

LN 16 BZT: def, ROC, rational, etc.

2023년 12월 5일 화요일 00:09

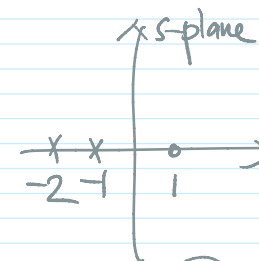
CM29 231207Th on ZTF, unilateral ZTF

Announcements

1. Messages from Instructor

- Goal of Life: Self-Fulfillment and Contribution to Collective (자아실현, 공동체 기여)
 - Attitude: Exceed the expectations! (기대를 넘어서라!)
 - Drive yourself by fear and dream.
- "아는 것이 힘이다. 배우고야 무슨 일이든 한다." - 심훈, 상록수-
- 학습 (배우고 익히기)
 - Theory Learning (이론 배우기)
 - Problem Solving (문제 풀기)
 - 배우기와 익히기의 균형
- How to Learn
 - Stop and Think; 복기가 chess master 를 만든다; Find commonality, similarity, and difference
- How to Practice

Purposeful practice with 3F: Time, Focused Action (무엇을 향상시키려는 훈련인가?; Push oneself little bit out of the COMFORTABLE ZONE), Experience, Lesson, Immediate Feedback (코치의 중요성), Fix, Growth
- How to solve? How to understand? ((S))implify and ((V))isualize:
 - essence, simplify; generalize; consider extremes; mental representation, representational examples;
 - Venn diagram, necessity and sufficiency, existence and uniqueness/plurality
 - An imperfect classification rule is better than no classification rule as long as you recognize the imperfection and update your rule.
 - block diagram, G(V,E); graph of a function, algebraic vs. geometric interpretations, mathematical vs. physical/engineering interpretations/implications



lex/ (9.23) $H(s) = \frac{s-1}{s^2+3s+2}$

$s \in \mathbb{C}$

$\begin{cases} \text{zero } 1\text{th} \\ \text{pole } 2\text{th} \end{cases}$

$\equiv \begin{cases} \text{zero } 1\text{th} @ 1 \\ \text{zero } 1\text{th} @ \infty \\ \text{pole } 2\text{th} \end{cases}$

$\mathbb{R}$

$\overline{\mathbb{R}} = \{x \in \mathbb{R}, \infty, -\infty\}$

$((f)) \frac{N(s)}{D(s)} \leftarrow \begin{matrix} \text{Numerator} \\ \text{Denominator} \end{matrix}$

$\frac{N(s)}{D(s)} \leftarrow N(s) \cdot \frac{1}{D(s)}$

2. 지난 강의 수정/보완 사항

- The idea of zero at infinity and pole at infinity for LT
- cf) extended real line

$\mathbb{R}, \overline{\mathbb{R}}, \mathbb{C}, \mathbb{C}$

3. HW, Quiz, Voluntary Project, and Exam

- Final Exam is approaching. Unless basic mandatory courses have conflicting schedules, it will be

K, K, L, L

3. HW, Quiz, Voluntary Project, and Exam

- Final Exam is approaching. Unless basic mandatory courses have conflicting schedules, it will be taken 7 pm to 11pm on Thursday Dec. 21 at LG 102. (No one has conflict with General Chemistry?)
- Voluntary Project (due by 12/28 18:00)
 - Topic must be approved before final exam. Topics related to this course material is acceptable.
 - Presentation file and presentation mp4 of length >20 minutes must be submitted.
 - No reuse, double use, etc.
 - To send, you must use POSTECH webmail and attach as Overzised Attachments.
 - Please check my confirmation reply. (I will be out of town from 26-28, so that my replay can be delayed.)
- There may be a supplementary CM. No attendance required.
- HW14 will be assigned. Due will be next Tuesday.

4. 강의 자료

- Handouts and annotated pdfs are downloadable from the non-PLMS class homepage at <http://cisl.postech.ac.kr/class/eece233/schedule.htm>.
- Recorded CMs are available from cisl.postech YouTube channel. Please place clickable time stamps.

5. 출석

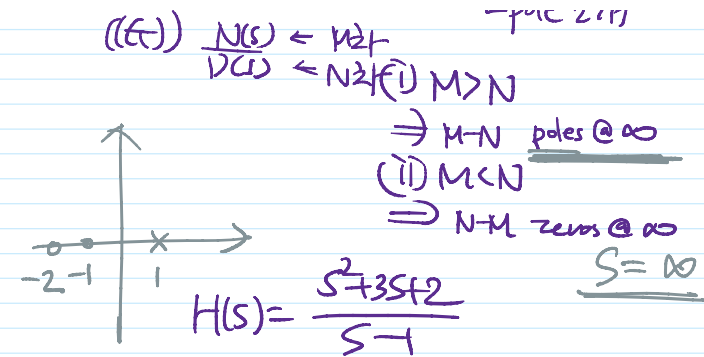
- 출결상태에 대해 전자출결시스템(<https://wpsec.postech.ac.kr/eams/login>)을 확인하세요.
정정사유가 있는 학생은 증빙을 제출해 주세요.

