EECE233 Signals and Systems

앞서 DTFT의 heuristic derivation을 보았고 이 lecture note에서는 examples와 properties를 배운다

Lecture Note #10

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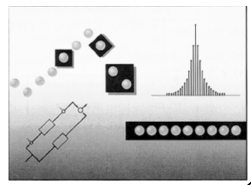
(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

CTFT일때와 유사점, 차이점에 주목하면서 일사천리로 진행한다. 휘릭~

- Covers O & W pp. 361-384
- Examples of the DT Fourier Transform
- Properties of the DT Fourier Transform
- The Convolution Property and its Implications and Uses



* These lecture notes are based on the lecture notes used in 6.003 taught by Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.

DT Fourier Transform Pair

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \qquad \text{— Analysis equation}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \text{— Synthesis equation}$$

Note for DTFT the *FT* and inverse *FT* are not symmetric. One is integration over a finite interval (2π), and the other is summation over infinite terms.

Convergence Issues

Synthesis Equation: None, since integrating over a finite interval

Analysis Equation: Need conditions analogous to CTFT, e.g.

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$ — Finite energy

or

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$ — Absolutely summable

Thm. DT Fourier−Plancherel TF exists if x[n] is an l^2 sequence. The inverse FP TF of it FP TF also exists.

Thm. DTFT exists iff x[n] is an l^1 sequence.

Examples

Parallel with the CT examples in Lecture #8

1) $x[n] = \delta[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n] e^{-j\omega n} = 1$$

2) $x[n] = \delta[n-n_o]$ — shifted unit sample

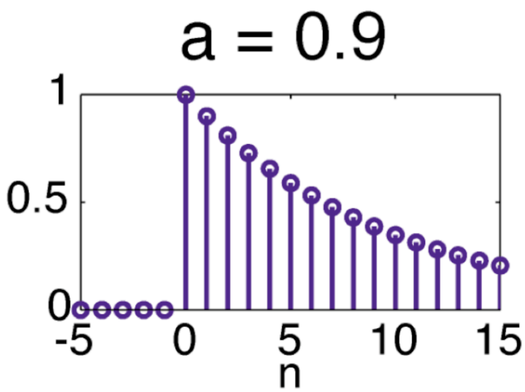
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n-n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

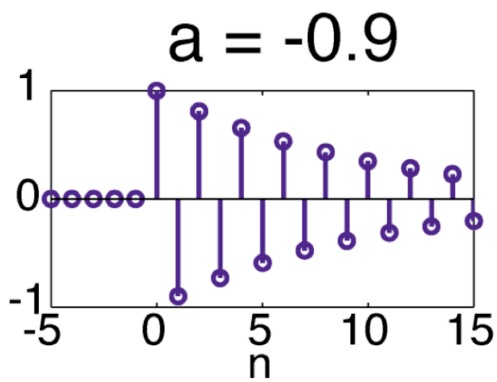
— Same amplitude (=1) as above, but with a *linear* phase $-\omega n_o$.

More Examples

right−sided

3) $x[n] = a^n u[n], \quad |a| < 1$ — Exponentially decaying function





주의.
exponential function z^n 은 모든 DT LTI의 eigenfunction의
후보 이지만 right xor left−sided real exponential은 아니다.

More Examples (cont.) $x[n] = a^n u[n], |a| < 1$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} \underbrace{\left(ae^{-j\omega}\right)}_{|ae^{-j\omega}|<1}{}^{n}$$

$$= \frac{1}{1 - ae^{-j\omega}} = \frac{1}{(1 - a\cos\omega) + ja\sin\omega}$$

Q. CT에서 j\omega의 유리함수였던 것과 유사하게
DT right−sided real exponential도 e^{j\omega} 의 rational
function으로 나온다. 그렇다면 주파수가 서로 다른 두 right−sided
real exponetial의 convolution은 ???

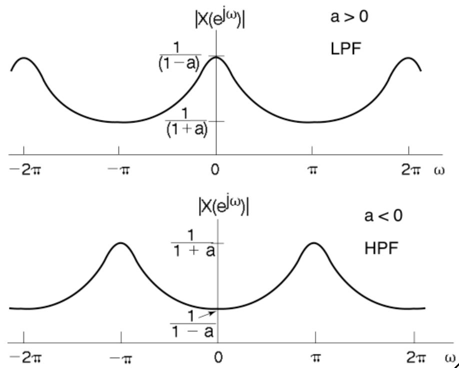
A. 두 right−sided real exponential의 weighted sum이 된다. (주파
수가 같은 경우는 답이 조금 달라지므로 유의한다. 뒤에 다시 나온다.)
왜냐하면 곱이 되었다가 다시 partial fraction expansion하면
weight 만 달라진채 합으로 나오기 때문이다.

