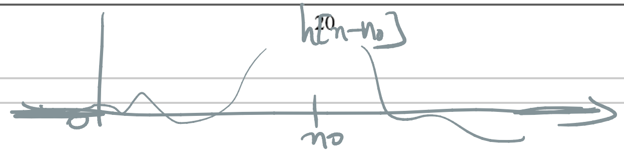
Reminder: Ideal Lowpass Filter



$$h[n]=\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\,d\omega=\frac{\sin\omega_c n}{\pi n}$$

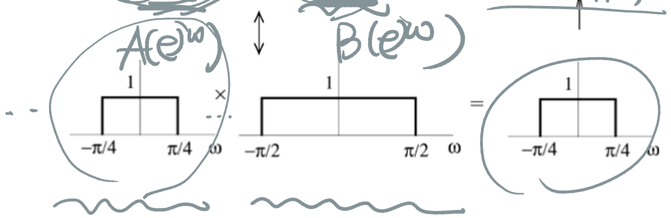


Not causal!

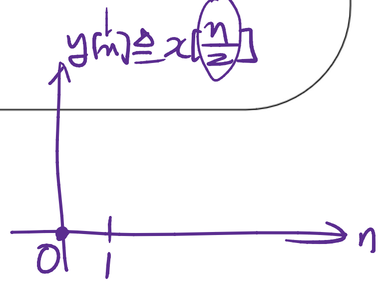


Example #3: $a[n]*b[n]=\sum_{l=-\infty}^{\infty}a[l]\,b[n-l] \overset{\mathcal{F}}{\longleftrightarrow} A(e^{j\omega})B(e^{j\omega})$

$$\frac{\sin(\pi n/4)}{\pi n}*\frac{\sin(\pi n/2)}{\pi n}=?\qquad \frac{\sin(\pi n/4)}{\pi n}$$



$y[n]$은 $x[\frac{n}{2}]$



Time Expansion

compression / dilation

Time Expansion
Recall CT property: $x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$

$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

Time scale is infinitesimally fine

a>1이면 빠르게 play…

But in DT: $x[n/2]$ makes no sense for odd n

$y[n] = x[2n]$ misses odd values of $x[n]$

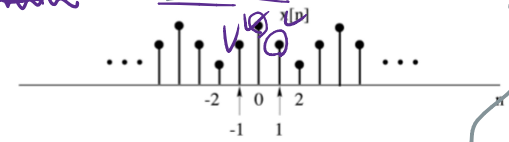
이 차이를 주목하자.

Def. DT downsampling이라 한다.

But we can "slow" a DT signal down by inserting zeros:

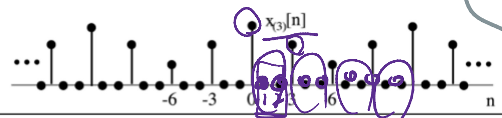
k — an integer ≥ 1

$x_{(k)}[n]$ — insert $(k - 1)$ zeros between successive values



time-domain상에서 느려지기는 하는데 frequency-domain상에서는 어떻게 될까? CT에서와 어떻게 다를까?

Insert two zeros in this example



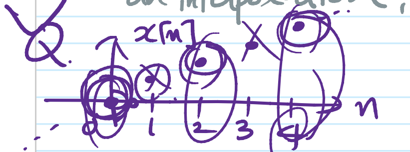
DT upsampling이라 부른다.

cf) $x[\lfloor \frac{n}{M} \rfloor]$ an interpolation!

$M \in \mathbb{N}$

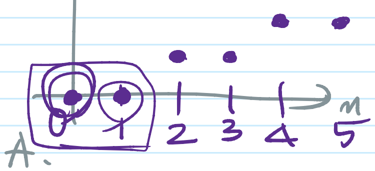
DT sampling ⟵ down sampling

Q.



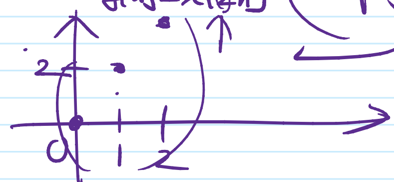
$y[n] = x[\lfloor \frac{n}{2} \rfloor]$?

A.



$y[n] = x[2n]$

$y[n] = x[3n]$



$x(n) \longleftrightarrow X(e^{j\omega})$

① $N\times$ downsampling $y[n] = x[Nn]$, $Y(e^{j\omega})$

② $N\times$ upsampling $z[n] = x_{(N)}[n]$, $Z(e^{j\omega})$

$k=3$ $n = 0, 3, 6, 9$
x 1 2 3 4

$x[\frac{n}{k}]$ $x_{(k)}[n]$

Time Expansion (continued)

$$z[n] = x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k \\ 0 & \text{otherwise} \end{cases}$$

— Stretched by a factor of k in time domain

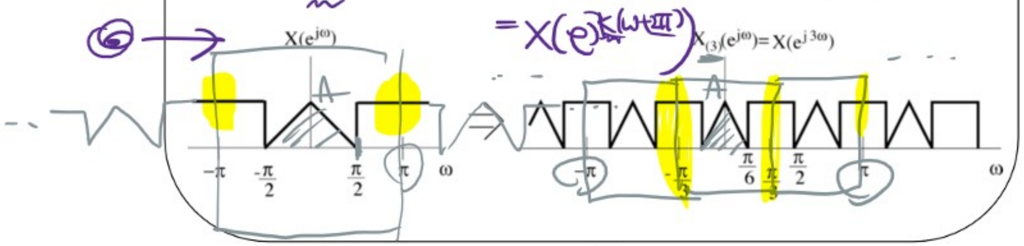
$n = mk$

$x_{(k)}[n] = \begin{cases} 0 & \text{otherwise} \end{cases}$ of k in time domain

$n = mk$

$$Z(e^{j\omega}) = X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n] e^{-j\omega n} \overset{n=mk}{=} \sum_{m=-\infty}^{+\infty} x_{(k)}[mk] e^{-j\omega mk}$$

$$= \sum_{m=-\infty}^{+\infty} x[m] e^{-j(k\omega)m} = X(e^{jk\omega})$$ — compressed by a factor of k in frequency domain



zero insertion을 하면 느려는 지는데 그 외의 효과는 없나?
MATLAB으로 소리를 들어보자.

Def. DT interpolation = DT upsampling followed by LP filtering