6. Overview (The numbers are the chapter numbers of the textbook)

1. Signals and Systems
2. Linear Time-Invariant Systems
3. Fourier Series Representation of Periodic Signals
4. Continuous-time Fourier Transform
5. Discrete-time Fourier Transform
6. Time and Frequency Characterization of Signals and Systems
9. Laplace Transform
10. Z-transform
7. Sampling
8. Communication Systems

7. Related Courses offered in 24-1

- EECE302 전자수학 A (확률 및 랜덤프로세스); 월수 2시; 신호및시스템



- EECE361/DISU361 전자파응용; 화목 11시; 전자기학 개론
- EECE441 디지털통신개론; 월수 11시; 신호및 시스템, 전자수학
- EECE442/NGCN301 통신및네트워크개론; 화목 2시
- EECE443/NGCN302 통신및네트워크실험; 월5시-10시, 1 seat available now
- EECE451 디지털신호처리개론; 화목 3시반; 신호및시스템
- 이번 학기말 통신 및 네트워크 융합부전공 신청 기간은 12월 4일부터 12일까지 (신청이전 1과목 이상 수강 필수)

8. Review

- Unilateral LT
 - Def. ... 0^- ...
 - Lemma. If $h(t)$ is known to be causal and $H(s)$ be rational, then the ROC needs not to be specified to find $h(t)$.
 - Use Case
 - causal CT system
 - ◆ with I/O relation in LCCDE
 - ◆ with causal input signal, and
 - ◆ not in the initial rest
 - ex) RLC circuit with initially charged capacitor, inductor, etc.
- Integration Property
 - Lemma. If $x(t)$ is causal, then $\int_{-\infty}^t x(\tau) d\tau$ has the ULT ...
- Differentiation Property
 - Lemma. If $x(t)$ is causal with initial conditions, then $d^n x(t)/dt^n$ has the ULT ...
 - ex)
 - ZIR = homogeneous solution
 - ZSR = particular solution
- Initial-value and Final-value Theorems
- ZTF
 - Bilateral vs. Unilatera
 - Def. $X(z) = \sum_{n=-\infty}^{\infty} z^{-n} x[n]$ in ROC
 - Relation b/w BZT and DTFT
 - unit circle
 - Properties of ROC

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\left. \begin{aligned} x(t) &= x(t) u(t) \\ y(t) &= y(t) u(t) \end{aligned} \right\} \text{인 경우}$$

$$y(t) = \int_{-\infty}^t x(\tau) d\tau \xrightarrow{\text{UL}} y(s) = \text{UL} (u(t) * x(t))$$

$$= \text{UL} \{u(t)\} \text{UL} \{x(t)u(t)\}$$

$$= \frac{1}{s} X(s)$$

$$y(t) = \frac{d}{dt} x(t) \xrightarrow{\text{UL}} y(s) = sX(s) - x(0^-)$$

$$y(t) = \frac{d^2}{dt^2} x(t) \xrightarrow{\text{UL}} y(s) = s(sX(s) - x(0^-)) - x'(0^-)$$

$$y(t) = \frac{d^n}{dt^n} x(t) \xrightarrow{\text{UL}} y(s) = s^n X(s) - s^{n-1} x(0^-) - s^{n-2} x'(0^-) - \dots - x^{(n-1)}(0^-)$$



9. Preview

- Properties of ROC (cont.)
- Inverse BZT
- Properties of BZT
- Design of a DT LTI system by using the BZT
- UZT and solving an LCCDE with initial conditions
 - Fibonacci sequence $u[n] = u[n-1] + u[n-2]$, for $n \geq 0$, $u[-1] = 1$, $u[-2] = 1$

$$\frac{d^3}{dt^3} x(t) \xrightarrow{\text{UL}} s^3 X(s) - s^2 x(0^-) - s x'(0^-) - x''(0^-)$$

- Design of a DT LTI system by using the BZT
- UZT and solving an LCCDE with initial conditions

▪ Fibonacci sequence $y[n] = y[n-1] + y[n-2]$, for $n \geq 0$, $y[-1] = 1$, $y[-2] = 1$

$$y[n] = y[n-1] + y[n-2]$$

$$y[-1] = 1, y[-2] = 1$$

$$a_1 = 1, a_2 = 1, a_{n+2} = a_{n+1} + a_n, n \geq 1$$

$$\Rightarrow y[n] = a_{n+2}$$

EECE233 Signals and Systems

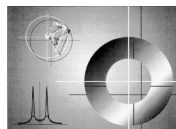
Lecture Note #16

Prof. Joon Ho Cho

(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

- Covers O&W pp. 741-763
- The (Bilateral) z -transform and its ROC
- Properties of the ROC
- Inverse z -Transform
- Examples

* These lecture notes are based on the lecture notes used in 6.003 taught by Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.



1

CT에서 FT의 확장인 LT를 배웠듯, DT에서 FT의 확장인 ZT를 배운다.

LT와 ZT은 많은 유사점을 가지므로 유사점과 차이점을 표로 만들어 숙지하자.

ZTF is very important for digital filter design, digital control, digital signal processing, digital communications, etc.

The z -Transform (another transform, any end?)

LT와의 유사점:

1. ZT은 1-D discrete-parameter signal을 2-D continuous-parameter

The z -Transform (another transform, any end?)

Motivation: Analogous to LT in CT

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$x[n] = z^n \longrightarrow y[n] = H(z)z^n$$

Eigenfunction
for DT LTI

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n} \quad (\text{assuming it converges})$$

(From Lecture #5)

The (Bilateral) z -Transform

$$x[n] \xleftrightarrow{Z} X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} = Z\{x[n]\} \quad - \quad z\text{-transform}$$

ZTF은 (... , $x[-1]$, $x[0]$, $x[1]$, ...) 을 z polynomial $\dots + x[-1]z^1 + x[0] + x[1]z^{-1}$ 로 나타낸 것이라 생각하면 간단하다.
물론 이를 z 의 함수로 보면서 ROC를 명시해야하겠다. 그런데 무한 terms면 rational이라 하지 않는다.

