EECE233 Signals and Systems

Lecture Note #12

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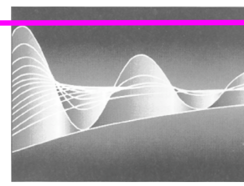
(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

- Covers O & W pp. 427-472
- Log-Magnitude and Phase
- Linear and Nonlinear Phase
- Ideal and Nonideal Frequency Response
- CT and DT Rational Frequency Responses
- DT First- and Second-Order Systems



* These lecture notes are based on the lecture notes used in 6.003 taught by Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.

이 lecture note 후반부에서는
LCCDE로 나타나는 CT LTI system과 DT LIT system에서 coefficient들을 어떻게 선택해야 원하는 response가 나타나는지에 대해 정성적(qualitatively)으로 배운다.

Log-Magnitude and Phase

$$x(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$\Downarrow$$

$$|Y(j\omega)| = |H(j\omega)| \cdot |X(j\omega)|$$

or: $\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$

and: $\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$

Cascading: $\longrightarrow \boxed{H_1(j\omega)} \longrightarrow \boxed{H_2(j\omega)} \longrightarrow$

$$\left.\begin{aligned} \log|H(j\omega)| &= \log|H_1(j\omega)| + \log|H_2(j\omega)| \\ \angle H(j\omega) &= \angle H_1(j\omega) + \angle H_2(j\omega) \end{aligned}\right\} \text{Easy to add}$$

magnitude response 대신 log magnitude response를 쓰기도 한다. 장점은, convolution property에 의해 output log magnitude는 input log magnitude 에 impulse response의 log magnitude의 '합'이 된다는 것이다. 그리고 phase response는 본래부터 '합'이 되었다.

(이렇게 곱을 합으로 나타낼 수 있기 때문에 log가 popular해 졌었다. 지금은 쓰이지 않지만, '계산자'라는 것을 이용하면 곱셈을 간단히 계산할 수 있다. 1950-60년대에는 공대에서 계산자를 사용한 계산대회가 열리기도 했었다.)

Plotting Log-Magnitude and Phase – Bode Plot

'보우디'라고 읽는다.

a) For real-valued signals and systems

$$\left.\begin{aligned} |H(-j\omega)| &= |H(j\omega)| \\ \angle H(-j\omega) &= -\angle H(j\omega) \end{aligned}\right\} \Rightarrow$$ Plot for $\omega \geq 0$, often with a *logarithmic* scale for frequency in CT

CT에서는 주파수도 log scale로 나타내기도 한다.

b) In DT, need only plot for $0 \leq \omega \leq \pi$ (with *linear* scale)

c) For historical reasons, log-magnitude is usually plotted in units of *decibels* (dB):

$$(1\ bel = 10\ decibels = \frac{\text{output power}}{\text{input power}} = 10)$$

1 meter = 100 centi meters인 것과 같은 원리.

$$20\log_{10}|H(j\omega)|$$

e.g.

$|H(j\omega)| = 1 \longrightarrow 0\,\text{dB}$

$|H(j\omega)| = \sqrt{2} \longrightarrow \sim 3\,\text{dB}$

$|H(j\omega)| = 2 \longrightarrow \sim 6\,\text{dB}$

$|H(j\omega)| = 10 \longrightarrow 20\,\text{dB}$ — 2 *bels*, 100 power gain

$|H(j\omega)| = 100 \longrightarrow 40\,\text{dB}$

반드시 기억: 3 dB의 power 또는 energy차이는 2배의 power 또는 energy차이이다.
10 log10 (2) = 10* 0.3010 = 3.010
따라서 power 2배는 약 3 dB차이를 만든다.

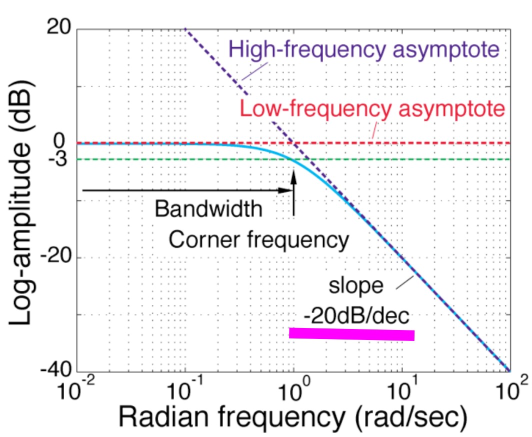
참고로, 음계에서 한 octave차이는 주파수로 2배이다. -> MATLAB 피아노 만들어보자. CTFS해 보면 악기별로 harmonic 성분들이 다르다.

