EECE233 Signals and Systems

Lecture Note #11

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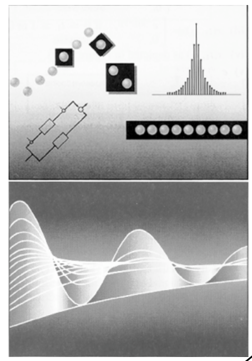
(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

Covers O & W pp. 385-427

- DTFT Properties and Examples
- Duality
- Magnitude/Phase of Fourier transforms



* These lecture notes are based on the lecture notes used in 6.003 taught by Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.

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Convolution Property Example

Example 5.13 (O & W p385)

$$h[n] = \alpha^n u[n], \qquad x[n] = \beta^n u[n], \quad |\alpha|, |\beta| < 1$$

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}, \quad X(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}$$

$$\Downarrow$$

$$y[n] = h[n] * x[n] \longleftrightarrow Y(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}} \cdot \frac{1}{1-\beta e^{-j\omega}}$$

– ratio of polynomials in $v = e^{-j\omega}$

$$\beta \neq \alpha : \quad Y(e^{j\omega}) \overset{PFE}{=} \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$$

A, B – determined by partial fraction expansion

$$y[n] = A\alpha^n u[n] + B\beta^n u[n].$$

$$\beta = \alpha : \quad Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

$$y[n] = ? \xrightarrow{ny[n] \leftrightarrow j\frac{dY(e^{j\omega})}{d\omega}} y[n] = (n+1)\alpha^n u[n]$$

2

Q. 왜 이 조건들이 필요한가?
A. 이 조건들이 violate되면 DTFT들이 존재하지 않는다.

앞서 CT에서는 j\omega 의 rational function이었다.
이제는 e^(−j\omega)의 rational function이다.
주의!!
e^(+j \omega)의 rational function으로 보지 않는다.

CT에서 이와 parallel한 것을 기억해 보자.

Q. 앞서 설명했듯, 주파수영역에서 원하는 시스템을 설계하여 inverse DTFT하여 구한 impulse response를 저장하여 software적으로 DT LTI system을 구현하는 방법은 저가의 빠른 시스템을 구현하는데 적합치 않다.
CT에서는 RLC 그리고 op-amp등을 써서 (또는 their equivalents를 써서) approximately 구현했는데 DT에서는 어떻게 하면 될까?
A. We use delay components (shift registers), multipliers, and adders.

DT LTI System Described by LCCDE's

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

From time-shifting property: $x[n-k] \longleftrightarrow e^{-jk\omega} X(e^{j\omega})$

$$\Downarrow$$

$\mathcal{FT}$ on both sides: $\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$

$$\Downarrow$$

Devide both sides by $\sum_{k=0}^{N} a_k e^{-jk\omega}$: $Y(e^{j\omega}) = \underbrace{\left[\frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}} \right]}_{H(e^{j\omega})} X(e^{j\omega})$

— Rational function of $e^{-j\omega}$, use PFE to get $h[n]$.

주의. LCCDE로 input output relation이 나타난다고 해서 모두 linear system은 아니다. zero input zero output이 만족되도록 initial conditions가 잘 정해져야 한다.

주의: 여기서는 x[n]과 y[n]의 DTFT가 모두 존재한다는 가정하에 변환을 하였다.

모순되는 결과(예를 들어 h[n]이 absolutely 또는 square summable하지 않게 나오는 경우)가 생기면 가정이 잘못된 것이다.

LCCDE에서 과거의 출력이 현재의 y[n]에 영향을 준다. = recursive system이다.

Example: First-order *recursive* system

$$y[n] - \alpha y[n-1] = x[n], \qquad |\alpha| < 1$$

with the condition of initial rest $\Leftrightarrow$ *causal*

$\mathcal{FT}$ on both sides: $(1 - \alpha e^{-j\omega}) Y(e^{j\omega}) = X(e^{j\omega})$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \alpha e^{-j\omega}}$$

$$\Updownarrow$$

$h[n] = \alpha^n u[n]$ – infinite duration

first-order=y[n-1]이 영향을 준다.

Q. 좌측과 같이 I/O relation이 LCCDE로 나타나는 causal DT LTI system의 impulse response를 구하시오.

Recall that a linear system is causal iff initial rest.

A. h[n] = ...

이렇게 impulse response가 infinite duration을 가지는 filter (DT system)을 IIR (Infinite Impulser Response) filter라 한다.

Q. 위 문제에서 causal 조건이 없으면 어떻게 될까?
A. $h[n] = -a^n u[-n-1]$ 도 impulse response 로서 가능하다.!!!!!

이 impulse response는 여전히 LCCDE를 만족한다. 즉, LCCDE가 주어지면 무조건 causal system 인 것은 아니다.

