

LN13 Bilaeral Laplace Transform

2023년 11월 17일 금요일 15:35

CM24 231123Th on Bilaeral Laplace Transform

Announcements

1. Messages from Instructor

- Goal of Life: Self-Fulfillment and Contribution to Collective (자아실현, 공동체 기여)
 - Attitude: Exceed the expectations! (기대를 넘어서라!)
- "아는 것이 힘이다. 배우고야 무슨 일이든 한다." -심훈, 상록수-
- 학습 (배우고 익히기)
 - Theory Learning (이론 배우기)
 - Problem Solving (문제 풀기)
 - 배우기와 익히기의 균형
- How to Learn
 - Stop and Think; 복기가 chess master 를 만든다; Find commonality, similarity, and difference
- How to Practice

Purposeful practice with 3F: Time, Focused Action (무엇을 향상시키려는 훈련인가?), Experience, Lesson, Immediate Feedback (코치의 중요성), Fix, Growth
- How to solve? How to understand? ((S))implify and ((V))isualize:
 - essence, simplify; generalize; consider extremes; mental representation, representational examples;
 - Venn diagram, necessity and sufficiency, existence and uniqueness/plurality
 - An imperfect classification rule is better than no classification rule as long as you recognize the imperfection and update your rule.
 - block diagram, $G(V,E)$; graph of a function, algebraic vs. geometric interpretations, mathematical vs. physical/engineering interpretations/implications

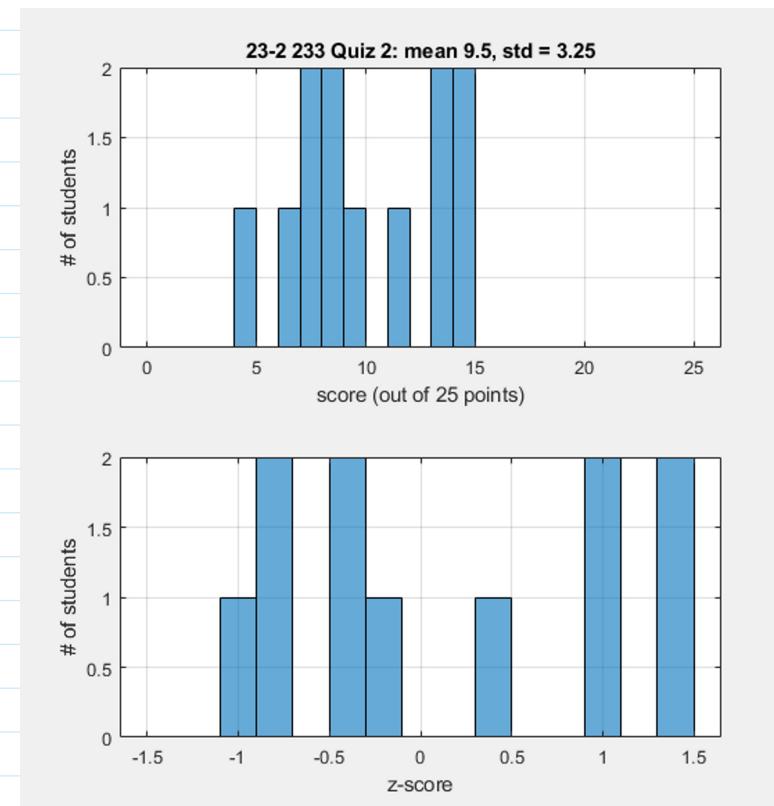
2. 지난 강의 수정/보완 사항

3. HW, Quiz, and Exam

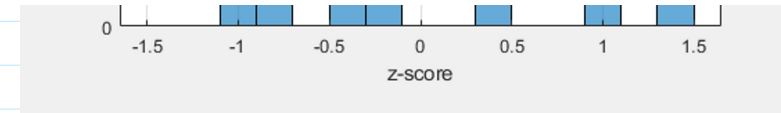
- Quiz 2
- **There may be a supplementary CM. No attendance required.** The recorded lecture will be uploaded as CM21.
- HW12 will be assigned by this Friday. Due will be next Tuesday.

4. 강의 자료

- Handouts and annotated pdfs are downloadable from the non-PLMS class homepage at <http://cisl.postech.ac.kr/class/eece233/schedule.htm>.
- Recorded CMs are available from cisl.postech YouTube channel. Please place **clickable time stamps**.



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5. 출석

- 출결상태에 대해 전자출결시스템(<https://wpsec.postech.ac.kr/eams/login>)을 확인하세요. 정정사유가 있는 학생은 증빙을 제출해 주세요.

