LN14 Inverse LT, LT Properties, System Function and LT, LT and Frequency Response

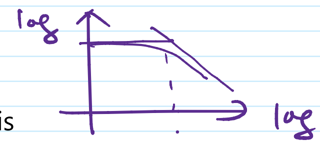
2023년 11월 17일 금요일 17:16

CM25 231128T on Inverse LT, LT Properties, System Function and LT, LT and Frequency Response

Announcements

1. Messages from Instructor
    - Goal of Life: Self-Fulfillment and Contribution to Collective (자아실현, 공동체 기여)
        - Attitude: Exceed the expectations! (기대를 넘어서라!)
    - "아는 것이 힘이다. 배우고야 무슨 일이든 한다." -심훈, 상록수-
    - 학습 (배우고 익히기)
        - Theory Learning (이론 배우기)
        - Problem Solving (문제 풀기)
        - 배우기와 익히기의 균형
    - How to Learn
        - Stop and Think; 복기가 chess master 를 만든다; Find commonality, similarity, and difference
    - How to Practice

      Purposeful practice with 3F: Time, Focused Action (무엇을 향상시키려는 훈련인가?; Push oneself little bit out of the COMFORTABLE ZONE), Experience, Lesson, Immediate Feedback (코치의 중요성), Fix, Growth
    - How to solve? How to understand? ((S))implify and ((V))isualize:
        - essence, simplify; generalize; consider extremes; mental representation, representational examples;
        - Venn diagram, necessity and sufficiency, existence and uniqueness/plurality
            - An imperfect classification rule is better than no classification rule as long as you recognize the imperfection and update your rule.
        - block diagram, G(V,E); graph of a function, algebraic vs. geometric interpretations, mathematical vs. physical/engineering interpretations/implications



log-mag $20\log_{10}|H(j\omega)|$

Q. $x_1(t) \xleftrightarrow{\mathcal{L}} X(s),\ s\in S$

$x_2(t) \xleftrightarrow{\mathcal{L}} X(s),\ s\in S$

$\Rightarrow x_1(t)=x_2(t),\ \forall t$?

2. 지난 강의 수정/보완 사항
    - Bode plots: asymptotic analysis
    - All pass system can be used to adjust phase response of a bandpass filter.

$y=x^n$

$\tilde{y}\ \boxed{\log_{10}y} = n\boxed{\log_{10}x}\ \tilde{x}$

3. HW, Quiz, Voluntary Project, and Exam
    - Final Exam is approaching. Unless basic mandatory courses have conflicting schedules, it will be taken 7 pm to 11pm on Thursday Dec. 21 at LG 102.
    - Voluntary Project
        - Topic must be approved before final exam. Topics related to this course material is acceptable.

A. ?

It can be proven that, if a function $F(s)$ has the inverse Laplace transform $f(t)$, then $f(t)$ is uniquely determined (considering functions which differ from each other only on a point set having Lebesgue measure zero as the same). This result was first proven by Mathias

11pm on Thursday Dec. 21 at LG 102.
- Voluntary Project
  - Topic must be approved before final exam. Topics related to this course material is acceptable.
  - Presentation file and presentation mp4 of length >20 minutes must be submitted.
  - No reuse, double use, etc.
- There may be a supplementary CM. No attendance required. The recorded lecture will be uploaded as CM21.
- HW13 will be assigned by this Friday. Due will be next Tuesday.

4. 강의 자료
   - Handouts and annotated pdfs are downloadable from the non-PLMS class homepage at http://cisl.postech.ac.kr/class/eece233/schedule.htm.
   - Recorded CMs are available from cisl.postech YouTube channel. Please place clickable time stamps.

5. 출석
   - 출결상태에 대해 전자출결시스템(https://wpsc.postech.ac.kr/eams/login)을 확인하세요. 정정사유가 있는 학생은 증빙를 제출해 주세요.

6. Overview (The numbers are the chapter numbers of the textbook)
   1. Signals and Systems
   2. Linear Time-Invariant Systems
   3. Fourier Series Representation of Periodic Signals
   4. Continuous-time Fourier Transform
   5. Discrete-time Fourier Transform
   6. Time and Frequency Characterization of Signals and Systems
   9. Laplace Transform
   10. Z-transform
   7. Sampling
   8. Communication Systems

7. Related Courses offered in 24-1
   - EECE302 전자수학 A (확률 및 랜덤프로세스); 월수 2시; 신호및시스템
   - EECE361/DISU361 전자파응용; 화목 11시; 전자기학 개론
   - EECE441 디지털통신개론; 월수 11시; 신호및 시스템, 전자수학
   - EECE442/NGCN301 통신및네트워크개론; 화목 2시
   - EECE443/NGCN302 통신및네트워크실험; 월5시-10시
   - EECE451 디지털신호처리개론; 화목 3시반; 신호및시스템

8. Review
- Limitation of FT

transform $f(t)$, then $f(t)$ is uniquely determined (considering functions which differ from each other only on a point set having Lebesgue measure zero as the same). This result was first proven by Mathias Lerch in 1903 and is known as Lerch's theorem.