More Examples (cont.) $x[n] = a^n u[n], |a| < 1$

$$\left|X(e^{j\omega})\right| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

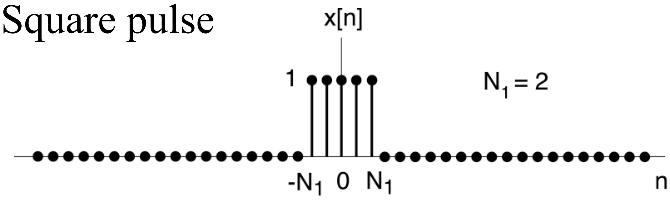
$$\omega = 0: \quad X(e^{j\omega}) = \frac{1}{\sqrt{1 - 2a + a^2}} = \frac{1}{1 - a}$$

$$\omega = \pi: \quad X(e^{j\omega}) = \frac{1}{\sqrt{1 + 2a + a^2}} = \frac{1}{1 + a}$$



Still More

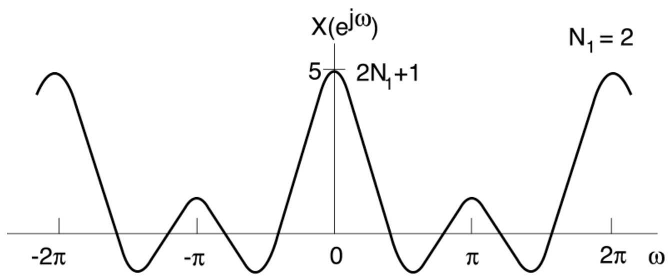
4) DT Square pulse



FIR (Finite-Impulse-Response) LPF

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \sum_{n=-N_1}^{N_1} (e^{-j\omega})^n = \frac{\sin\left[\omega\left(N_1 + 1/2\right)\right]}{\sin[\omega/2]} = X(e^{j(\omega - 2\pi)})$$

Similar to a *sinc*, but periodic with a period 2π.



DT Dirichlet kernel이다.

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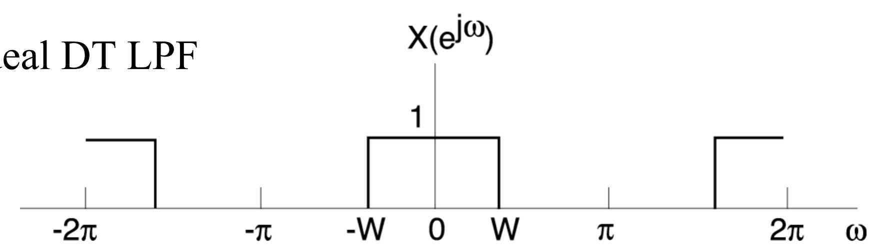
여기서 N_1이 무한대로 가면 주기 2\pi인 comb signal이 얻어진다?!

이것이 DTFT의 key assumption이다.

Still More ...

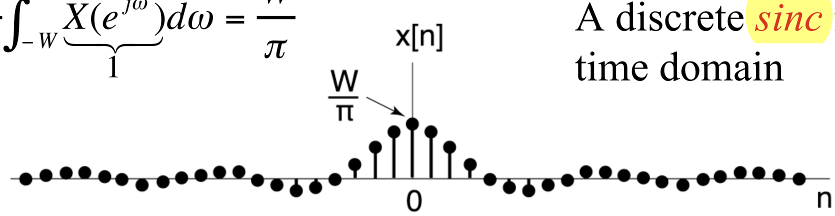
5) Ideal DT LPF



$$x[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n} d\omega = \frac{\sin Wn}{\pi n}$$

$$x[0] = \frac{1}{2\pi}\int_{-W}^{W} \underbrace{X(e^{j\omega})}_{1} d\omega = \frac{W}{\pi}$$

A discrete *sinc* in time domain



즉, continuous sinc이든discrete sinc이든 이를 impulse response로 갖는 LTI system 은 ideal LPF이다.