1. adder, constant multiplier, delay components로 구성된 discrete-time circuit => causal하며 LCCDE로 나타남.

그러나

2. LCCDE로 linear 함 does not necessarilly imply causal system.

Remind that

x(t) and y(t) are periodic with same period

=>

the periodic convolution between x(t) and y(t) has the CTFS coefficients that are term by term products of the CTFS coefficients of x(t) and y(t)

DTFT Multiplication Property

$$y[n] = x_1[n]\cdot x_2[n] \longleftrightarrow Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega}) \quad - \text{ periodic convolution}$$

각각의 DT 신호의 DTFT는 주기 2\pi인 주기신호이고 곱의 DTFT는 이들 주기신호의 periodic convolution이다.

Proof :
$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_1[n]\cdot x_2[n]e^{-j\omega n}$$

Use synthesis Eq. for $x_1[n]$:
$$= \sum_{n=-\infty}^{\infty} \underbrace{\left(\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\right)}_{x_1[n]} x_2[n]e^{-j\omega n}$$

Exchange order of $\sum$ and $\int$:
$$= \frac{1}{2\pi}\int_{2\pi}\left\{X_1(e^{j\theta}) \underbrace{\sum_{n=-\infty}^{\infty} x_2[n]e^{-j(\omega-\theta)n}}_{X_2(e^{j(\omega-\theta)})}\right\}d\theta$$

$$= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \qquad \text{QED}$$

비교하면,
DT의 경우 TD에서의 convolution은 FD에서의 곱이고 TD에서의 곱은 FD에서의 periodic convolution이다.
CT의 경우 TD에서의 convolution은 FD에서의 곱이고 TD에서의 곱은 FD에서의 convolution이다.
공통점과 차이점에 유의!!!

Calculating Periodic Convolutions

Suppose we integrate from -π to π:

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

can be written as :
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

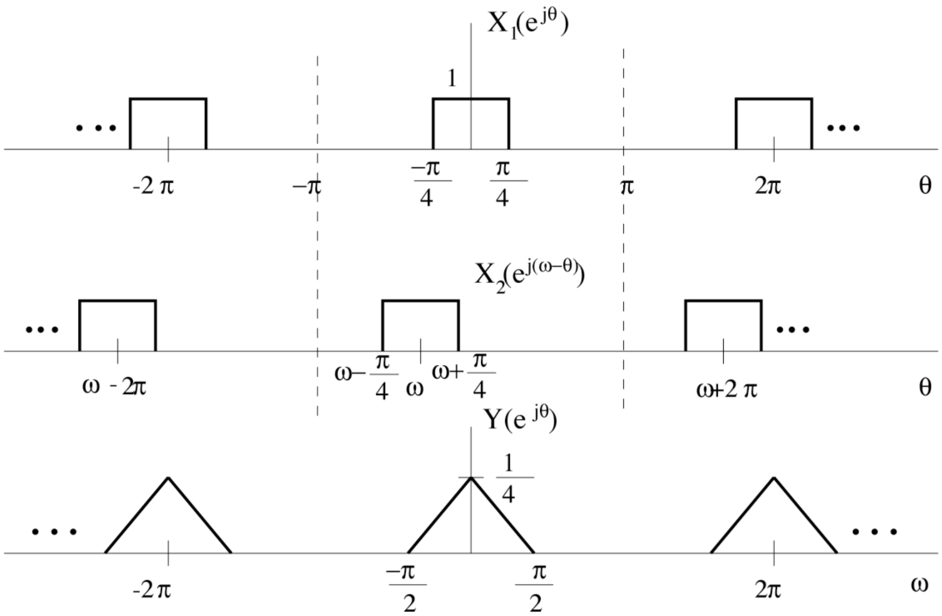
∈
Changes a periodic convolution to a regular convolution.

where

$$\hat{X}_1(e^{j\theta}) = \begin{cases} X_1(e^{j\theta}) & , \quad |\theta| \le \pi \\ 0 & , \quad \text{otherwise} \end{cases}$$

Example: $y[n] = \left[\dfrac{\sin(\pi n/4)}{\pi n}\right]^2 = x_1[n]\cdot x_2[n]\ ,\quad x_1[n] = x_2[n] = \dfrac{\sin(\pi n/4)}{\pi n}$

$$Y(e^{j\omega}) = \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega})$$



DTFT이야기

이제는 CT/DT FS/FT이야기

Duality in Fourier Analysis

Fourier Transform is highly *symmetric*

duality = '쌍대성'