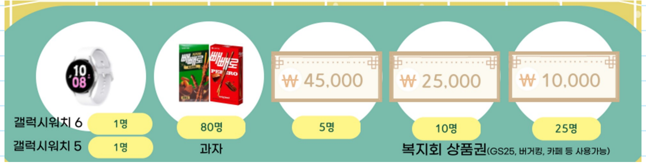
출처: <https://en.wikipedia.org/wiki/Inverse_Laplace_transform>

8. Review

- Limitation of FT
  - control of an unstable system
  - Recall BIBO stable iff …
  - In the world of functions not generalized functions, Fourier transform exists iff L^1.
    - FT does not work with an unstable system.
    - Q. Why did we consider complex sinusoidal signal?
    - A. e^st can be an eigenfunction of a CT LTI system.
    - Can we go back to the complex exponential function not limited to the imaginary axis?
- Laplace transforms
  - bilateral vs. unilateral
- Bilateral LT
  - Motivation. A complex exponential function e^(st) can be an eigenfunction of an LTI system if the system function has s in its ROC.
  - Def. The bilateral LT is a mapping with
    - input: a time function from R into C
    - output: a function of s from the ROC into C.
  - Def. of the ROC for the BLT of a time function.
    - ROC is the domain of X(s)
  - Def. Two functions are the same if …
    - Importance of the domain of a mapping rule
  - LT 의 mapping rule vs domain; rational mapping rule?; RLC circuit, amp, adder, constant scale,
  - e^s_0t -> H(s_0) e^s_0t
  - est input??? No generalized DT alllowed as output of BLT but allowed as input!!!
  - e^|t| no LT? 1/(s^2-1) wo ROC?
  - Examples.
  - Properties of ROC of LT

9. Preview

- inverser LT



이번 학기말에는 융합부전공 신청 기간은 12월 4일부터 12일까지

$e^{st} \to \boxed{h(t)} \to H(s)e^{st}$, where $H(s)=\int_{-\infty}^{\infty} h(t)e^{-st}dt$ exists.

$\mathbb{C}$

$x(t) \to \boxed{LT} \to X(s) = \frac{1}{s+a}$

$Re\{s\} > Re\{-a\}$

$x: \mathbb{R} \to \mathbb{C}$

$x(t)$

$\Omega = \mathbb{R}$

$X: ROC \to \mathbb{C}$

$X(s)$

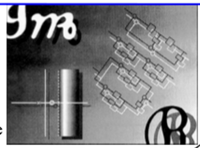
$\Omega = ROC \subset \mathbb{C}$

EECE233 Signals and Systems

EECE233 Signals and Systems

Lecture Note #14
Prof. Joon Ho Cho
(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

- Covers O&W pp. 670-698
- Inverse Laplace Transforms
- Examples
- Laplace Transform Properties
- The System Function of an LTI System
- Geometric Evaluation of Laplace Transform and Frequency Response



* These lecture notes are based on the lecture notes used in 6.003 taught b Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.

$x(t)=\sum_i a_i e^{s_i t}$ $s=\sigma+j\omega$

$y(t)=\sum_i H(s_i)a_i e^{s_i t}$

$e^{st} \rightarrow \boxed{H(s)} \rightarrow H(s)e^{st}$

$e^{j\omega t} \rightarrow \boxed{H(j\omega)} \rightarrow H(j\omega)e^{j\omega t}$

여기까지는 signal의 Laplace transform

여기부터는 LTI system과 Laplace transform

1

우리가 s-domain에서 analysis 와 design을 한 후 다시 time-domain으로 넘어 오기 위해서는 inverse transform이 필요하다.
page 2에서 보는 바와 같이 inverse Laplace transform을 이해하기 위해서는 complex variables, functions 등에 대한 지식이 필요한데
다행히도 이 과목에서는 inverse transform이 쉽게 구해지는 것들 만을 다룬다.

Inverse Laplace Transform

$$X(s)=\int_{-\infty}^{+\infty} x(t)e^{-st}dt\,,\quad s=\sigma+j\omega\in ROC$$

$$=\mathfrak{F}\{x(t)e^{-\sigma t}\}$$

$\boxed{X(\sigma+j\omega)}=\mathcal{F}\{x(t)e^{-\sigma t}\}$

Fix $\sigma\in$ ROC and apply the inverse Fourier transform

$$x(t)e^{-\sigma t}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\boxed{X(\sigma+j\omega)}e^{j\omega t}d\omega \qquad \text{FT Synthesis Eq.}$$

$$\Downarrow$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(\sigma+j\omega)e^{(\sigma+j\omega)t}d\omega$$

But $s=\sigma+j\omega\ (\sigma\ \textit{fixed})\ \Rightarrow ds=jd\omega$ $d(\sigma+j\omega)$

$$\Downarrow$$

$$x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds=\frac{1}{2\pi j}\oint_C X(s)e^{st}ds$$

— Will not use in EECE233

우리는 Fourier transform과 inverse Fourier transform에 대해 알고 있고
또한 Laplace transform은 x(t) e^{-\sigma t}의 Fourier transform이라는 것도 알고 있다. 따라서 ....

contour integral이다. -_-;마지막 등호 이전까지는 이해할 수 있어야 한다.