1. ZT은 1-D discrete-parameter signal을 2-D continuous-parameter signal로 transform한 것으로 볼 수 있다.

이 새로운 signal의 domain은 1-D complex plane (z -plane) 의 부분 집합이다.

2. 서로 다른 두 신호가 같은 모양의 algebraic mapping rule을 가질 수가 있다. (물론 ROC는 다르지만...)

3. LT에서와 마찬가지로 ROC를 명시해야 한다.

The z -Transform — DT Laplace Transform

Recall:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \stackrel{t=n\Delta t}{=} \lim_{\Delta t \rightarrow 0} \sum_{n=-\infty}^{\infty} \underbrace{x(n\Delta t) \cdot \Delta t}_{x[n]} \underbrace{(e^{s\Delta t})^{-n}}_z$$

$\Downarrow$ DT, Δt is now finite

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where

$$z = e^{s\Delta t}$$

$\Downarrow$

$0^n = 0$ for n nonnegative,
not defined elsewhere.

차이점

1. 가장 두드러진 차이점은 z^{-n} 을 곱하므로 '0으로 나눌수 없다'와 관련된 여러 다른 상황이 생긴다는 것. LT에서는 항상 nonzero값을 곱했다.

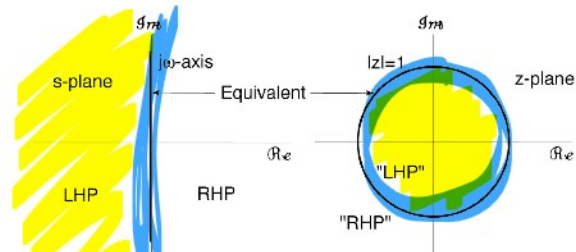
2. '0으로 나눌수 없다'가 중요한데 이를 헷갈리게하는 conventino이 있다.

3. 그외 기타 차이 여럿....

The Relations Between $\mathcal{ZT}$ and $\mathcal{LT}$, s -plane and z -plane

$$e^{s\Delta t} = z$$

$j\omega$ -axis in s -plane ($s = j\omega$) $\Leftrightarrow |z| = |e^{j\omega\Delta t}| = 1$ — a unit circle in z -plane



- A vertical line in s -plane, $Re(s) = \text{constant} \Leftrightarrow |e^{s\Delta t}| = \text{constant}$, a circle in z -plane.
- LHP in s -plane, $Re(s) < 0 \Rightarrow |z| = |e^{s\Delta t}| < 1$, inside the $|z| = 1$ circle, special case, $Re(s) = -\infty \Leftrightarrow |z| = 0$.
- RHP in s -plane, $Re(s) > 0 \Rightarrow |z| = |e^{s\Delta t}| > 1$, outside the $|z| = 1$ circle, special case, $Re(s) = +\infty \Leftrightarrow |z| = \infty$.

The ROC and the Relation Between $\mathcal{ZT}$ and $\mathcal{DTFT}$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{+\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{+\infty} (x[n]r^{-n})e^{-j\omega n} \\ &= \mathcal{F}\{x[n]r^{-n}\} \end{aligned}$$

$$ROC \in \left\{ z = re^{j\omega} \text{ at which } \sum_{n=-\infty}^{+\infty} |x[n]r^{-n}| < \infty \right\}$$

- depends only on $r = |z|$, just like the ROC in s -plane

LT에서와 같이:

이런 의미에서는 FT이 ZT보다 일반적이다. 그러나...

$$\sum_{n=-\infty}^{\infty}$$

- depends only on $r = |z|$, just like the **ROC** in s -plane only depends on $\text{Re}(s)$
- Unit circle ($r = 1$) in the ROC $\Rightarrow$ DTFT $X(e^{j\omega})$ exists

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Example #1 $x[n] = a^n u[n]$ — right-sided

$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$

$= \frac{1 - (az^{-1})^{\infty}}{1 - az^{-1}} = ?$

If $|az^{-1}| < 1$, i.e. $|z| > |a|$

Then $X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$

That is, ROC $|z| > |a|$, "RHP"

$z \neq 0$ $\frac{z}{z-a} \rightarrow \frac{1}{1-az^{-1}}, |az^{-1}| < 1$

$a^n u[n] z^{-n} = (az^{-1})^n u[n]$

$\lim_{n \rightarrow \infty} (az^{-1})^n = 0$ if $|az^{-1}| < 1$

Unit Circle z -plane

$z = a$ pole, $z = 0$ zero

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가장 간단한 DT signal로 geometric sequence를 생각해 보자.

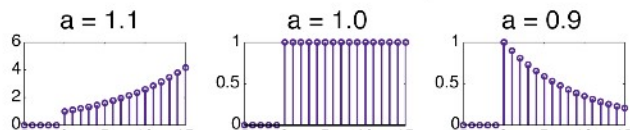
$x(t) = e^{at} u(t)$ 의 LT에서와는 달리

pole at $z=a$ 와 zero at $z=0$ 가 동시에 나타난다!!!!

ROC가 0을 포함하지 않으므로 z 를 분자분모에 곱하여 오른쪽과 같이 z 의 polynomial로 분자분모 나타내면 pole과 zero의 위치를 쉽게 파악할 수 있다.

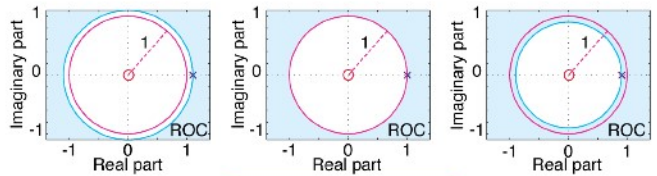
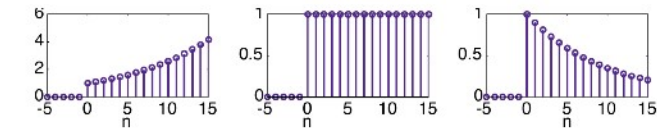
Example #1 (cont.)

$$x[n] = a^n u[n] \xleftrightarrow{z} X(z) = \frac{1}{1 - az^{-1}} \text{ for } |z| > |a|.$$



The signals are all right-sided and consequently the ROCs are all outside the circles.