CTFT: *Both* time and frequency are continuous and in general *aperiodic*

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$$

Same operation

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt$$

$$\Downarrow$$

Suppose $f(*)$ and $g(*)$ are two functions such that they are a Fourier-transform pair (***Eg***. f is a square and g is a *sinc*),

$$x_1(t) = f(t) \leftrightarrow X_1(j\omega) = g(\omega).$$

만약에 f(t)의 CTFT가 g(\omega)라면

Then for another time-domain function such that,

$$x_2(t) = g(t)$$

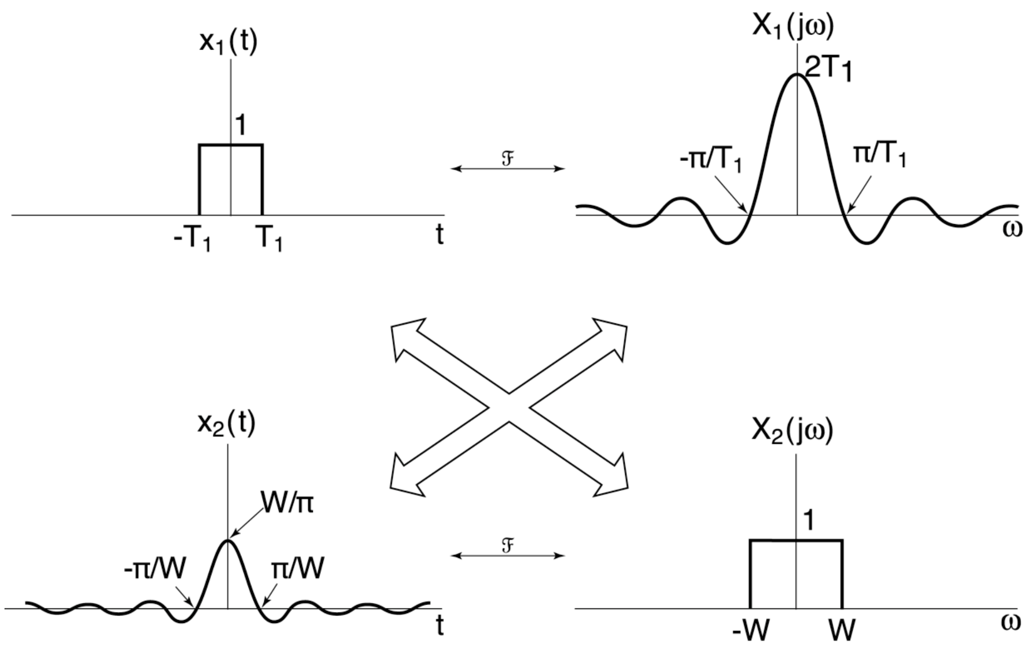
then without doing any math, we should know that

$$X_2(j\omega) = 2\pi f(-\omega)$$

g(t)의 CTFT는 2\pi f(−\omega)이다.

Example of CTFT duality

Square pulse in either time or frequency domain



Duality between CTFS and DTFT

CTFS

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = x(t+T) \qquad \omega_o = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

Periodic in time $\longleftrightarrow$ *discrete* in frequency

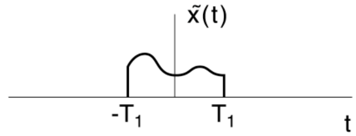
DTFT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

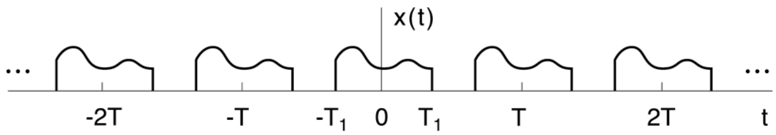
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} = X(e^{j(\omega+2\pi)})$$

Discrete in time $\longleftrightarrow$ periodic in frequency

Illustration: *Periodic* in time $\longleftrightarrow$ *discrete* in frequency

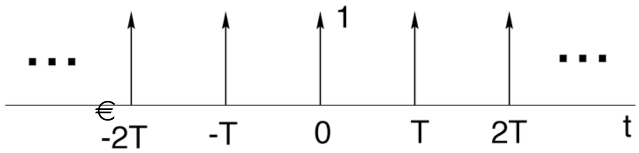


periodic extension in TD ...