2

그러나 이 식으로부터 쉽게, 미분기의 system function이 s 로 나타나고 즉 unit doublet의 LT이 s이고 2계미분기의 system function은 s^2이고 ... 등등을 알 수 있다.

How Do We Perform Inverse Laplace Transform?

- In EECE233, we will only deal with Laplace transform that are:
  1) Rational, *i.e.* $X(s) = N(s)/D(s)$;
  And/or:
  2) exponential, *i.e.* $X(s) = e^{sT}$.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^{sT} = \sum \frac{(sT)^n}{n!}$

delay를 다루기 위해서...

- For case 2), use shift property

$y(t) = x(t-T) \longleftrightarrow e^{-sT}X(s)$ (Similar to the FT property $x(t-T) \longleftrightarrow e^{-j\omega T}X(j\omega)$

$h(t) = \delta(t-T)$

- For case 1), use PFE.

- For case 1) & 2), *i.e.* $X(s) = e^{-sT}\underbrace{(N(s)/D(s))}_{X_1(s)\leftrightarrow x_1(t)}$

Then $\updownarrow$

$$x(t) = x_1(t)\big|_{t\to t-T} = x_1(t-T)$$

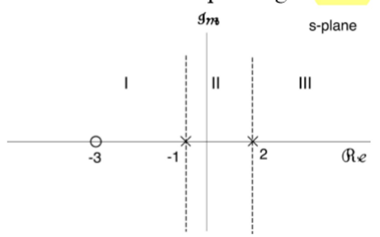
3

Inverse Laplace Transforms Via Partial Fraction Expansion and Properties

Example: $X(s) = \frac{s+3}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$ ROC $= \phi$.

$A = -\frac{2}{3}$, $B = \frac{5}{3}$

Three possible ROC's — corresponding to three *different* signals



어떤 이들은 4개의 가능한 신호가 있다고도 한다. 어느 경우든, t=0에서 discontinuity가 생긴다.

Recall $\frac{1}{s+a}$, $\mathrm{Re}(s) < -a \longleftrightarrow -e^{-at}u(-t)$ left - sided

and $\frac{1}{s+a}$ $\mathrm{Re}(s) > -a \longleftrightarrow e^{-at}u(t)$ right - sided

delay가 없는 rational의 경우 분자를 분모로 나눠 상수 및 s의 polynomial, 그리고 분자의 차수가 분모의 차수보다 낮은 rational function으로 나타내면, 유리함수 부분은 PFE하여 역

Recall $\frac{1}{s+a}$, $\mathrm{Re}(s) < -a \longleftrightarrow -e^{-at}u(-t)$ left - sided

and $\frac{1}{s+a}$, $\mathrm{Re}(s) > -a \longleftrightarrow e^{-at}u(t)$ right - sided

4

분자의 차수가 분모의 차수보다 낮은 rational function으로 나타내면, 유리함수 부분을 PFE하여 얻은 각 term은 언제나 causal 아니면 anti-causal signal로 inverse LT된다.

Example (cont.)

ROC I: — Left-sided signal.

$$x(t) = -Ae^{-t}u(-t) - Be^{2t}u(-t)$$

$$= \left[\frac{2}{3}e^{-t} - \frac{5}{3}e^{2t}\right]u(-t) \quad \text{Diverges as } t \to -\infty$$

$x(t) = \frac{2}{3}e^{-t}u(-t)$

$+\frac{5}{3}e^{2t}u(t)$

$\frac{-\frac{2}{3}}{s+1}$

$\frac{\frac{5}{3}}{s-2}$

ROC II: — Two-sided signal, has Fourier Transform.

$$x(t) = -\left[\frac{2}{3}e^{-t}u(t) + \frac{5}{3}e^{2t}u(-t)\right] \to 0 \text{ as } t \to \pm\infty$$

ROC III: — Right-sided signal.

$$x(t) = Ae^{-t}u(t) + Be^{2t}u(t)$$

$$= \left[-\frac{2}{3}e^{-t} + \frac{5}{3}e^{2t}\right]u(t) \quad \text{Diverges as } t \to +\infty$$

5

여기서 보는 바와 같이 ROC에 따라 3가지 (크게는 4가지) 다른 신호로 inverse transform된다.

암기 tip:

1/(s−pole)은
"e to the pole t u t"
또는
"−e to the pole t u −t"로
inverse LT된다.