6. Overview (The numbers are the chapter numbers of the textbook)

- Signals and Systems
- Linear Time-Invariant Systems
- Fourier Series Representation of Periodic Signals
- Continuous-time Fourier Transform
- Discrete-time Fourier Transform
- Time and Frequency Characterization of Signals and Systems
- 9. Laplace Transform**
- 10. Z-transform**
- Sampling
- Communication Systems

7. Related Courses offered in 24-1

- EECE302 전자수학 A (확률 및 랜덤프로세스); 월수 2시; 신호및시스템
- EECE361/DISU361 전자파응용; 화목 11시; 전자기학 개론
- EECE441 디지털통신개론; 월수 11시; 신호및 시스템, 전자수학
- EECE442/NGCN301 통신및네트워크개론; 화목 2시
- EECE443/NGCN302 통신및네트워크실험; 월5시-10시
- EECE451 디지털신호처리개론; 화목 3시반; 신호및시스템

8. Review

- Miscellaneous
 - LTI system
 - magnitude and phase responses
 - vs. log-magnitude and phase responses
 - B, dB
 - Bode plot
 - How to read a log-log plot
 - Linear-phase system vs. Nonlinear-phase system
 - Def. A CT/DT LTI system is a linear phase system if Otherwise, the system is of nonlinear phase
 - Caution. In DT LTI, arbitrary linear phase is NOT allowed.

Handwritten notes and diagrams:

- A diagram showing a sphere with a grid, labeled "log₁₀ energy" and "power reference power".
- The equation $H(j\omega) = |H(j\omega)| e^{j\angle H(j\omega)}$ is written.
- A block diagram showing a sequence of four blocks: $H_1 \rightarrow H_2 \rightarrow H_3 \rightarrow H_4$.
- The text "Bode plot" is written below the block diagram.

무엇이든 물어보세요!

차세대 통신 및 네트워크 융합부전공

삼성전자 산학제도

상담부스 이벤트

11.27(월) ~ 12.1(금)

참여대상 학부생이면 (전공무관) 누구나 참여가능
설문지만 작성하면 **뽑기 이벤트** 참여 기회!

장소 학생회관 1층 **운영시간** 10:30~16:00

문의사항 LG연구동 416호 | 054-279-5115

홈페이지 www.ngcn.postech.ac.kr

갤럭시워치 6	1명	₩ 45,000	5명	₩ 25,000	10명	₩ 10,000	25명
갤럭시워치 5	1명	80명	과자				

복지회 상품권(GS25, 버거킹, 커피 등 사용가능)

1000 = 1000 mm

- Linear-phase system vs. Nonlinear-phase system

- Def. A CT/DT LTI system is a linear phase system if Otherwise, the system is of nonlinear phase
- Caution. In DT LTI, arbitrary linear phase is NOT allowed.

- All-pass system

- Def. A CT/DT LTI system is an all-pass system if ...

- Def. Given a nonlinear-phase system, the group delay at ω_0 is defined as ...

- Ideal vs. non-ideal LPFs

- Q. How to choose coefficients to design a desirable frequency response?

- A1. CT/DT rational frequency responses with real coefficients

- 1st-order system

- 2nd-order system

- A2. Laplace and Z-TFs.

Doae p. 1

$$|m = 1000 \text{ mm}$$

$$100 \text{ cm}$$

$$10 \text{ dm}$$

$$1B = 10 \text{ dB}$$

$$10 \log_{10} \text{ (circled)}$$

9. Preview

- Limitation of FT

- control of an unstable system
- Recall BIBO stable iff ...
- In the world of functions not generalized functions, Fourier transform exists iff L^1 .
 - FT does not work with an unstable system.
 - Q. Why did we consider complex sinusoidal signal?
 - A. e^{st} can be an eigenfunction of a CT LTI system.
 - Can we go back to the complex exponential function not limited to the imaginary axis?

- Laplace transforms

- bilateral vs. unilateral

- Bilateral LT

- Def. Two functions are the same if ...
 - Importance of the domain of a mapping rule
- Def. of bilateral LT and its ROC
- Examples.
- Properties of ROC of LT

EECE233 Signals and Systems

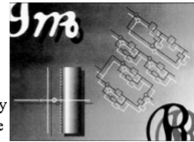
Lecture Note #13

Prof. Joon Ho Cho

(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

- Covers O & W pp. 654-670
- Motivation and Definition of the Bilateral Laplace Transform
- Examples of Laplace Transforms and Their Regions of Convergence (ROC)
- Properties of ROC's

* These lecture notes are based on the lecture notes used in 6.003 taught by Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.



지금까지 우리는 time domain상에서의 design과 analysis, 그리고 frequency domain상에서의 design과 analysis에 대해 배웠다.

그러나, 원하는 시스템을 제대로 디자인 하기 위해서는 아직 부족함을 느낀다.

지금부터는 LCCDE로 CT LTI 시스템을 디자인하는데 필요한 좀 더 강력한 transform을 배운다.

이 LN13에서는 signal에 대해 주로 다루고 다음 LN14의 9페이지부터는 system에 대해 다룬다.

1

Laplace transform(LT)이 유용한 이유는 Fourier transform처럼 convolution property가 있기 때문이다. $Y(s) = H(s)X(s)$
그러나, 일반적인 신호의 inverse transform은 쉽게 구하지 못한다.
다행히

rational function of s 로 LT이 나오는 경우는 쉽게 inverse를 구할 수 있는데 마침 LCCDE로 I/O관계가 나타나는 시스템은 s 의 rational function으로 impulse response의 LT이 나타난다.