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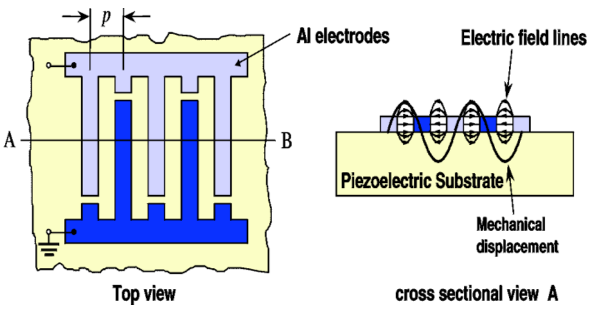
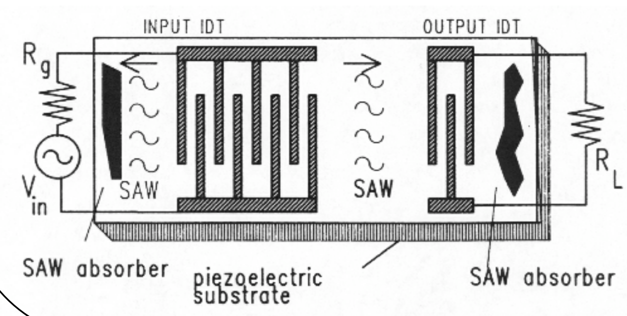
Eg. Compact BPF in your cell phone and TV

SAW (Surface-Acoustic-Wave) filters



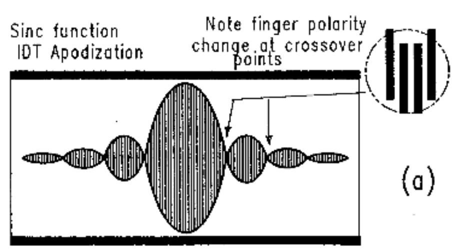


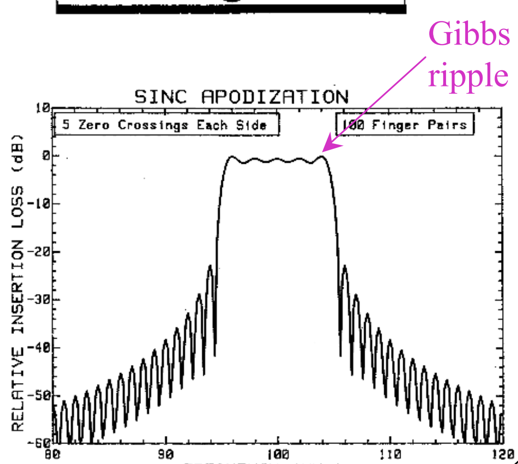
Operation principle:

- Convert electrical signals into sound (acoustic waves).
- At output, pick up the propagating sound signals after a delay $t = x/c_s$, where x is the distance traveled and c_s is the speed of sound.
- This is a good example that time = space with a scaling factor.

SAW Bandpass Filters (cont.)

- Since $h[n]$ is proportional to the overlap length of the electrode pair, we can design and fabricate any $h[n]$ desired. Here we use a modulated *sinc* (multiplied with $\cos\omega_o n$) which results in a bandpass filter.
- Note the passband ripples due to the Gibbs' phenomenon.





CT에서와 마찬가지로 DT에서도 DTFS와 DTFT와의 관계를 알아보자.

DTFT's of Sums of Complex Exponentials

Recall CT result: $x(t) = e^{j\omega_0 t} \longleftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$

What about DT: $x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = ?$

a) We expect an impulse (of area 2π) at $\omega = \omega_o$
b) But $X(e^{j\omega})$ must be periodic with period 2π, In fact

$$X(e^{j\omega}) = 2\pi \sum_{m=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi m)$$

Note: The integration in the synthesis equation is over 2π period, only need $X(e^{j\omega})$ in *one* 2π period. Clearly,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{2\pi \sum_{m=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi m)}_{X(e^{j\omega})} e^{j\omega n} d\omega = e^{j\omega_o n}$$

DTFT of Periodic Signals

$$x[n] = x[n+N]$$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N}$$

Finite sum

From last page:

$$e^{jk\omega_o n} \longleftrightarrow 2\pi \sum_{m=-\infty}^{+\infty} \delta(\omega - k\omega_0 - 2\pi m)$$

$$X(e^{j\omega}) = \sum_{k=<N>} a_k \left[2\pi \sum_{m=-\infty}^{+\infty} \delta(\omega - k\omega_0 - 2\pi m)\right]$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right) \overset{a_k = a_{k+N}}{=} X(e^{j(\omega+2\pi)})$$