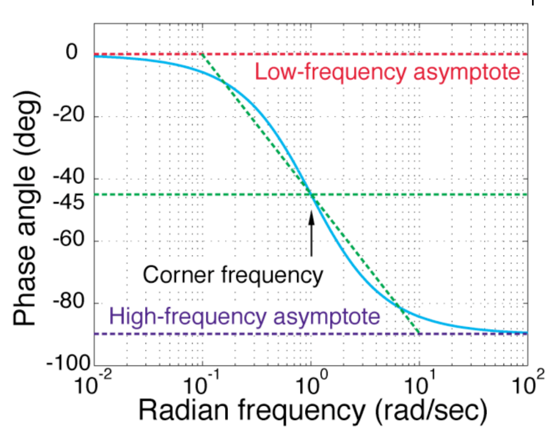
Approximate Bode plots for insightful analysis

For a frequency response $\quad H(j\omega) = \dfrac{1}{j\omega + 1}$

For small \omega, H(jomega) \approx 1

For large \omega, H(j\omega) \approx 1/(jomega)





1에 log를 취하면 0.

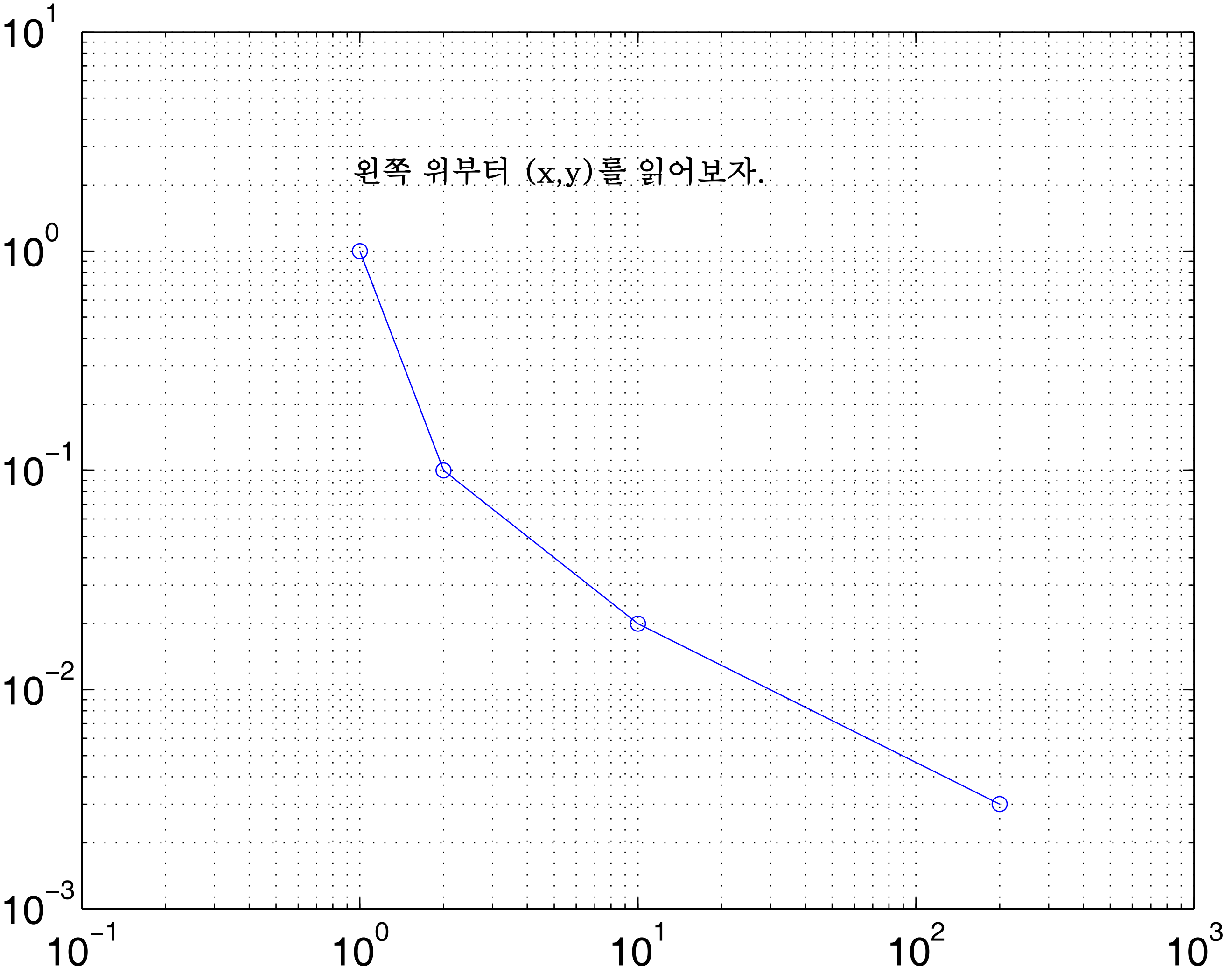
Q. h(t)는 ?