The third one is DT Fourier transformable and its ROC contains the unit circle.



ROC outside $|z|=1$
unstable

$$u[n] \xleftrightarrow{z} \frac{1}{1-z^{-1}}$$

include $|z|=1$
stable

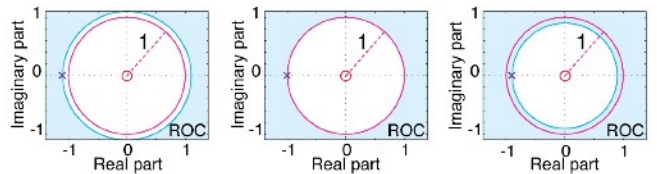
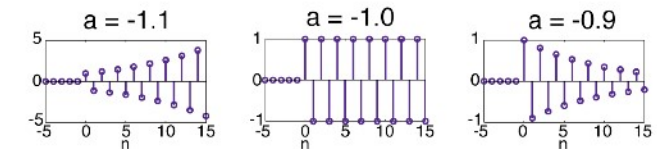
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Fourier transformable
and its ROC contains the
unit circle.

DT LTI's BIBO stability
iff
absolutely summable iff
Fourier transformable.

Example #1 (cont.) — What if $a < 0$?

$$x[n] = a^n u[n] \xleftrightarrow{z} X(z) = \frac{1}{1-az^{-1}} \text{ for } |z| > |a|.$$



ROC outside $|z|=1$
unstable

include $|z|=1$
stable

8

The signals are all
right-sided and
consequently the ROCs
are all outside the
circles.

The third one is DT
Fourier transformable
and its ROC contains
the unit circle.

$$\frac{1}{1-az^{-1}} \quad \left[\begin{array}{l} \text{pole @ } a \\ \text{zero @ } 0 \end{array} \right]$$

Example #2: $x[n] = -a^n u[-n-1]$ — left-sided

LT에서와 마찬가지로 ZT
에서도

Example #2: $x[n] = -a^n u[-n-1]$ — left-sided

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

$$= - \sum_{n=-\infty}^{-1} a^{-n} z^{-n} = - \sum_{n=0}^{\infty} (a^{-1} z)^n$$

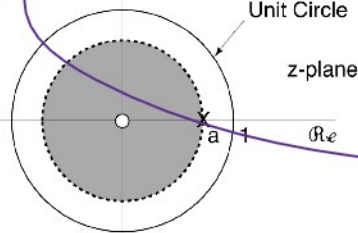
$$= 1 - \frac{1 - (a^{-1} z)^{\infty}}{1 - a^{-1} z} = ?$$

If $|a^{-1} z| < 1$, i.e. $|z| < |a|$

$$\text{Then } X(z) = 1 - \frac{1}{1 - a^{-1} z} = \frac{a^{-1} z}{a^{-1} z - 1}$$

$$= \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

Same $X(z)$ as in Ex #1, but different ROC.



LT에서와 마찬가지로 ZT에서도 서로 다른 두 신호가 같은 algebraic form의 ZT를 가질 수 있다. 물론 ROC는 다르지만...

원점이 ROC에 포함되므로 가운데도 칠해야 한다.

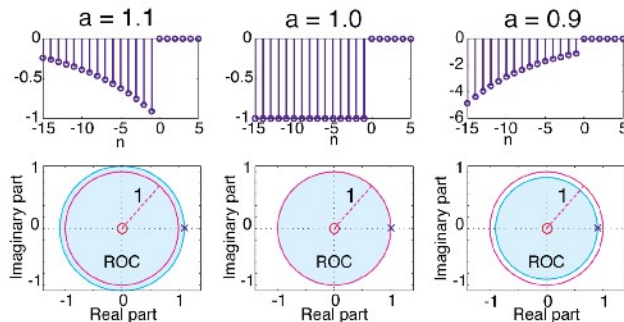
위식과 이 식의 등식은 ROC 내에서 성립한다.

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이 등식은 ROC내에서 성립하지 않는다. 0에서 성립하지 않기 때문이다. 하지만 편의상 $x[n]$ 의 ZT이 $1/(1 - az^{-1})$ 이고 ROC가 $|z| < |a|$ 라고 말한다. 조금 문제가 있는 표현인데 ZT와 관련하여서는 이런 식으로 표현하는 것이 convention이다. 다행인것은 pole과 zero를 찾기위해 $N(z)/D(z)$ 로 바꾸어보면 $z=0$ 은 zero이지 pole이 아니다. 따라서 여전히, "pole zero의 위치를 찾기 위해서는 $N(z)/D(z)$ 로 바꾸어 찾으면 된다"는 말이 옳다. 다만 pole과 zero at infinity는 따로 주의해야 한다.

Example #2 (cont.)

$$x[n] = -a^n u[-n-1] \xleftrightarrow{Z} X(z) = \frac{1}{1 - a z^{-1}} \text{ for } |z| < |a|.$$



ROC include $|z|=1$

stable

inside $|z|=1$

unstable

Observation: ROC includes $|z|=1 \Rightarrow$ stable.

이런 표현이 convention임을 명심하지 않으므로 ZT를 공부할 때 꼭 봐야 할 것이다.