Properties of Laplace Transforms

- Many parallel properties of the CTFT, but for Laplace transforms we need to determine implications for the ROC

- For example:

$\mathcal{R}$ $\quad\quad$ $\mathcal{R}_1$ $\quad\quad$ $\mathcal{R}_2$

<u>Linearity</u> $\quad ax_1(t) + bx_2(t) \longleftrightarrow aX_1(s) + bX_2(s)$

$\mathcal{R} \supset \mathcal{R}_1 \cap \mathcal{R}_2$

ROC at least the intersection of ROC's of $X_1(s)$ and $X_2(s)$

Linearity $ax_1(t) + bx_2(t) \longleftrightarrow aX_1(s) + bX_2(s)$

ROC at least the intersection of ROC's of $X_1(s)$ and $X_2(s)$

$\mathcal{R}_1$ $\mathcal{R}_2$

ROC can be bigger (due to pole-zero cancellation)

If $X_1(s)$ & $X_2(s)$ are rational, then ....

E.g. $x_1(t) = x_2(t)$ and $a = -b$

Then $a\,x_1(t) + b\,x_2(t) = 0 \rightarrow X(s) = 0$

$\Rightarrow$ ROC entire s-plane

Time Shift

$$x(t-T) \longleftrightarrow e^{-sT}X(s) \text{ , same ROC as } X(s)$$

E.g.,

$$\frac{e^{3s}}{s+2} \text{ , } \mathrm{Re}(s) > -2 \longleftrightarrow ?$$

$$\frac{e^{-sT}}{s+2} \text{ , } \mathrm{Re}(s) > -2 \longleftrightarrow e^{-2t}u(t)\Big|_{t\rightarrow t-T}$$

$$\downarrow T = -3$$

$$\frac{e^{3s}}{s+2} \text{ , } \mathrm{Re}(s) > -2 \longleftrightarrow e^{-2(t+3)}u(t+3)$$

Time-Domain Differentiation

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds, \qquad \frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds$$

$$\Downarrow$$

$$\frac{dx(t)}{dt} \longleftrightarrow sX(s), \text{ with ROC containing the ROC of } X(s)$$

ROC could be bigger than the ROC of $X(s)$, if there is pole-zero cancellation. *E.g.*

앞서 페이지 2에서 설명했듯, contour integral로 된 inverse LT식은 time-domain differentiation property를 설명하는데 유용하다.

미분의 FT이 j\omega X(j\omega)가 되는 것과 같다.

Q1. 적분기의 FT은?

ROC could be bigger than the ROC of $X(s)$, if there is pole-zero cancellation. *E.g.*

$$x(t) = u(t) \longleftrightarrow \frac{1}{s}, \quad \mathrm{Re}(s) > 0$$

$$\frac{dx(t)}{dt} = \delta(t) \longleftrightarrow 1 = s \cdot \frac{1}{s} \quad \mathrm{ROC} = \text{entire } s-\text{plane}$$

X (j\omega)가 되는 것과 같다.

Q1. 적분기의 FT은?
LT은?
Q2. the identity system의 LT은?

s-Domain Differentiation

$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}dt$

$$-tx(t) \longleftrightarrow \frac{dX(s)}{ds}, \quad \text{with same ROC as } X(s) \quad \left(\text{Derivation is similar to } \frac{d}{dt} \leftrightarrow s\right)$$

$$E.g. \quad te^{-at}u(t) \longleftrightarrow -\frac{d}{ds}\left[\frac{1}{s+a}\right] = \frac{1}{(s+a)^2}, \quad \mathrm{Re}(s) > -a$$

s-domain diff property는 pole이 중근일때 어떻게 inverse Laplace transform을 할 것이가?
에 대한 답을 준다.

–

A1. 주의! u(t) 의 LT은 1/s 이고 ROC는 RHP이다. 그러나 u(t)의 FT은 1/(j\omega)가 아니고 1/(j\omega +pi \delta(\omega)이다. 이는 0.5 sgn(t)의 FT이 1/(j\omega)이기때문에
u(t)는 1/(j\omega) +\pi delta(\omega)가 되기 때문이다.
주의 2! RLC circuit의 phasor analysis 에서는 언제나 적분기를 1/(j\omega)로 바꾸었는데, 그 이유는 이때는 입력이 e^{j\omega t}인 것만 고려하였고 특히 omega가 0가 아닌 교류회로의 해석만을 생각했기 때문이었다.

Convolution Property

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * x(t)$$

For $\quad x(t) \longleftrightarrow X(s), \quad y(t) \longleftrightarrow Y(s), \quad h(t) \longleftrightarrow H(s)$

Then $\quad Y(s) = H(s) \cdot X(s)$

- This is the main reason why Laplace transform is useful in solving problems involving LTI systems, just like the Fourier transform. *However*, because Laplace transform and its inverse transform are not *symmetric*, the reverse relation

$$x(t) \cdot h(t) \xleftrightarrow{\times} X(s) * H(s)$$

is *not* true, different from Fourier transform. Thus, $\mathcal{L}$-transform is *not* useful in modulation and sampling.