It is clear that $x(t) = \tilde{x}(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT)$ — sampling function

$$\sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Any periodic signal $x(t)$ is a convolution of $x(t)$ in one period with

$$\sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$\Downarrow$

Illustration (continued):

$$\underbrace{X(j\omega)}_{\text{Also discrete}} = \tilde{X}(j\omega) \times \underbrace{\sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta\left(\omega - \frac{k2\pi}{T}\right)}_{\text{Discrete}}$$ — also a sampling function

Also discrete

Discrete

$$= \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \tilde{X}\left(j\frac{k2\pi}{T}\right) \delta\left(\omega - \frac{k2\pi}{T}\right)$$

Poisson sum Formula

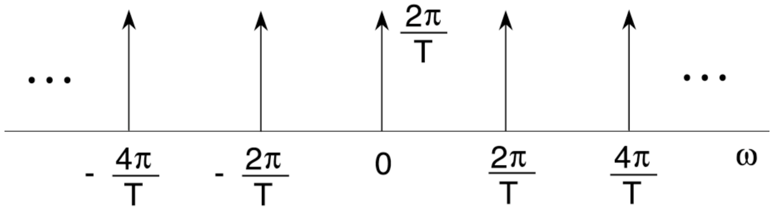
$$\xleftrightarrow{FT} \quad x(t) = \sum_{k=-\infty}^{\infty} \frac{\tilde{X}\left(j\frac{k2\pi}{T}\right)}{T} e^{j\frac{k2\pi}{T}t}$$

sampling function의 CTFS는 1/T times sum of all harmonics.
따라서 CTFT는 2\pi/T sum of delta functions with period 2\pi/T

Illustration (continued):

$$\underbrace{X(j\omega)}_{\text{Also discrete}} = \tilde{X}(j\omega) \times \underbrace{\sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta\left(\omega - \frac{k2\pi}{T}\right)}_{\text{Discrete}}$$ — also a sampling function

$$\sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta\left(\omega - \frac{k2\pi}{T}\right)$$



From the symmetry of Fourier Transform, the reverse is also true:

$$\textit{Discrete} \text{ in time} \xleftrightarrow{\text{DTFT}} \textit{periodic} \text{ in frequency}$$

DTFS

Discrete & *periodic* in time $\longleftrightarrow$ *periodic* & *discrete* in frequency

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = x[n+N] \qquad \omega_o = \frac{2\pi}{N}$$

$$a_k = \frac{1}{N}\sum_{k=<N>} x[n] e^{-jk\omega_0 n} = a_{k+N}$$

Same mathematical operation

$$\Downarrow$$

Suppose $f[*]$ and $g[*]$ are two functions such that they are a **FT** pair,

$$x_1[n] = f[n] \leftrightarrow a_k = g[k].$$

Then for another time-domain function such that,

$$b_k \leftrightarrow x_2[n] = g[n]$$

then without doing any math, we should know that

$$b_k = f[-k]/N$$

f[n]의 DTFS coefficient가 g[k]이면

g[n]의 DTFS coefficient는 f[−k]/N이다.

Magnitude and Phase of FT, and Parseval Relation

No phase information in the TD and FD.

CT:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = |X(j\omega)| e^{j\angle X(j\omega)}$$

Parseval Relation:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} \frac{1}{2\pi} \underbrace{|X(j\omega)|^2}_{\text{Energy distribution in } \omega} d\omega$$

DT:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\angle X(e^{j\omega})}$$

Parseval Relation:

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \int_{2\pi} \frac{1}{2\pi} |X(e^{j\omega})|^2 d\omega$$