The signals are all left-sided and consequently the ROCs are all inside the circles.

The first one is DT Fourier transformable and its ROC contains the unit circle.

$$x_1(t) = e^{-at} u(t) \leftrightarrow X_1(s) = \frac{1}{s+a}, \text{ ROC}_1$$

$$x_2(t) = e^{-at} u(-t) \leftrightarrow X_2(s) = \frac{1}{s+a}, \text{ ROC}_2$$

$$\frac{(a^{-1} z)^{-1}}{1 - (a^{-1} z)^{-1}} = \frac{a^{-1} z}{1 - a^{-1} z} = \frac{z}{a - z}$$

$$= \frac{z}{z - a}$$

$$|a^{-1} z| < 1 \Rightarrow |z| < |a|$$

$$X(z) = \frac{1}{1 - a z^{-1}}, |z| < |a|$$

$$0, z=0$$

$$f(x) = x = \frac{1}{(1/x)} = 1/(1/x)$$

Rational z-Transform

$x[n]$ = linear combination of exponentials for $n > 0$ and for $n < 0$



$X(z)$ is rational

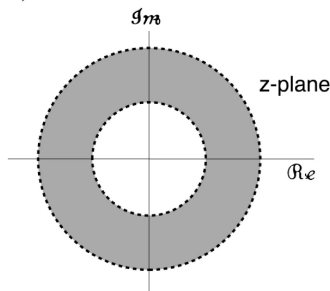
$$X(z) = \frac{N(z)}{D(z)} \quad \begin{array}{l} \swarrow \text{Polynomials in } z \\ \searrow \end{array}$$

— characterized (except for a prefactor) by its **poles** and **zeros**

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Properties of the ROC

- 1) The ROC of $X(z)$ consists of a ring in the z -plane centered about the origin (**equivalent to a vertical strip in the s -plane**)



- 2) The ROC does **not** contain any poles (same as in $\mathcal{L}\mathcal{T}$).

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a_n complex number 일때, $\sum_{n=-\infty}^{\infty} a_n z^n$ 은 any rearrangement가 수렴하면 즉 $\sum_{n=-\infty}^{\infty} |a_n| |z|^n$ 이 수렴하면 수렴하는 것으로

$$\{z \in \mathbb{C}, |z| > |a|\} \quad \begin{array}{l} z = \infty \\ z = -\infty \end{array}$$

pole zero의 위치를 알고 싶으면, 일단 기본적으로는 LT에서와 마찬가지로 정의하므로 반드시 분자 분모가 모두 z (not z^{-1})의 polynomial이 되게 한후, 즉 분모 분자에 z^M, z^N 을 곱한 후 인수분해한다.

만약 분모의 차수가 크면 " $|z| = \infty$ 에서 그 차수차만큼의 zero를 갖는다"라고 말하고

만약 분자의 차수가 크면 " $|z| = \infty$ 에서 그 차수차만큼의 pole을 갖는다"라고 말한다. 이경우 $|z| = \infty$ 는 ROC에 포함될 수가 없다.

이렇게 $|z| = \infty$ 에서의 pole과 zero를 정의하면, z^{-1} 에 대해 rational인 ZT는 항상 같은 갯수의 pole과 zero를 갖는다.!!!!!!!!!!!!!!

사실 LT에서도 pole and zero at infinity를 이상과 같이 정의하는 것이

a_n 이 complex number일때, $\sum_{n=-\infty}^{\infty} a_n$ 은 any rearrangement가 수렴할때, 즉 $\sum |a_n|$ 이 수렴할때 수렴하는 것으로 정의되므로, (absolute summability = sum exists for any rearrangement) 어느 한 점 $z=c$ 가 ROC에 포함되면, $|z|=|c|$ 도 ROC에 포함된다.

$z=re^{j\theta}$ 로 생각해 보라.

사실 LT에서도 pole and zero at infinity를 이상과 같이 정의할 수 있다.

ROC Properties (continued)

(3) If $x[n]$ is of **finite duration**, then the ROC is the **entire** z -plane, except possibly at $z=0$ and/or $z=\infty$.

Why?

From
$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}$$

$$= x[N_1] z^{-N_1} + x[N_1+1] z^{-(N_1+1)} + \dots + x[N_2] z^{-N_2}$$

e.g.

$\delta[n] \longleftrightarrow 1$	ROC all z	CT counterparts $\delta(t) \longleftrightarrow 1$
$\delta[n-1] \longleftrightarrow z^{-1}$	ROC $z \neq 0$	$\delta(t-T) \longleftrightarrow e^{-sT}$ $\text{Re}(s) \neq -\infty$
$\delta[n+1] \longleftrightarrow z$	ROC $z \neq \infty$	$\delta(t+T) \longleftrightarrow e^{+sT}$ $\text{Re}(s) \neq +\infty$

앞서 pole at infty 와 zero at infty의 정의에 따라

$1/z$ 는

pole at 0

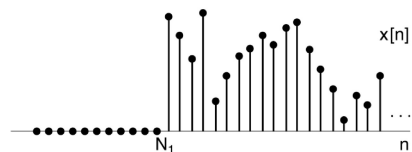
zero at ∞

앞서 pole at infty 와 zero at infty의 정의에 따라 z 는 pole at ∞ 와 zero at 0를 갖는다.

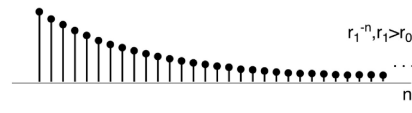
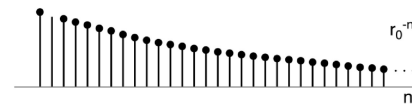
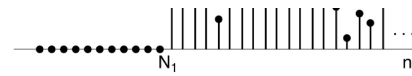
를 갖는다.