따라서 LT는 communication and signal processing에서 보다
control에서 많이 이용된다.
설령 comm. and signal processing에서 사용될 때에도
FT도 가능한 경우에 주로 사용된다.

사용된다.

Convolution Property (cont.)

$\mathcal{R}_Y ? \ \mathcal{R}_H \ \mathcal{R}_X$

ROC of $Y(s) = H(s)X(s)$, at least the overlap of the ROC's of $H(s)$ & $X(s)$

a) ROC could be empty if there is no overlap between the two ROCs

E.g. $x(t) = e^{t}u(t)\ Re(s) > 1$ and $h(t) = -e^{-t}u(-t)\ \ Re(s) < -1$

b) ROC could be larger than the overlap of the two.

E.g. $x(t)*h(t) = \delta(t) \Rightarrow$ ROC entire s-plane.

결론은, ROC는 그때그때 달라요...−_−;

The System Function of an LTI System

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

$h(t) \longleftrightarrow H(s)$ — the system function

The system function characterizes the system

$\Downarrow$

System dynamics correspond to properties of $H(s)$ and its ROC

A first example:

System is stable $\longleftrightarrow \int_{-\infty}^{+\infty} |h(t)|dt < \infty \Leftrightarrow$ ROC of $H(s)$ includes the $j\omega$ axis

$\Rightarrow$ The frequency response $H(j\omega)$ which is the Fourier transform of

ROC도 물론 specify되었을 때

일반적으로 LCCDE로 나타난 CT LTI system은 causal한 것으로 본다. (보통 initial rest조건을 줌으로써 해결한다.)

특별히 causal하지 않은 경우는 specify한다.

Fourier Plancherel이 아니 Fourier transform 이 존재할 필요충분조건이 absolute integrability이다

System is stable $\longleftrightarrow \int_{-\infty}^{\infty} |h(t)|dt < \infty \Leftrightarrow$ ROC of $H(s)$ includes the $j\omega$ axis

$\Rightarrow$ The frequency response $H(j\omega)$, which is the Fourier transform of $h(t)$, is well defined.

Fourier Plancherel이 아닌 Fourier transform 이 존재할 필요충분조건이 absolute integrability이다. (주의: inverse transform 의 존재여부와는 상관없다.)

LCCDE로 input-output relation이 주어진 CT LTI system이 있을때 이를 미분기 적분기 adder amp등으로 구현할 때 무작정 구현할 때는 component수가 많더라도 FT나 LT로 rational function으로 나타낸 후, 분자분모의 같은 factor를 나눠버리면 차수가 낮은 equivalent LCCDE를 얻을 수도 있고 이의 block digram을 그려서 구현하면 component수가 줄어든다!!! 자세한 것은 Lecture note #15의 page 9이후에...

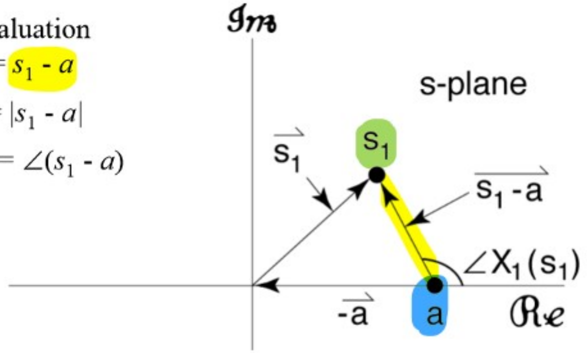
Geometric Evaluation of Rational Laplace Transforms

Example #1: $X_1(s) = s - a$ A first-order zero

Graphic evaluation of $X_1(s_1) = s_1 - a$

$\Rightarrow |X_1(s_1)| = |s_1 - a|$

& $\angle X_1(s_1) = \angle(s_1 - a)$



이제 Laplace transform이 Fourier transform보다 높은 차원에서 신호및시스템을 이해하게 해 줌을 확인해보자.

first order zero 즉 a가 분자의 중근이 아니라는 말.

Q. 이 그림으로부터 X(j\omega)가 어떤 모양을 가질지 geometrical하게 생각하여 예측해 보자.

Example #2: A first-order pole

$$X_2(s) = \frac{1}{s-a} = \frac{1}{X_1(s)}$$

$$\Rightarrow |X_2(s)| = \frac{1}{|X_1(s)|} \quad \left(\log|X_2(s)| = -\log|X_1(s)|\right)$$

$$and \quad \angle X_2(s) = -\angle X_1(s)$$

Example #3: A higher-order rational Laplace transform

$$and \quad \angle X_2(s) = -\angle X_1(s)$$

Example #3: A higher-order rational Laplace transform

$$X(s) = M\frac{\prod_{i=1}^{R}(s-\beta_i)}{\prod_{j=1}^{P}(s-\alpha_j)}$$

$$|X(s)| = |M|\frac{\prod_{i=1}^{R}|s-\beta_i|}{\prod_{j=1}^{P}|s-\alpha_j|}$$

$$and \quad \angle X(s) = \angle M + \sum_{i=1}^{R}\angle(s-\beta_i) - \sum_{j=1}^{P}\angle(s-\alpha_j)$$

a monic polynomial

$1 \cdot x^3 + 2x^2 - x - 1$

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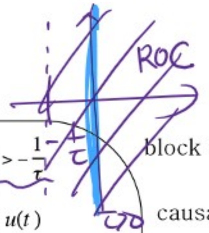