Both the amplitude and phase of $H(j\omega)$ change rapidly around the "corner frequency" $\omega = 1$.

MATLAB에서 plot(x,y) 대신 semilogx(x,y), semilogy(x,y), loglog(x,y) 명령을 쓰면 된다.

linear-linear에서는 쌍곡선loglog 에서는 기울기가 음수인 직선

10배되면 20dB떨어진다. log magnitude가 20 dB per decade로 감소한다. 즉 주파수 10배마다 파워 또는 에너지가 20 dB (즉 100배씩) 감소한다.



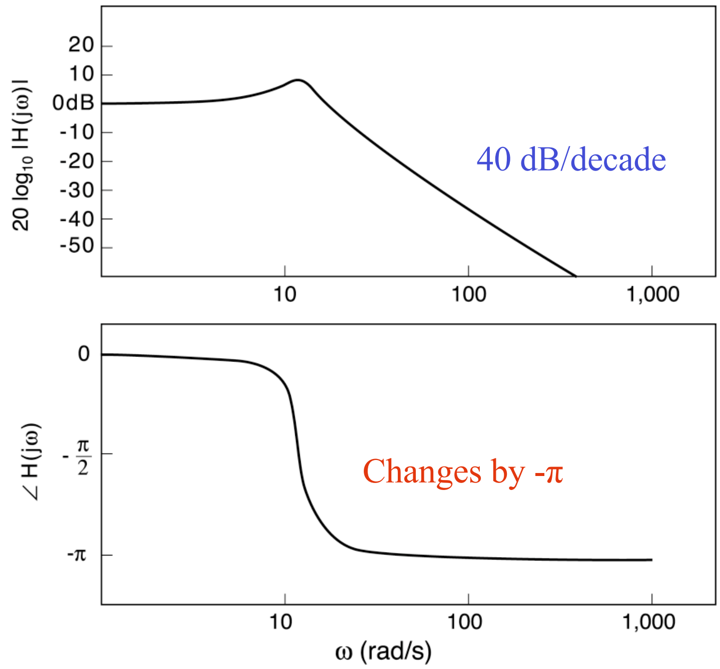
왼쪽 위부터 (x,y)를 읽어보자.
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}

Q. 10^0과 10^1사이의 점선의 갯수는? (양끝제외)

A Typical Bode plot for a second-order system

$20\log|H(j\omega)|$ and $\angle H(j\omega)$ vs. $\log\omega$

E.g. LCR circuit



여기서 주목할 것은,
first order
2nd order system들이
각각
−20 dB/decade
−40 dB/decade
등 loglog plot에서
직선으로 나타난다는 것
이다.

그리고 이들 시스템을
cascade하면 log
magnitude는 더해지는
데 직선은 더해도 직선이
므로 log−magnitude로
생각하면 합성 시스템의
response를 쉽게 이해
할 수 있게 된다는 장점
이 있다.

Linear Phase

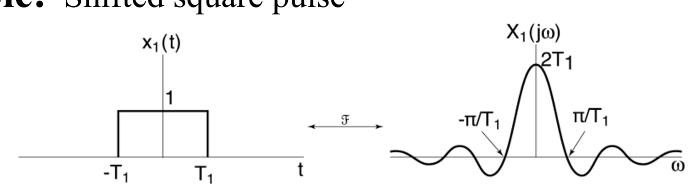
CT $\quad x(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$

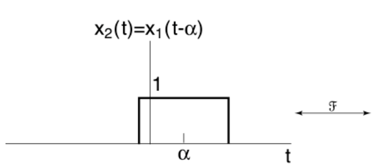
$$H(j\omega) = e^{-j\omega\alpha} \Rightarrow |H(j\omega)| = 1\ ,\ \angle H(j\omega) = -\alpha\omega \quad \text{— Linear in } \omega$$

$$Y(j\omega) = e^{-j\omega\alpha}X(j\omega) \xleftrightarrow{time-shift} y(t) = x(t-\alpha)$$

Result: Linear phase ⇔ simply a rigid shift in time, *no* distortion
Nonlinear phase ⇔ distortion as well as shift