More Properties!

(4) If $x[n]$ is a **right-sided** sequence, and if $|z| = r_o$ is in the ROC, then all finite values of z for which $|z| > r_o$ are also in the ROC.



pole at infinity가 가능하므로 주의한다.



$$\sum_{n=N_1}^{\infty} x[n] r_o^{-n}$$

converges slower than

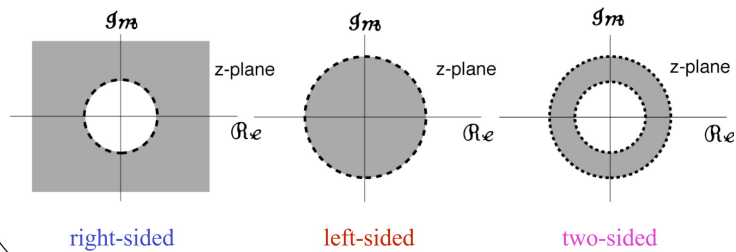
$$\sum_{n=N_1}^{\infty} x[n] r_1^{-n}$$

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Side by Side

- (5) If $x[n]$ is a **left-sided** sequence, and if $|z| = r_o$ is in the ROC, then all finite values of z for which $0 < |z| < r_o$ are also in the ROC.
- (6) If $x[n]$ is **two-sided**, and if $|z| = r_o$ is in the ROC, then the ROC consists of **a ring** in the z -plane including the circle $|z| = r_o$.

Quiz: What types of signals do the following ROC correspond to?



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Example #3

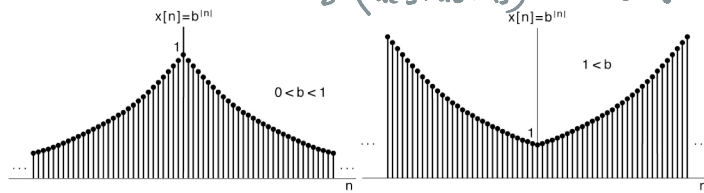
$$x[n] = b^{|n|} \quad b > 0$$

$$= b^{|n|} (u[n] + u[-n-1]) = b^n u[n] + (-b^{-n} u[-n-1])$$

Example #2

$$x[n] = b^{|n|} \quad b > 0$$

$$= b^{|n|} (u[n] + u[-n-1]) = b^n u[n] + b^{-n} u[-n-1]$$



Rewrite $x[n] = b^n u[n] + b^{-n} u[-n-1]$

From:

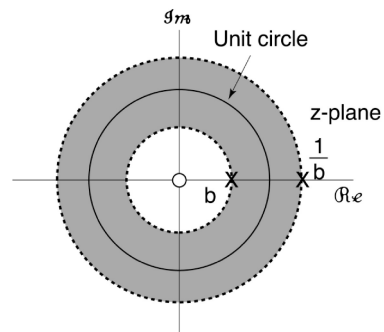
$$b^n u[n] \longleftrightarrow \frac{1}{1 - bz^{-1}} \quad |z| > b$$

and $b^{-n} u[-n-1] \longleftrightarrow \frac{-1}{1 - b^{-1}z^{-1}} \quad |z| < \frac{1}{b}$

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Example #3 (continued)

$$X(z) = \frac{1}{1 - bz^{-1}} + \frac{-1}{1 - b^{-1}z^{-1}} \quad b < |z| < \frac{1}{b}$$



Clearly, ROC does *not* exist if $b > 1 \Rightarrow$ No z -transform for $b^{|n|}$.

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Inverse z -Transforms

$$X(z) = X(re^{j\omega}) = \mathcal{F}\{x[n]r^{-n}\}, \quad z = re^{j\omega} \in \text{ROC}$$

$$x[n]r^{-n} = \mathcal{F}^{-1}\{X(re^{j\omega})\} = \frac{1}{2\pi} \int_{2\pi} X(re^{j\omega}) e^{j\omega n} d\omega$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(re^{j\omega}) \underbrace{r^n e^{j\omega n}}_{z^n} d\omega$$

$$z = re^{j\omega} \Rightarrow dz = jre^{j\omega} d\omega \Rightarrow d\omega = \frac{1}{j} z^{-1} dz$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz \quad \text{--- Will not use in EECE233}$$

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$$x[n]e^{-st} = x[n]e^{-(\sigma+j\omega)t} = x[n]e^{-\sigma t} e^{-j\omega t}$$

Example #4 Q. 아래와 같이 $X(z)$ 가 주어지는 DT신호를 구하라.

$$X(z) = \frac{3z^2 - \frac{5}{6}z}{(z - \frac{1}{4})(z - \frac{1}{3})} = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})} = \frac{A}{1 - \frac{1}{4}z^{-1}} + \frac{B}{1 - \frac{1}{3}z^{-1}}$$

Partial Fraction Expansion Algebra:

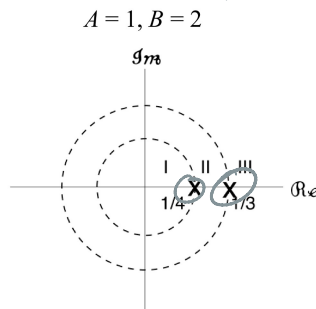
$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}$$

$$\Downarrow \quad \quad \quad \Downarrow \quad \quad \quad \Downarrow$$

$$x[n] = x_1[n] + x_2[n]$$

Note, unique for z -transform:

- 1) When finding poles and zeros, express $X(z)$ as a function of z .
- 2) When doing inverse z -transform using PFE, express $X(z)$ as a



z 의 polynomial로 인수분해 해서는 PFE를 해서 inverse ZT를 할 수 없다. 따라서 z^{-1} 의 polynomial을 인수분해 해서 PFE해야 한다. 그래서 왼쪽 것을 오른쪽 모양으로 바꾼 후 PFE한다.