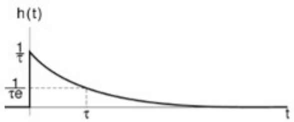
First-Order System

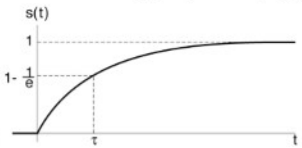
$$H(s) = \frac{1}{s\tau+1} = \frac{1/\tau}{s+1/\tau}, \quad \mathrm{Re}(s) > -\frac{1}{\tau}$$

block diagram을 그려보자.

causal이다.

$$h(t) = \frac{1}{\tau}e^{-t/\tau}u(t) \qquad s(t) = [1-e^{-t/\tau}]\,u(t)$$

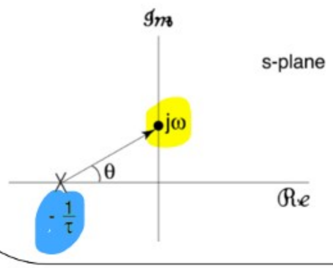




Graphical evaluation of $H(j\omega)$:

$$H(j\omega) = \frac{1/\tau}{j\omega+1/\tau}$$

$j\omega - \left(-\frac{1}{\tau}\right)$



$$|H(j\omega)| = \frac{1/\tau}{\sqrt{\omega^2+(1/\tau)^2}} = \begin{cases} 1 & \omega = 0 \\ 1/\sqrt{2} & \omega = 1/\tau \\ \sim 1/\omega\tau & \omega >> 1/\tau \end{cases}$$

$$\angle H(j\omega) = -\theta = -\tan^{-1}\left(\frac{\omega}{1/\tau}\right) = -\tan^{-1}(\omega\tau)$$

$$= \begin{cases} 0 & \omega = 0 \\ -\pi/4 & \omega = 1/\tau \\ -\pi/2 & \omega >> 1/\tau \end{cases}$$

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Bode Plot of the first-order system

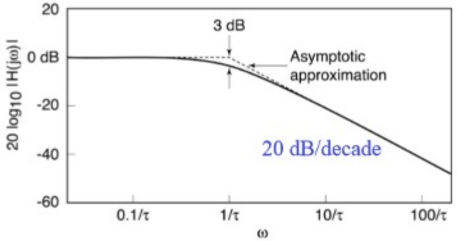
$$H(j\omega) = \frac{1/\tau}{\ldots}$$

Bode Plot of the first-order system

$$H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau}$$

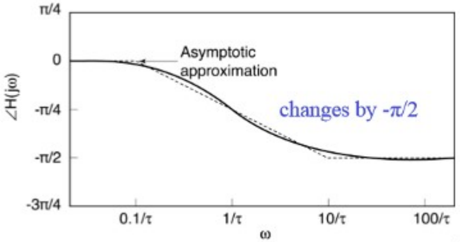
$$|H(j\omega)| = \frac{1/\tau}{\sqrt{\omega^2 + (1/\tau)^2}}$$

$$= \begin{cases} 1 & \omega = 0 \\ 1/\sqrt{2} & \omega = 1/\tau \\ \sim 1/\omega\tau & \omega >> 1/\tau \end{cases}$$



$$\angle H(j\omega) = -\theta = -\tan^{-1}(\omega\tau)$$

$$= \begin{cases} 0 & \omega = 0 \\ -\pi/4 & \omega = 1/\tau \\ -\pi/2 & \omega >> 1/\tau \end{cases}$$



1st, 2nd order system 이 중요한 이유는

LCCDE with real coefficients로 나타내어지는 수 많은 CT LTI system의 system function을 factoring하든가 PFE 하면 1st와 2nd order system 들의 곱또는 합으로 나타나기 때문이다.

(물론 중근이 있으면 그 이상의 차수도 필요하다.)

$$\sum_{k=0}^{N} a_k y^{(k)}(t) = \sum_{k=0}^{M} b_k x^{(k)}(t)$$

$$H(s) = \frac{\text{real } s \text{ poly}}{\text{real } s \text{ poly}}$$

$$= \sum c_k s^k + M\frac{N(s)}{D(s)}$$

$$= \sum c_k s^k + M\frac{\Pi(s-\ )}{\Pi(s-\ )}$$

$$\begin{cases} s = a + jb \\ s = a - jb \end{cases}$$

지금까지는 1st order system

이제부터는 2nd order system

Second-Order System (causal)

E.g. *LCR* circuit, a mechanical system with a spring and friction

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \text{ROC} \quad Re(s) > Re(pole)$$