↑

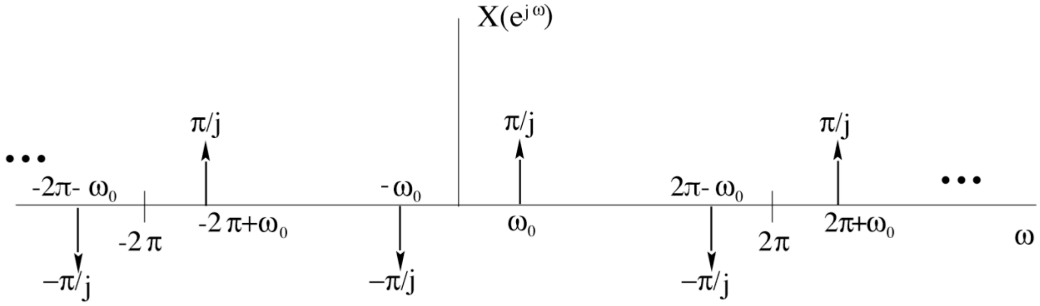
Infinite sum

중요:
주기 DT signal의 DTFS coefficients와 DTFT 의 관계에 대해 확인하자.

Example #1: DT sine function

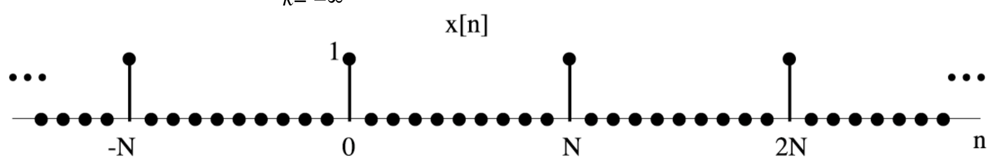
$$x[n] = \sin \omega_0 n = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \frac{\pi}{j} \sum_{m=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi m) - \frac{\pi}{j} \sum_{m=-\infty}^{+\infty} \delta(\omega + \omega_0 - 2\pi m)$$



Example #2: DT periodic impulse train

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN] \qquad \omega_o = 2\pi / T$$

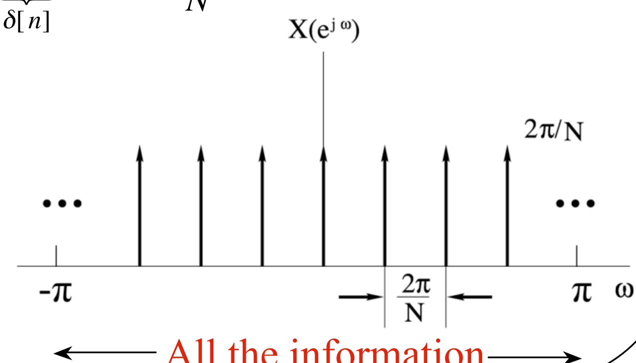


$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{x[n]}_{\delta[n]} e^{-jk\omega_0 n} = \frac{1}{N}$$

$$\Downarrow$$

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

— Sampling function again!



Properties of the DT Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n} \qquad \text{— Analysis equation}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n} d\omega \qquad \text{— Synthesis equation}$$

1) Periodicity: $X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$ — Different from CTFT

2) Linearity: $ax_1[n] + bx_2[n] \longleftrightarrow aX_1\left(e^{j\omega}\right) + bX_2\left(e^{j\omega}\right)$

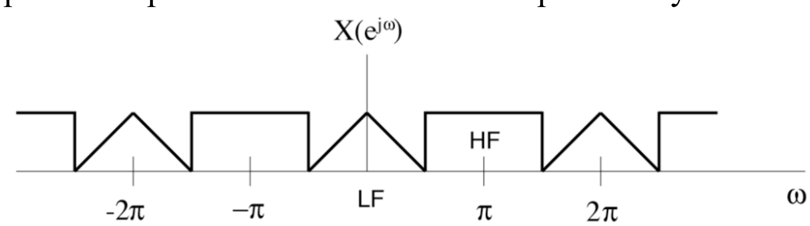
More Properties

3) Time Shifting $x[n-n_0] \longleftrightarrow e^{-j\omega n_0} X\left(e^{j\omega}\right)$