To find poles and zeros, 이 모양이 좋다.

To do inverse ZT, 먼저 z^{-1} 의 rational

- 1) When finding poles and zeros, express $X(z)$ as a function of z .
- 2) When doing inverse z -transform using PFE, express $X(z)$ as a function of z^{-1} .



To do inverse ZT, 먼저 z^{-1} 의 rational function으로 나타낸 후 PFE을 한다.

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ROC III: $|z| > \frac{1}{3}$ — right - sided signal

$$x_1[n] = (1/4)^n u[n]$$

$$x_2[n] = 2 \cdot (1/3)^n u[n]$$

ROC II: $\frac{1}{4} < |z| < \frac{1}{3}$ — two - sided signal

$$x_1[n] = (1/4)^n u[n]$$

$$x_2[n] = -2 \cdot (1/3)^n u[-n-1]$$

ROC I: $|z| < \frac{1}{4}$ — left - sided signal

$$x_1[n] = -(1/4)^n u[-n-1]$$

$$x_2[n] = -2 \cdot (1/3)^n u[-n-1]$$

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Inversion by Identifying Coefficients in the Power Series When $X(z)$ Has Finite Terms

$$X(z) = \sum_{n=-\infty}^{+\infty} \underbrace{x[n]}_{\text{coefficient of } z^{-n}} z^{-n}$$

Example #5:

$$X(z) = 3z^3 - z + 2z^{-4}$$

Handwritten notes: $x[n] = 3$ for $n=-3$, $x[n] = -1$ for $n=-1$, $x[n] = 2$ for $n=4$. The expression is also written as $3z^3z + \frac{2}{z^4} = 3z^4 + \frac{2}{z^4}$.

언제나 위와 같이 z^{-1} 의 rational function이 되는 것은 아니다. 예를 들어 z^{-1} 의 finite length polynomial이 될 경우가 있다. 이럴 경우의 inverse는 매우 간단하다.

$z=0$ 와 $z=\infty$ 를 제외한 모든 z -plane이 ROC이다.

Example #5:

$$X(z) = 3z^3 - z + 2z^{-4}$$

$x[-3] = 3$
 $x[-1] = -1$
 $x[4] = 2$
 $x[n] = 0$ for all other n

$3z^3 - z + \frac{2}{z^4}$
 $x[n] = \begin{cases} 3 & n = -3 \\ -1 & n = -1 \\ 2 & n = 4 \\ 0 & \text{otherwise} \end{cases}$

4 poles @ 0,
 3 poles @ ∞

$z=0$ and $z=\infty$ 를 제외한 모든 z -plane이 ROC이다.

— A finite-duration DT sequence

21

Infinite terms가 있는 신호는

- 1) right-sided
 - 2) left-sided
 - 3) two-sided
- 로 나누어 ROC의 특성을 정리해보고

Finite terms가 있는 신호는

- 1) causal
- 2) anti-causal
- 3) neither

로 나누어 ROC의 특성을 확인해 보라. 특히 ROC가 $z=0$ 과 $z=\infty$ 를 포함하는지 여부에 주의하라.

Lecture Note #17

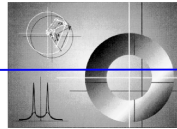
Prof. Joon Ho Cho

(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T.

Weiss, J. White, and A. Willsky)

- Covers O&W pp. 763-797
- Properties of z -Transforms
- **System** Functions for DT LTI Systems
- Geometric Evaluation of z -Transforms and DT Frequency Responses
- First- and Second-Order Systems
- System Function Algebra and Block Diagrams
- **Unilateral** z -Transforms

* These lecture notes are based on the lecture notes used in 6.003 taught by Prof. Qing Hu at M.I.T. These notes are adopted for EEECE233 under the permission of Prof. Qing Hu.



DT signals and ZT

DT LTI **systems** and ZT,

특히 LCCDE로 나타나는
DT LTI system

initial condition이 있는...

이를 이용하면 고등학교에서
배운 점화식의 일반항을 구
하는 문제중 많은 문제를 쉽
게 풀 수 있다.

1

Q. $X(z) = 1 - 3z^{-1} + 2z^{-2}$ 의 pole과 zero의 위치는?

A. z^{-1} 으로 묶어보면, $z=1$ or 2 에서 zero이고 $z=0$ 에서 2중 pole임을 알 수 있다.

Properties of z -Transforms

(1) Time Shifting

$$x[n] \xleftrightarrow{z} X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$x[n - n_0] \xleftrightarrow{z} z^{-n_0} X(z)$$

delay by n_0

The **rationality** of $X(z)$ is unchanged, **different** from $\mathcal{L}\mathcal{T}$. ROC unchanged except for the possible addition or deletion of the origin or infinity

$n_0 > 0 \Rightarrow \text{ROC } z \neq 0$ (maybe)
 $n_0 < 0 \Rightarrow \text{ROC } z \neq \infty$ (maybe)

z^{-n_0} 는 n_0 만큼의
delay를 나타낸다.

(2) z -D main Differentiation

$$nx[n] \xleftrightarrow{z} -z \frac{dX(z)}{dz}$$

Derivation

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$\frac{dX(z)}{dz} = - \sum_{n=-\infty}^{+\infty} nx[n]z^{-n-1}$$

$$z \frac{dX(z)}{dz} = - \sum_{n=-\infty}^{+\infty} nx[n]z^{-n}$$

QED

2

z 가 곱해지면 0에서의 pole이
cancel되거나 infinity에서
pole이 생긴다.

z^{-1} 가 곱해지면 0에서의
zero가 cancel되거나 infinity
에서 zero가 생긴다.