Motivation for the Laplace Transform

- CT Fourier transform enables us to do a lot of things, e.g.
 - Solving LCCDE's
 - Analyzing frequency response of LTI systems
 - Sampling
 - Modulation
 - ⋮
- Why do we need yet another transform?
- Fourier transform **cannot** handle **unstable** signals/systems, i.e.

$$\int_{-\infty}^{\infty} |x(t)| dt = \infty$$

Because the eigenfunction $e^{j\omega t}$ in Fourier transform has unit amplitude $|e^{j\omega t}| = 1$. For example:

Thm. A CT LTI system is BIBO stable

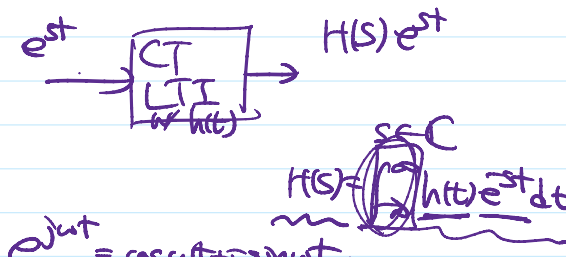
$$\text{iff } \int_{-\infty}^{\infty} |h(t)| dt < \infty$$

Lemma. A CT signal is Fourier transformable

$$\text{iff } \int_{-\infty}^{\infty} |x(t)| e^{j\omega t} dt < \infty$$

Some real-world systems are better modeled by unstable systems.

We may want to design a proper electrical system that will stabilize the overall system when

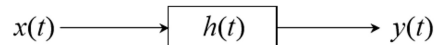


Because the eigenfunction $e^{j\omega t}$ in Fourier transform has unit amplitude $|e^{j\omega t}| = 1$. For example:

2

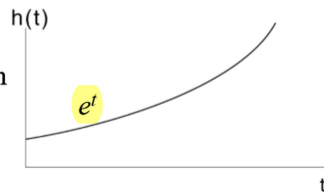
a proper physical system that will stabilize the overall system when connected to the unstable system.

Motivation for the Laplace Transform (continued)



And $h(t) = e^t \cdot u(t)$
— an unstable causal system

$$H(j\omega) = ?$$



Consequently, we cannot analyze this system using the easy method of

$$Y(j\omega) = H(j\omega)X(j\omega).$$

Do we have to go back to $y(t) = h(t) * x(t)$?

3

Remind that a CT LTI system is BIBO stable iff $h(t)$ is absolutely integrable.

예를 들어 right or left-sided exponential function $e^{-at}u(t)$ 또는 $e^{at}u(-t)$ 의 경우 a 가 음수이면 Fourier transform은 존재하지 않는다.

Motivation for the Laplace Transform (continued)

- In many applications, we do need to deal with **unstable** systems, e.g.
 - Stabilizing an inverted pendulum
 - Stabilizing an airplane or space shuttle
 - ⋮
 - Instability is **desired** in some applications, e.g. oscillators and lasers
- How do we analyze unstable signals/systems?
Recall from LN #5, eigenfunction property of LTI systems:

Laplace transform이 중요한 이유는, Fourier transform으로 다룰 수 없는 신호도 다룰 수 있다는 것과

Fourier transform이 존재하는 신호의 경우, Laplace transform에서 Fourier transform을 구할 수 있다는 점이다.

따라서 Fourier transform이 존재하는 신호만 놓고 보더라도 Laplace transform은 보다 높은 차원에서 신호를 이해할 수 있게 해 준다.

2차원적인 지구표면에 붙어서 태양계를 논하는 것이 Fourier transform이라면 3차원 우주공간에서 태양계를 논하는 것이 Laplace transform이라 비유할 수 있겠다.

즉, 주어진 physical system이 CT LTI system으로 modeling이 되는 BIBO unstable system인데

여기서 보조 시스템을 달아서 overall stable한 CT LTI system을 만들고 싶은 경우가 많다.

이를 위해서는 먼저, time-domain외에서도

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

$$z^n \rightarrow \boxed{\text{DT LTI}} \rightarrow H(z) z^n$$

$$z \in \mathbb{C}$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

- How do we analyze unstable signals/systems?
Recall from LN #5, eigenfunction property of LTI systems:

$$e^{st} \xrightarrow{\quad} \boxed{h(t)} \xrightarrow{\quad} H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st} dt \quad (\text{assuming this converges})$$

- e^{st} is an eigenfunction of *any* LTI systems,
- $s = \sigma + j\omega$ can be complex in general

4

이를 위해서는 먼저, time-domain외에서도 unstable CT LTI system을 다룰 수 있는 tool이 있으면 좋겠고 물론 이 새로운 tool은 BIBO stable한 CT LTI도 함께 다룰 수 있으면 좋겠다.