$H(0) = 1$ $s = j\omega,\ \omega = 0$

$D/4 = (\zeta\omega_n)^2 - \omega_n^2$
$= (\zeta^2 - 1)\omega_n^2$

$s^2 + as + b$

$0 < \zeta < 1 \Rightarrow$ complex poles — *Under*damped

$p = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n$

$\zeta = 1 \Rightarrow$ double pole at $s = -\omega_n$ — *Critically* damped

$p = -\omega_n$

$\zeta > 1 \Rightarrow$ 2 poles on negative real axis — *Over*damped

$p = -\zeta\omega_n \pm \sqrt{\zeta^2-1}\,\omega_n < 0$

D/4, D discriminant

이렇게 불리는 이유를 먼저 time domain에서 step response를 보면서 이해하자.

따라서 다음 페이지 전에 먼저 page 19를 보라.

$\zeta^2 - 1 > 0$

서다두실, 허근, 중근, 서다두실

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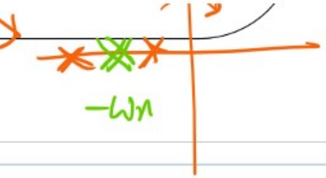


$P = -\zeta\omega_n \pm \sqrt{(\zeta^2-1)\omega_n^2}$

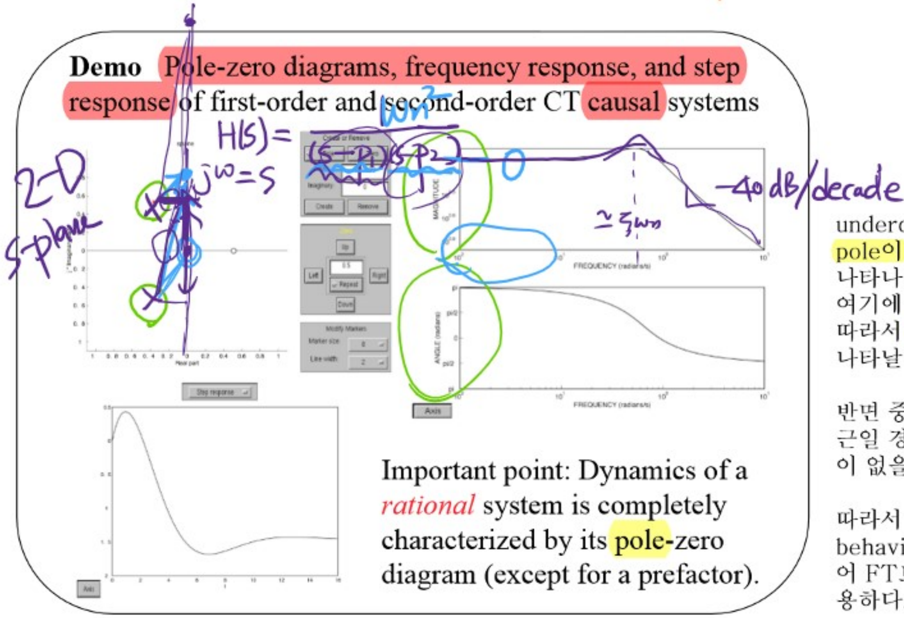
실중근

>0

$\pm j\sqrt{1-\zeta^2}\,\omega_n$

Demo Pole-zero diagrams, frequency response, and step response of first-order and second-order CT causal systems

$H(s) = \frac{\omega_n^2}{(s-p_1)(s-p_2)}$



Important point: Dynamics of a *rational* system is completely characterized by its pole-zero diagram (except for a prefactor).

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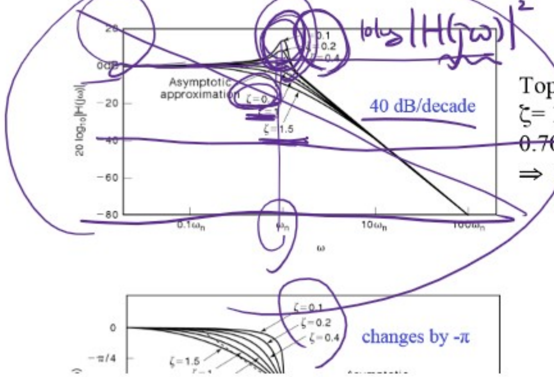
underdamped일 경우 pole이 conjuage pair로 나타나는데 여기에 u(t)를 가하면 따라서 oscillatory성분이 나타날 수 밖에 없다.

반면 중근이거나 서다두실 근일 경우 oscilatory성분이 없을 수밖에 없다.

따라서 time domain상의 behavior를 설명하는데 있어 FT보다 LT가 보다 유용하다.