Convolution Property and System Functions

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

$Y(z) = H(z)X(z)$, ROC at least the intersection of the ROC's of $H(z)$ and $X(z)$, can be bigger if there are pole/zero cancellation. *e.g.*

$$H(z) = \frac{1}{z-a} \quad |z| > a$$

$$X(z) = z-a \quad z \neq \infty$$

$$\Rightarrow Y(z) = 1 \quad \text{ROC all } z$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \quad \text{— The System Function}$$

FIR impulse response를 갖는 시스템에 finite duration signal을 가했을때 출력 구하는 것에 바로 이용할 수 있다.

3

CAUSALITY

- (1) $h[n]$ right-sided $\Rightarrow$ ROC is the exterior of a circle possibly including $z = \infty$:

$$H(z) = \sum_{n=N_1}^{\infty} h[n]z^{-n}$$

If $N_1 < 0$, then the term $h[N_1]z^{-N_1} \rightarrow \infty$ at $z = \infty$

$\Rightarrow$ ROC outside a circle, but does not include ∞ .

$$\text{Causal} \Leftrightarrow N_1 \geq 0$$

$\Downarrow$

A DT LTI system with system function $H(z)$ is causal $\Leftrightarrow$ the ROC of $H(z)$ is the exterior of a circle including $z = \infty$

non-causal FIR filter를 생각해 보라. pole 이 infinity에 있을 수 있으므로... 주의 한다.

$$h[n] = \delta[n+1] + \delta[n] + \delta[n-1]$$

예: 신호 $x[n] = \delta[n+1]$ 은 causal이 아니고 $X(z) = z$ 로 infy에서 pole를 갖는다. 따라서 ROC는 infy를 제외한 전 영역이 된다.

사실 $|z| = \infty$ 라는 위치가 C내에 존재하는 것은 아니므로 엄밀한 의미에서는 ROC가 entire z-plane이지만

$$h[n] = \delta[n+1] + \delta[n] + \delta[n-1] \xleftrightarrow{\text{BZT}} H(z) = 1 + z^{-1} = \frac{z+1}{z}, \text{ ROC} = \{z \in \mathbb{C} : |z| > 0\} \Rightarrow z \neq 0$$

zeros @ $\frac{\pm j\sqrt{3}}{2}$

$$H(z) = \frac{z+1}{z} = \frac{z+1}{z}$$

↑ pole @ ∞
↑ pole @ 0

$$\text{ROC} = \{z \in \mathbb{C} : |z| > 0\} \Rightarrow z \neq 0$$

A DT LTI system with system function $H(z)$ is **causal** $\Leftrightarrow$ the ROC of $H(z)$ is the exterior of a circle **including** $z = \infty$

$$\exists r \geq 0 \text{ s.t. } \text{ROC} = \{z \in \mathbb{C} : |z| > r\}$$

사실 $|z| = \infty$ 라는 위치가 C내에 존재하는 것은 아니므로 엄밀한 의미에서는 ROC가 entire z-plane이지만 ZT을 논할때는 일반적으로 $|z| = \infty$ 에 대해 언급한다.

DT LTI Systems Described by LCCDE's

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

Use the time-shift property,

$$\left(\sum_{k=0}^N a_k z^{-k} \right) Y(z) = \left(\sum_{k=0}^M b_k z^{-k} \right) X(z)$$

$$\Downarrow$$

$$Y(z) = H(z) X(z)$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} \quad \text{--- Rational in } z^{-1}$$

ROC: Depends on Boundary Conditions, left-, right-, or two-sided.
For Causal Systems $\Rightarrow$ ROC is outside the outermost pole

수학적으로 이 시스템은 linear일수도 아닐수도, causal일수도 아닐수도, stable할수도 아닐수도 있다.

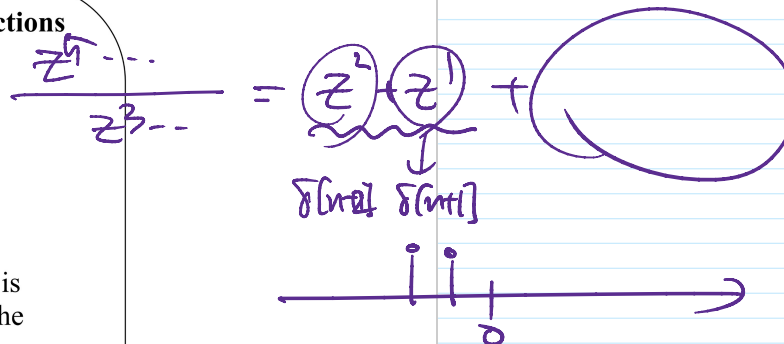
5

Causality for Systems with Rational System Functions

$$H(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \dots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \dots + a_1 z + a_0}$$

$\Downarrow$ No poles at ∞ , if $M \leq N$

A DT LTI system with rational system function $H(z)$ is causal $\Leftrightarrow$ the ROC is the exterior of a circle outside the outermost pole, **including** ∞ . Thus



A DT LTI system with rational system function $H(z)$ is causal $\Leftrightarrow$ the ROC is the exterior of a circle outside the outermost pole, **including** ∞ . Thus

$$N \geq M$$

분자의 차수가 높으면 분자를 분모로 나누었을때, z 의 polynomial이 나온다. 이 들이 non-causal한 term 들을 만들어 낸다.