The (Bilateral) Laplace Transform

(Can be viewed as an extension of the Fourier Transform)

$$x(t) \xleftrightarrow{\quad} X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st} dt = \mathcal{L}\{x(t)\}$$

$s = \sigma + j\omega$ is a complex variable — Now we explore the full range of s
First comments:

$$(1) X(s) = X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t} dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

(2) A critical issue in dealing with Laplace transform is convergence:

- $X(s)$ generally exists only for *some* values of s , located in what is called **the region of convergence (ROC)**

$$\text{ROC} \in \{s = \sigma + j\omega \text{ so that } \int_{-\infty}^{+\infty} |x(t)e^{-\sigma t}| dt < \infty\}$$

Depends only on σ
not on ω

(3) If $s = j\omega$ is in the ROC (i.e. $\sigma = 0$), then

$$\mathcal{L}\{x(t)\} \Big|_{s=j\omega} = X(s) \Big|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

반면, 적분 구간이 non-negative t 만인 Laplace transform은 unilateral LT라 부른다.

Fourier transform과 마찬가지로 Laplace transform도 t 의 함수를 다른 변수의 함수로 바꾸는 것이다. 즉, 함수를 함수로 바꾸는 변환이다.

그런데, 이 새로운 함수의 정의역이 있을 것이다. 이것이 ROC이다.

Q. Laplace transform이 general한가 Fourier transform이 general한가?

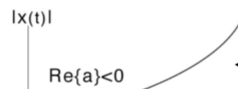
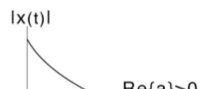
A. (1)에 따르면, Laplace transform of $x(t)$ 는 $x(t)$ times $e^{-\sigma t}$ 의 Fourier transform으로 볼 수 있다. 이런 점에서는 Fourier transform이 Laplace transform보다 일반적이다.

그러나, 많은 경우... 그 반대로 이야기 한다. (3)을 보라.. 즉, $X(s)$ 를 $x(t)$ 의 Laplace transform으로 정의하고 $s = j\omega$ 에서 이를 evaluate한 것을 Fourier transform으로 볼 수 있다. 그런 점에서 Laplace transform이 Fourier transform보다 일반적이다.

Example #1:

$$x_1(t) = e^{-at}u(t)$$

(a — an arbitrary real or complex number)



Unstable, no FT

LT는 s 의 함수가 되는데, 어떤 s 값에서는 적분이 수렴하고 즉 LT가 정의되고 어떤 s 값에서는 적분이 정의되지 않거나 발산하여 LT가 정의되지 않을 수 있다

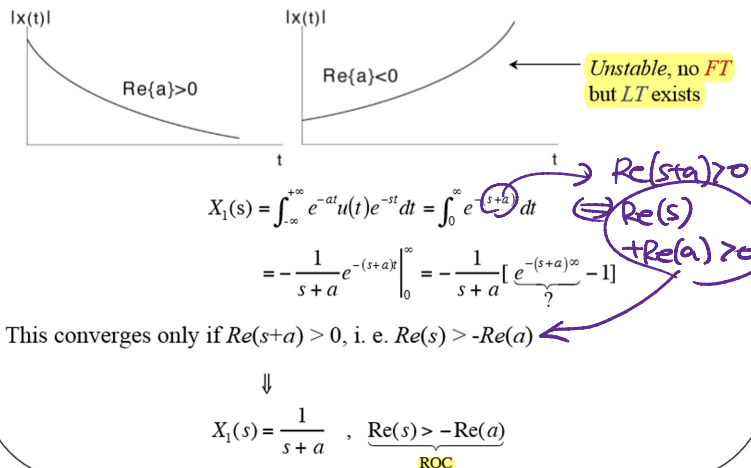
Q. FT. LT general

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st} dt$$

$$s \in \mathbb{C}$$

$$s = j\omega$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t} dt$$



어떤 s값에서는 적분이 정의되지 않거나 발산하여 LT가 정의되지 않을 수 있다.

ROC가 중요한 이유는, 서로 다른 두 CT signal $x_1(t)$ 와 $x_2(t)$ 가 서로 다른 ROC를 가지면서도 같은 모양의 s의 함수 $X_1(s)$, $X_2(s)$ 를 가질 수 있기 때문이다.

함수 $X(s)$ 가 정의되는 domain을 specify하는 것은 대단히 중요하다.

See Example #1 and Example #2

Example #2:

$$x_2(t) = -e^{-at} u(-t)$$

$$\begin{aligned}
 X_2(s) &= -\int_{-\infty}^{+\infty} e^{-at} u(-t) e^{-st} dt \\
 &= -\int_{-\infty}^0 e^{-(s+a)t} dt \quad \text{Handwritten: } \text{Re}(s+a) < 0 \\
 &= \frac{1}{s+a} e^{-(s+a)t} \Big|_{-\infty}^0 = \frac{1}{s+a} [1 - e^{-(s+a)\infty}]
 \end{aligned}$$

This converges only if $\text{Re}(s+a) < 0$, i. e. $\text{Re}(s) < -\text{Re}(a)$

$$\Downarrow \\
 X_2(s) = \frac{1}{s+a}, \quad \text{Re}(s) < -\text{Re}(a) \quad \text{Same as } X_1(s), \text{ but a different ROC}$$

Key Point (and a key difference with FT): Need both $X(s)$ and ROC to uniquely determine $x(t)$. No such an issue of ROC for FT.