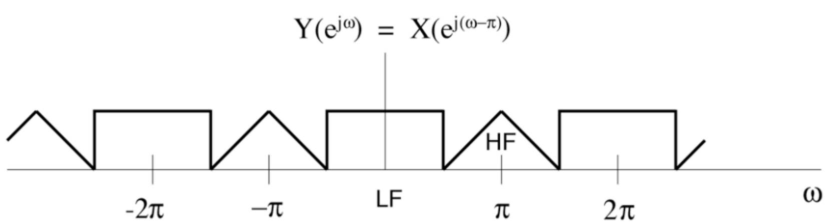
4) Frequency Shifting $e^{j\omega_0 n} x[n] \longleftrightarrow X\left(e^{j(\omega-\omega_0)}\right)$

— Important implications in DT because of periodicity

Example



$\omega_0 = \pi$, $y[n] = e^{j\pi n} x[n] = (-1)^n x[n]$



Still More Properties

5) Time Reversal $x[-n] \longleftrightarrow X(e^{-j\omega})$

6) Conjugate Symmetry

Both are the same as CTFT.

€ $x[n]$ real $\Rightarrow X(e^{-j\omega}) = X^*(e^{j\omega})$

$$\Downarrow$$

$\left|X(e^{j\omega})\right|$ and $\mathcal{R}e\left\{X(e^{j\omega})\right\}$ are even functions

$\angle X(e^{j\omega})$ and $\mathcal{I}m\left\{X(e^{j\omega})\right\}$ are odd functions

and

$x[n]$ real and even $\Leftrightarrow X(e^{j\omega})$ real and even

$x[n]$ real and odd $\Leftrightarrow X(e^{j\omega})$ purely imaginary and odd

Is There No End to These Properties?

7) Differentiation in Frequency

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$\frac{d}{d\omega}X(e^{j\omega}) = -j\sum_{n=-\infty}^{+\infty} nx[n]e^{-j\omega n}$$

$$\Downarrow$$

$$nx[n] \longleftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$$

Differentiation in frequency

8) Parseval's Relation

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi}\left|X(e^{j\omega})\right|^2 d\omega$$

Total energy in time domain

Total energy in frequency domain

Def. CT에서와 마찬가지로 energy spectral density라 불린다.

The Convolution Property

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

$$\Downarrow$$

Frequency response $H\left(e^{j\omega}\right)$ = DTFT of the unit sample response

Example #1: $x[n] = e^{j\omega_0 n} \longleftrightarrow X\left(e^{j\omega}\right) = 2\pi \sum_{k=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi k)$

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right) \cdot 2\pi \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \omega_0 - 2\pi k\right)$$

$$= 2\pi \sum_{k=-\infty}^{+\infty} H\left(e^{j(\omega_0 + 2\pi k)}\right) \delta\left(\omega - \omega_0 - 2\pi k\right)$$

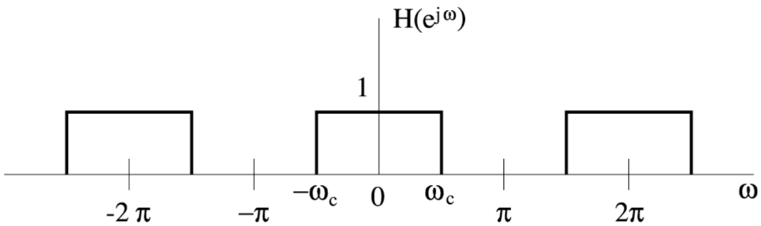
$$\overset{H \text{ periodic}}{=} H\left(e^{j\omega_0}\right) \cdot 2\pi \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \omega_0 - 2\pi k\right)$$

$$\Downarrow$$

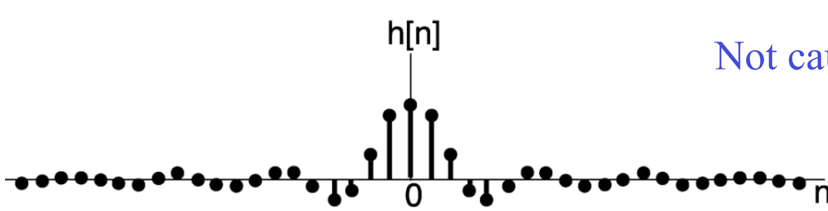
$$\boxed{y[n] = H\left(e^{j\omega_0}\right) e^{j\omega_0 n}}$$

Now, we have a crude tool to design a digital filter.
1. Find the frequency response of the desired filte
2. Apply Inverse DTFT to find h[n]. Done!
MATLAB에서는 h[n]을 저장하여 convolution하는 것으로 구현할 수 있는데, 이렇게 software적으로 하면 느리다. 빠른 system을 구현하려면 hardware적으로 구현해야한다. 어떻게?????