Bode Plot of a Second-order System

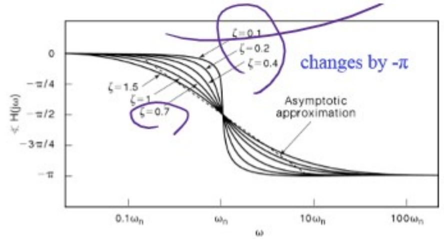


Top is flat when $\zeta = 1/\sqrt{2} = 0.707$

$\Rightarrow$ a LPF for $\omega < \omega_n$

underdampted일 경우 \omega_n 근처의 frequency component가 살아 남는다.

근사적으로 보아 underdamped에서는 omega_n성분이 있으므로 oscillatory behavior가 일어난다.



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underdamped에서는 omega_n성분이 있으므로 oscillatory behavior가 일어난다.

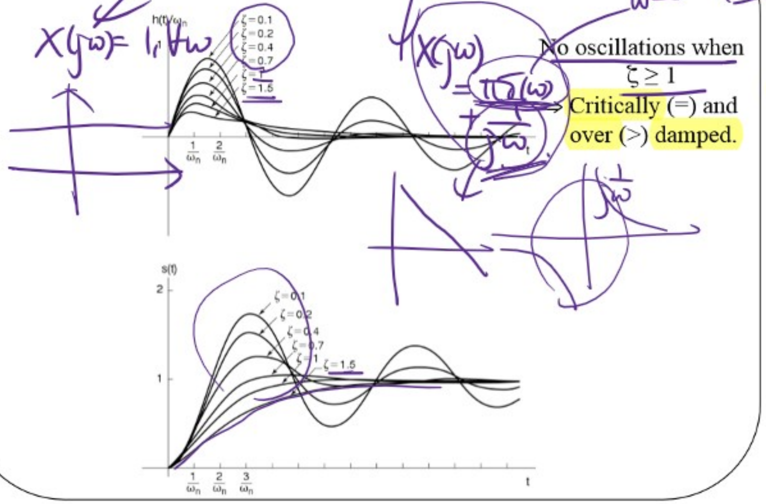
$x(t)=\delta(t)$ $x(t)=u(t)$

Unit-impulse and Unit-step Response of a Second-order System

$X(j\omega)=1, \forall\omega$

$X(j\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$

$\omega=0$ $|\omega|\to\infty$



No oscillations when $\zeta \geq 1$
⇒ Critically (=) and over (>) damped.

반면 critically 와 overdamped경우에는 \omega_n성분이 많이 attenuate된다. 따라서 underdampled 경우 보다 response가 천천히 변할 것이다.

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under/critial/over damp라는 이름의 유래는 이 step response에서.... underdamped의 경우 빠르게 움직이나 대신 overshoot하고 또 oscillate 한다. 반면....
step response의 LT은 system function에 1/s곱한 것이다. PFE하면 1/s성분과 1/(s-근) 성분들이 나온다. 따라서 근이 어디 있는냐에 따라 oscillation이 있으면서 u(t)로 가는지 없으면서 가는지 등이 결정된다.

First-order All-Pass System

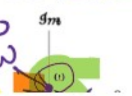
$s=j\omega$

$H(j\omega)=\frac{j\omega-a}{j\omega+a}$

$$H(s) = \frac{s-a}{s+a}, \quad \mathrm{Re}(s) > -a \quad (a > 0)$$

$|H(j\omega)|=1, \forall\omega$

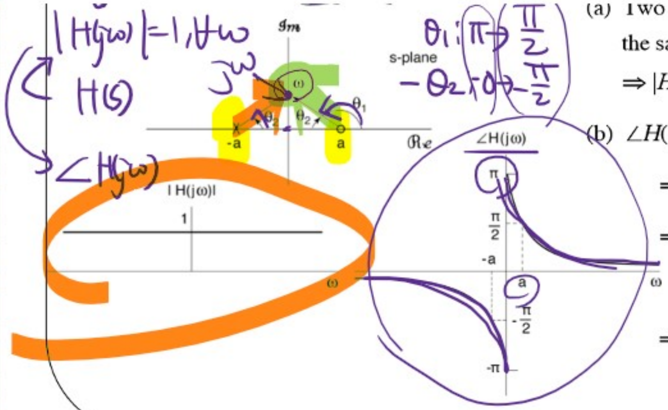
(a) Two vectors have the same length
$\Rightarrow |H(j\omega)| = 1$



$\theta_1 : \pi \to \frac{\pi}{2}$
$-\theta_2 : 0 \to -\frac{\pi}{2}$

먼저 block diagram을 그려보자.

다음 pole-zero diagram을 그려서 geometrical하게 생각하여 왜 이 system이 all pass인지 확인해 보자



(a) Two vectors have the same length

$\Rightarrow |H(j\omega)| = 1$

(b) $\angle H(j\omega) = \theta_1 - \theta_2$

$= (\pi - \theta_2) - \theta_2$

$= \pi - 2\theta_2$

$$= \begin{cases} \pi & \omega = 0 \\ \pi/2 & \omega = a \\ \sim 0 & \omega >> a \end{cases}$$

그래서 geometrical하게 생각하여
왜 이 system이 all pass인지 확인해 보자.