예를 들어
 $f(x) = x$, for $x > 0$
 not defined, elsewhere
 와
 $g(x) = x$, for $x < 0$
 not defined elsewhere
 는 명백히 다른 함수이다.

마찬가지로
 $X_1(s) = \frac{1}{s+a}$, for $\text{Re}(s+a) > 0$
 not defined elsewhere
 와
 $X_2(s) = \frac{1}{s+a}$, for $\text{Re}(s+a) < 0$
 not defined elsewhere
 는 s의 함수로서 정의역이 다른, 다른 함수들이다.

a=1

Graphical Visualization of the ROC

Example #1

Example #2

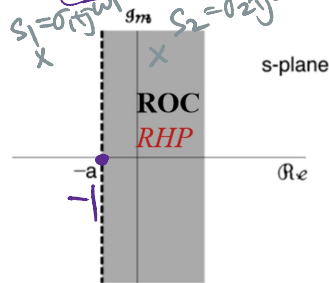
$x_1(t) = e^{-at} u(t)$ — right-sided signal, $x_2(t) = -e^{-at} u(-t)$ — left-sided signal

Example #1

$x_1(t) = e^{at}u(t)$ — right-sided signal, $x_2(t) = -e^{at}u(-t)$ — left-sided signal

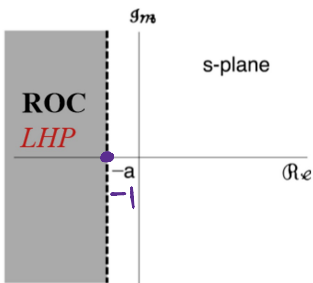
$$X_1(s) = \frac{1}{s+a}, \quad \text{Re}(s) > -\text{Re}(a)$$

$s_1 = \sigma_1 + j\omega_1$
 $s_2 = \sigma_2 + j\omega_2$



Example #2

$$X_2(s) = \frac{1}{s+a}, \quad \text{Re}(s) < -\text{Re}(a)$$



$$\sigma = -1$$

$$\sigma + j\omega = 0$$

$$\sigma = -1$$

8

암기: right-sided exponential
has right-sided ROC

암기: left-sided exponential
has left-sided ROC

Rational Transforms

- Many (but by no means all) Laplace transforms of interest to us are rational functions of s (e.g. Examples #1 & #2, in general, LTIs described by LCCDEs), i. e.

$$X(s) = \frac{N(s)}{D(s)}, \quad N(s), D(s) \text{ — polynomials in } s$$

- Roots of $N(s)$ = zeros of $X(s)$
- Roots of $D(s)$ = poles of $X(s)$
- Any $x(t)$ consisting of a linear combination of complex exponentials for $t > 0$ and for $t < 0$ (i. e. as in Example #1 and #2) has a rational Laplace transform.

사실, LTI를 LCCDE로 나타내어 RLC등을 써서 디자인 하기 위해 Laplace transform을 배운다.

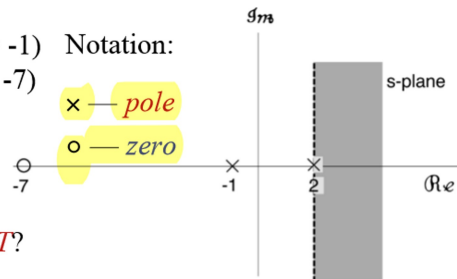
앞서 LCCDE의 Fourier transform으로부터 $j\omega$ 의 rational function으로써의 transfer function을 구해 보았다. 간단히 생각하면 $j\omega$ 대신에 s 로 쓴 것으로 볼 수 있다.

9

Example #3 $x(t) = 3e^{2t}u(t) - 2e^{-t}u(t)$

$$\begin{aligned} X(s) &= \int_0^{\infty} [3e^{2t} - 2e^{-t}]e^{-st} dt \\ &= 3 \underbrace{\int_0^{\infty} e^{-(s-2)t} dt}_{\text{Requires } \operatorname{Re}(s) > 2} - 2 \underbrace{\int_0^{\infty} e^{-(s+1)t} dt}_{\text{Requires } \operatorname{Re}(s) > -1} \\ &= \frac{3}{s-2} - \frac{2}{s+1} = \frac{s+7}{(s-2)(s+1)} \quad \operatorname{Re}(s) > 2 \end{aligned}$$

$\Rightarrow$ two poles ($p_1=2, p_2=-1$) & one zero ($z_1=-7$) Notation:

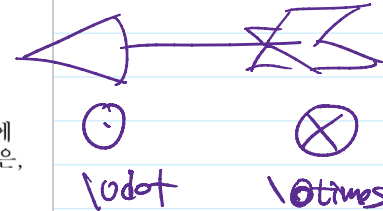


Q: Does $x(t)$ have FT?