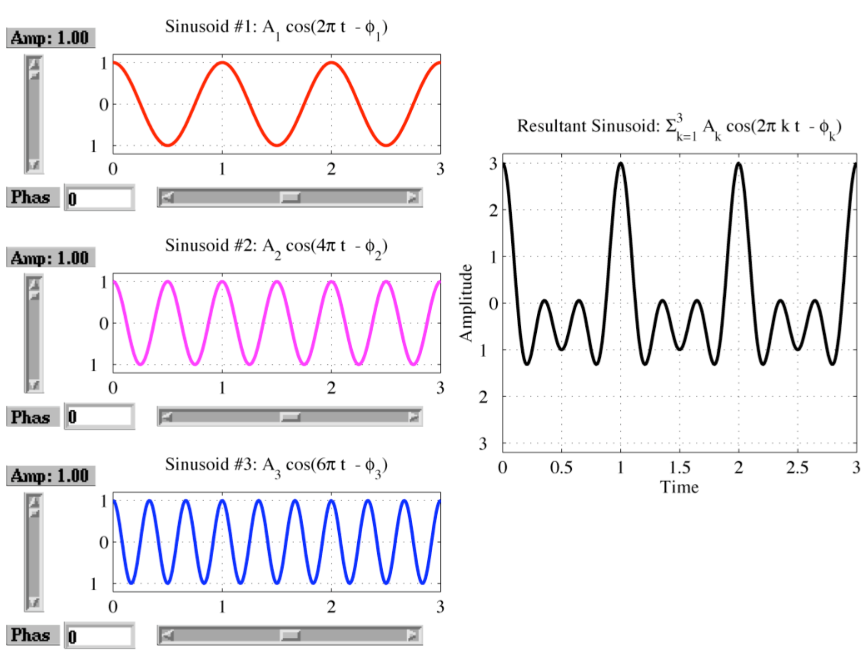
Effects of Phase

- *Not* on signal energy distribution as a function of frequency
- *Can* have dramatic effect on signal shape/character
  — Constructive/Destructive interference
- Is that important?
  — Depends on the signal and the context, not as important as the amplitude in hearing, but very important for image processing.

전혀 관계없지는 않다. 예를 들어 linear phase shift 가 일어나면 signal이 delay가 되고 이는 우리 귀가 알아챈다.

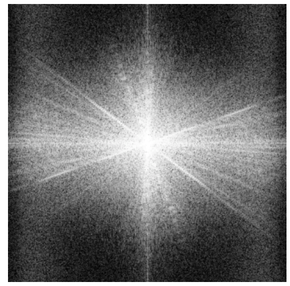
다만 우리귀는 short time Fourier transform을 하는 것으로 생각할 수 있고 이렇게 짧은 시간동안 서로 다른 phase를 가진 tone신호들이 들어오면 phase는 알아듣지 못한다.

Demo: 1) Effect of phase on Fourier Series

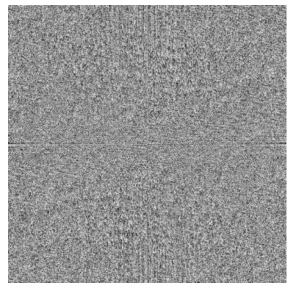


Demo: 2) Effect of phase on image processing

Original image of a boat

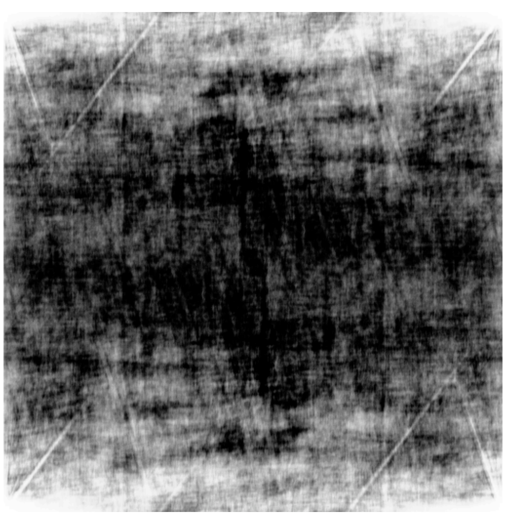


Amplitude of $\mathcal{FT}$



Phase of $\mathcal{FT}$

Demo:2) Effect of phase on image processing (cont.)



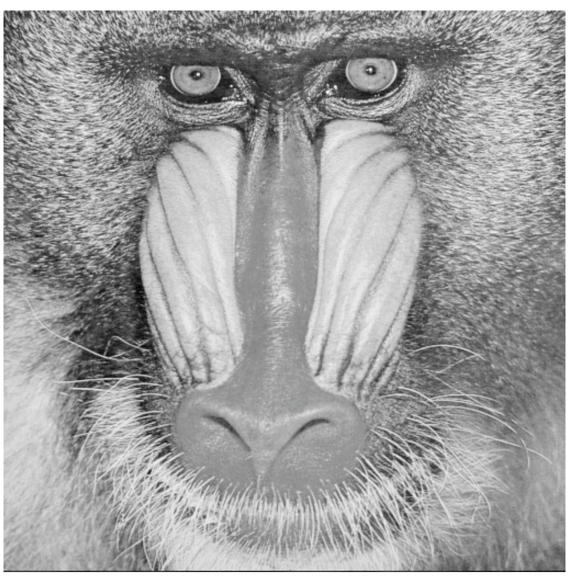
Constructed with the amplitude of $\mathcal{FT}$ of the boat, but with a uniform phase.

Constructed with the phase of $\mathcal{FT}$ of the boat, but with a uniform amplitude.

Demo:2) Effect of phase on image processing (cont.)

This image was constructed with the phase of $\mathcal{FT}$ of the boat, but with amplitude from a very different image — you wouldn't have guess it.

Demo:2) Effect of phase on image processing (cont.)



The amplitude information for the previous image is from our old friend! Cannot make a monkey out of a boat!
Next lecture covers O & W pp. 427-472.