Reminder: Ideal Lowpass Filter



$$h[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{\sin \omega_c n}{\pi n}$$

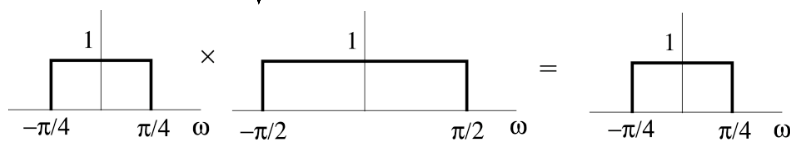


Not causal!

Example #3:

$$\frac{\sin(\pi n/4)}{\pi n} * \frac{\sin(\pi n/2)}{\pi n} = ?$$

$$\updownarrow \qquad\qquad\qquad\qquad \uparrow$$



Time Expansion

Time Expansion
Recall CT property: $x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\left(\frac{\omega}{a}\right)\right)$

Time scale is infinitesimally fine

a>1이면 빠르게 play...

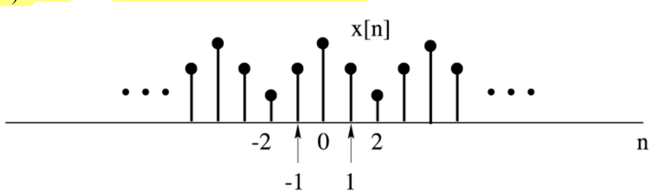
But in DT: $x[n/2]$ makes no sense for odd n
$x[2n]$ misses odd values of $x[n]$

이 차이를 주목하자.

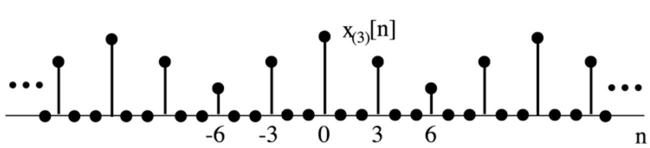
Def. DT downsampling이라 한다. 여기서는 2x downsampling.

But we can "slow" a DT signal down by inserting zeros:
k — an integer ≥ 1
$x_{(k)}[n]$ — insert (k - 1) zeros between successive values

Def. DT upsampling이라 부른다.



Insert two zeros in this example



Q. time−domain상에서 느려지기는 하는데 frequency−domain상에서는 어떻게 될까? CT에서와 어떻게 다를까?

Time Expansion (continued)

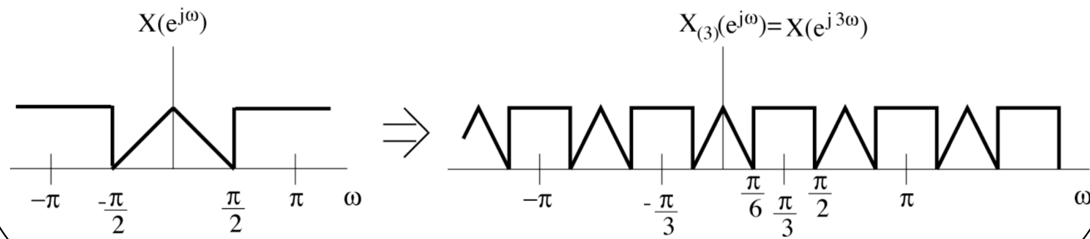
$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k \\ 0 & \text{otherwise} \end{cases}$$

— Stretched by a factor of k in time domain

$$X_{(k)}\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} \overset{n=mk}{=} \sum_{m=-\infty}^{+\infty} x_{(k)}[mk]e^{-j\omega mk}$$

$$= \sum_{m=-\infty}^{+\infty} x[m]e^{-j(k\omega)m} = X\left(e^{jk\omega}\right)$$

— compressed by a factor of k in frequency domain



zero insertion을 하면 느려는 지는데 그 외의 효과는 없나?
MATLAB으로 소리를 들어보자.

Def. DT interpolation = DT upsampling followed by LP filtering