$X(s)$ 의 마지막 꼴에서
또는 pole-zero plot에
서 쉽게 알 수 있는 것은,

right-sided인지 left-
sided인지 모르겠으나
 e^{2t}
그리고
 e^{-t}
성분이 있다는 것이다.

"s의 pole times t가
exponent가 된다."

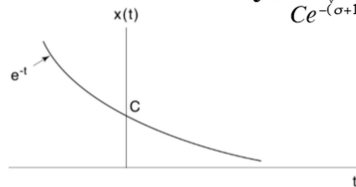


"FT가 존재한다" = " ω 의 함수로서의 FT가 모든 ω 에 대해 존재한다"
이므로 이 질문은
다른 말로, ROC 가 $\sigma = j\omega$ 축을 포함하느냐 하지 않느냐라는 말이다.

Laplace Transforms and ROC's

- First of all, some signals do not have Laplace transforms

(a) $x(t) = Ce^{-t}$ for all t since $\int_{-\infty}^{\infty} \underbrace{|x(t)e^{-\sigma t}|}_{Ce^{-(\sigma+1)t}} dt = \infty$ for all σ



(b) $x(t) = e^{j\omega_e t}$ for all t

정확히 말해,
"많은 신호들의
Laplace transform의
ROC가 공집합이 아니지만,
어떤 신호들은
ROC가 공집합이다."

Laplace transform은
CT signal중
right 또는 left -sided
exponential function을 다
루기 편하고
CT system중 LCCDE로
나타난 system을 다루기
편하다.

(b) $x(t) = e^{j\omega_s t}$ for all t

$$\int_{-\infty}^{+\infty} |x(t)e^{-\sigma t}| dt = \int_{-\infty}^{+\infty} e^{-\sigma t} dt = \infty \quad \text{for all } \sigma$$

$X(s)$ is defined only in **ROC**, by definition,
 $X(s) \neq \infty \Rightarrow$ No $\delta(s)$ is allowed, different from **FT**

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통신에서는 complex sinusoidal signals가 중요하다. FT이 보다 더 자주 쓰인다.

CT system중 LCCDE로 나타난 system을 다루기 편하다.

key restriction in FT를 LT에서는 $X(s)$ 로 generalized function을 허용하지 않는다!!!!

Properties of the ROC

Secondly, the ROC can take on only a small number of different forms

- 1) The ROC consists of a collection of lines parallel to the $j\omega$ -axis in the s -plane (i.e. the ROC only depends on σ). Why?

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt = \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty \quad \text{depends only on } \sigma = \text{Re}(s)$$

- 2) If $X(s)$ is rational, then the ROC does not contain any poles. Why?

Poles are places where $D(s) = 0$

$$\Rightarrow X(s) = \frac{N(s)}{D(s)} = \infty \quad \text{Not convergent.}$$

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A. Lebesgue integral에서

improper integral

$$\int_{-\infty}^{\infty} x(t) dt \text{ 는}$$

$$\int_{-\infty}^{\infty} [x(t)]^+ dt$$

와

$$\int_{-\infty}^{\infty} [x(t)]^- dt$$

가 각각 수렴할 때,

$$\int_{-\infty}^{\infty} [x(t)]^+ dt$$

—

$$\int_{-\infty}^{\infty} [x(t)]^- dt$$

로 정의된다. 즉,

$x(t)$ 가 absolutely integrable할 때 정의된다.

More Properties

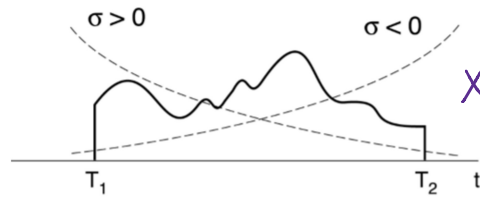
- 3) If $x(t)$ is of finite support and is absolutely integrable, then the ROC is the entire s -plane.

$$\sigma > 0$$

$$\sigma < 0$$

$$s = \sigma + j\omega$$

then the ROC is the **entire** s-plane.



$$s = \sigma + j\omega$$

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

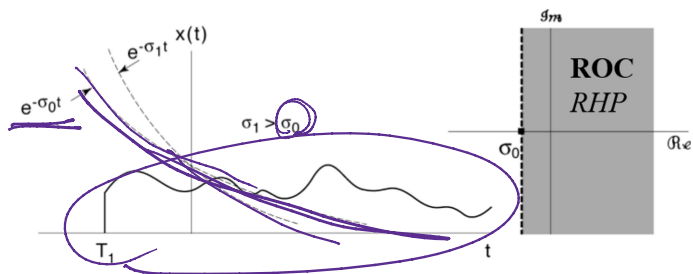
$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \underbrace{\int_{T_1}^{T_2} x(t) e^{-st} dt}_{\text{A finite integration}}$$

$$< \infty \text{ if } \int_{T_1}^{T_2} |x(t)| dt < \infty$$

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ROC Properties that Depend on which Side you are On

- 4) If $x(t)$ is **right-sided** (i.e. if it is zero *before* some time), and if $\text{Re}(s) = \sigma_0$ is in the ROC, then all values of s on the **right side** of the vertical line $\text{Re}(s) = \sigma_0$ are also in the ROC.

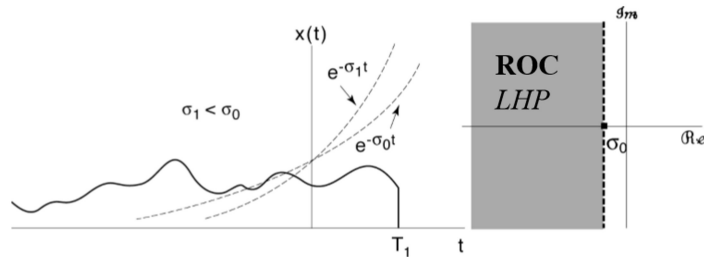


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더 빨리 감쇠시키
므로
적분이 존재할 수
밖에 없다.
comparison test!
의 적용.

ROC Properties that Depend on which Side you are On

- 5) If $x(t)$ is **left-sided** (i.e. if it is zero *after* some time), and if $\text{Re}(s) = \sigma_0$ is in the ROC, then all values of s on the **left side** of the vertical line $\text{Re}(s) = \sigma_0$ are also in the ROC.

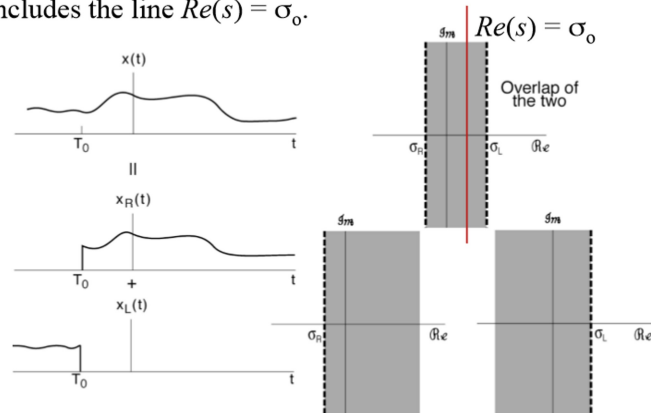


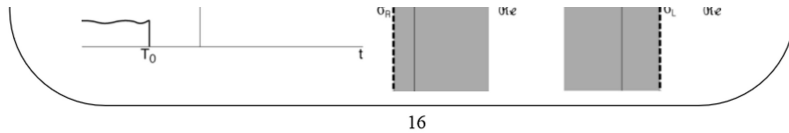
앞서와 마찬가지로
더 빨리 감쇠시키므로
적분이 존재할 수밖에
없다.
comparison test!의
적용.

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Still More ROC Properties

- 6) If $x(t)$ is **two-sided** and if the line $\text{Re}(s) = \sigma_0$ is in the ROC, then the ROC consists of a **strip** in the s -plane that includes the line $\text{Re}(s) = \sigma_0$.



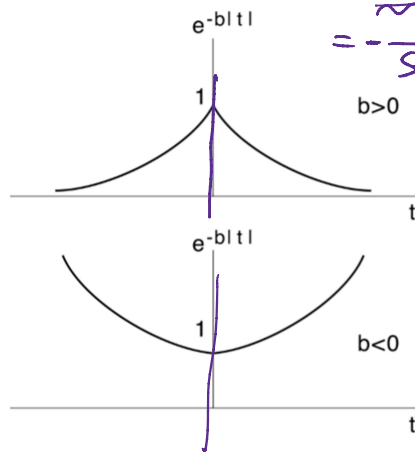


Example:

$$x(t) = e^{-b|t|} = \underbrace{e^{-bt} u(t)}_{Ri} + \underbrace{e^{bt} u(-t)}_{Le}$$

$$= -\frac{1}{s+b} + \frac{1}{s-b} = \frac{2b}{s^2 - b^2}$$

ROC?



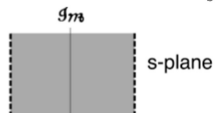
$$f(x) = \begin{cases} x, & \text{for no } x \in \mathbb{R} \end{cases}$$

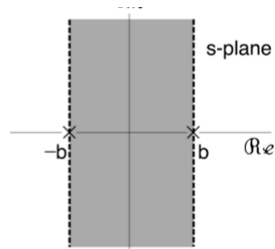
$f: \mathbb{R} \rightarrow \mathbb{R}$

Example (continued):

$$\begin{aligned} x(t) &= e^{bt} u(-t) + e^{-bt} u(t) \\ &\quad \uparrow \quad \quad \quad \uparrow \\ &= -\frac{1}{s-b}, \operatorname{Re}\{s\} < b \quad \quad \frac{1}{s+b}, \operatorname{Re}\{s\} > -b \end{aligned}$$

Overlap only if $b > 0 \Rightarrow X(s) = \frac{-2b}{s^2 - b^2}$, with ROC



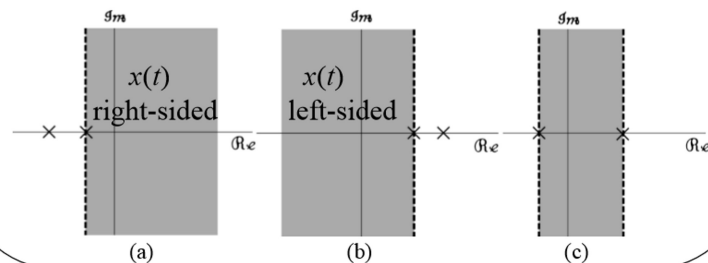


What if $b < 0$? $\Rightarrow$ No overlap $\Rightarrow$ No Laplace Transform

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Properties, Properties

- 7) If $X(s)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.
- 8) Suppose $X(s)$ is rational, then
 - (a) If $x(t)$ is right-sided, the ROC is to the right of the rightmost pole.
 - (b) If $x(t)$ is left-sided, the ROC is to the left of the leftmost pole.



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$X(s)$ 가 rational이면 PFE으로 $X(s)$ 를 $1/(s-a_i)$ 들의 linear combination으로 나타낼 수 있다. (중근일때는?? 분모가 더 높은 차수일때는?)
즉,
 $X(s)$ 가 rational일 때는 PFE으로 모든 property를 쉽게 설명할 수 있다.
분자의 차수가 크면 일단 분모로 나눠서 차수를 낮춘후에 PFE하면 된다. 이때 1, s, s^2 , .. 등은 각각 $u_0(t) = \delta(t)$, $u_1(t)$ unit doublet, ... 등으로 바뀌면 된다.

s가 unit doublet 임은 inverse LT식에서 쉽게 알수가 있다.

ROC 확인도 쉽다.(?!)

- 9) If ROC of $X(s)$ includes the $j\omega$ -axis, then FT of $x(t)$ exists.

Example:

$$X(s) = \frac{(s+3)}{(s+1)(s-2)}$$

Two poles $\Rightarrow$ three possible ROC's

$j\omega$

s-plane

분모의 차수가 높고, 중근이 없다. $\rightarrow$ 따라서 PFE으로 나타낼 수 있다.

사실은 ROC

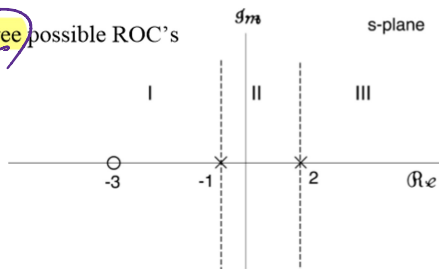
$$\begin{aligned} & \frac{(s+3)}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} \\ & \rightarrow A = -1, B = 2 \\ & \rightarrow -e^{-t}u(t) + 2e^{2t}u(t) \end{aligned}$$

Example.

$$A(s) = \frac{1}{(s+1)(s-2)}$$

분모의 차수가 높고, 중근이 없다. $\rightarrow$ 따라서 PFE로 나타낼 수 있다.

Two poles $\Rightarrow$ three possible ROC's



$x(t)$ is right-sided $\Rightarrow$ ROC: III
 $x(t)$ is left-sided $\Rightarrow$ ROC: I
 $x(t)$ has a Fourier transform $\Rightarrow$ ROC: II

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사실은 ROC가 공집합인 것까지 포함하여 four possible ROCs이지만 정의역이 공집합인 함수식이 의미가 있을까?

$f: \mathbb{R} \rightarrow \mathbb{R}$
 $f(x) = x$, for no $x \in \mathbb{R}$

Example (cont.)

ROC I: — Left-sided signal.

$$\begin{aligned}
 x(t) &= -Ae^{-t}u(-t) - Be^{2t}u(-t) \\
 &= \left[\frac{2}{3}e^{-t} - \frac{5}{3}e^{2t} \right] u(-t) \quad \text{Diverges as } t \rightarrow -\infty
 \end{aligned}$$

ROC II: — Two-sided signal, has Fourier Transform.

$$x(t) = -\left[\frac{2}{3}e^{-t}u(t) + \frac{5}{3}e^{2t}u(-t) \right] \rightarrow 0 \text{ as } t \rightarrow \pm \infty$$

ROC III: — Right-sided signal.

$$\begin{aligned}
 x(t) &= Ae^{-t}u(t) + Be^{2t}u(t) \\
 &= \left[-\frac{2}{3}e^{-t} + \frac{5}{3}e^{2t} \right] u(t) \quad \text{Diverges as } t \rightarrow +\infty
 \end{aligned}$$

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여기서 보는 바와 같이 ROC에 따라 3가지 (크게는 4가지) 다른 신호로 inverse transform된다.

암기 tip:

$1/(s-\text{pole})$ 은
 "e to the pole t u t"
 또는
 "-e to the pole t u -t"로
 inverse LT된다.