Example: Shifted square pulse





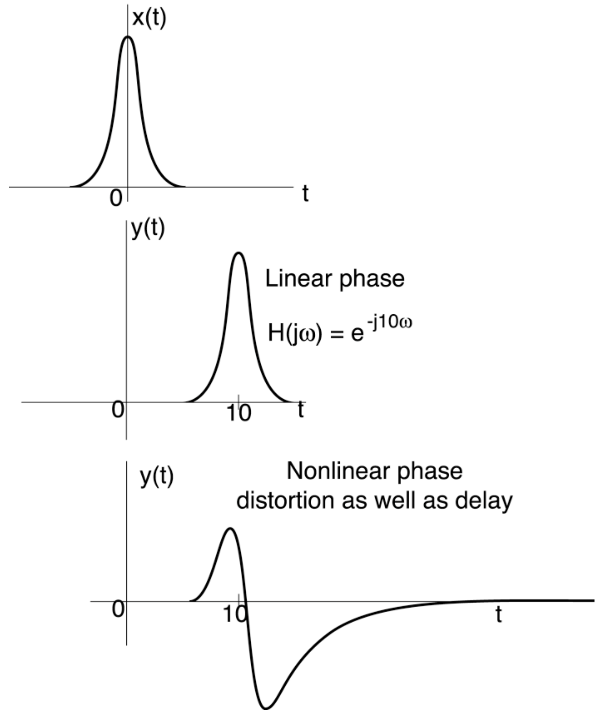
? $\quad X_2(j\omega) = X_1(j\omega)e^{-j\omega\alpha}$

Effect of Linear and Nonlinear Phase on the Output Signal

Input

Output from a system with linear phase *E.g.* SAW devices and telecom systems.

Output from a system with nonlinear phase



Linear Phase in DT systems

DT $\quad y[n] = x[n - n_0] \longleftrightarrow Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$

$$H(e^{j\omega}) = e^{-j\omega n_0} \Rightarrow \left|H(e^{j\omega})\right| = 1 \ , \ \angle H(e^{j\omega}) = -n_0\omega$$

Example: *FIR* (Finite-Impulse Response) filters

$$h[n] = \sum_{k=-N_1}^{N_2} a_k \delta[n-k] \qquad - \quad \text{Finite duration}$$

A special case : $\quad N_1 = 0, N_2 = 2n_o, a_k = 1, \longrightarrow$ A $2n_o + 1$ – point averager

$$h[n] = \sum_{k=0}^{2n_o} \delta[n-k] \overset{FT}{\longleftrightarrow} \frac{\sin[\omega(n_o + 1/2)]}{\sin(\omega/2)} e^{-j\omega n_o}$$

$$\uparrow$$

Note: $\quad h[n+n_o] = \sum_{k=-n_o}^{n_o} \delta[n-k] \overset{FT}{\longleftrightarrow} \dfrac{\sin[\omega(n_o + 1/2)]}{\sin(\omega/2)}$

Question: What about $H(e^{j\omega}) = e^{-j\omega\alpha}, \ \alpha \neq$ integer?

FIR filter의 antonym은 IIR filter 이다.
IIR = infinite impulse response

ex/ h[n]=alpha^n u[n]

Q. TD에서 어떤 신호인지 알기위해 inverse transform해 보면?
A. H(e^{j \omega})는 반드시 \omega의 주기함수로 주기 2\pi여야 한다. 따라서 e^{-j \omega \alpha} 에서 \alpha가 integer가 아니면 DTFT가 아니다.

All-Pass Systems

$(|H(j\omega)| = 1 = |H(e^{j\omega})|)$

CT

$$H(j\omega) = e^{-j\alpha\omega} \qquad \text{– Linear phase}$$

$$H(j\omega) = \frac{a - j\omega}{a + j\omega} \qquad \text{– Nonlinear phase}$$

$$\text{but all - pass} \quad |H(j\omega)| = \sqrt{\frac{a^2+\omega^2}{a^2+\omega^2}} = 1$$

DT

$$H(e^{j\omega}) = e^{-j\omega n_o} \qquad \text{– Linear phase}$$

$$H(e^{j\omega}) = \frac{1 - e^{-j\omega}}{1 - e^{j\omega}} \qquad \text{– Nonlinear phase}$$

$$\text{but all - pass} \quad |H(e^{j\omega})| = \sqrt{\frac{(1-\cos\omega)^2 + \sin^2\omega}{(1-\cos\omega)^2 + \sin^2\omega}} = 1$$

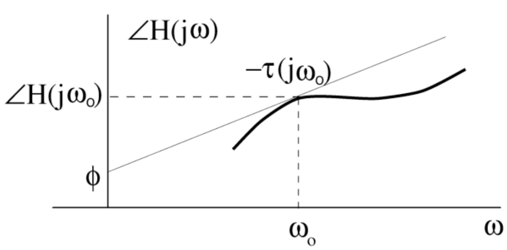
all−pass system = frequency flat filters such as identity, linear phase, nonlinear−phase systems

<<−−> >

frequency selective filters including LPF, BPF, HPF

How do we handle signal delay when the phase is nonlinear?

Group Delay



When the signal is narrowband, then

$$-\frac{d\angle H(j\omega)}{d\omega} \text{ instead of } -\frac{\angle H(j\omega)}{\omega}$$

reflects the time delay.

For frequencies “near” ω_o

$$\angle H(j\omega) \approx \angle H(j\ \omega) - \tau(\omega_o)(\omega - \omega_o) = \phi - \tau(\omega_o)\cdot\omega$$

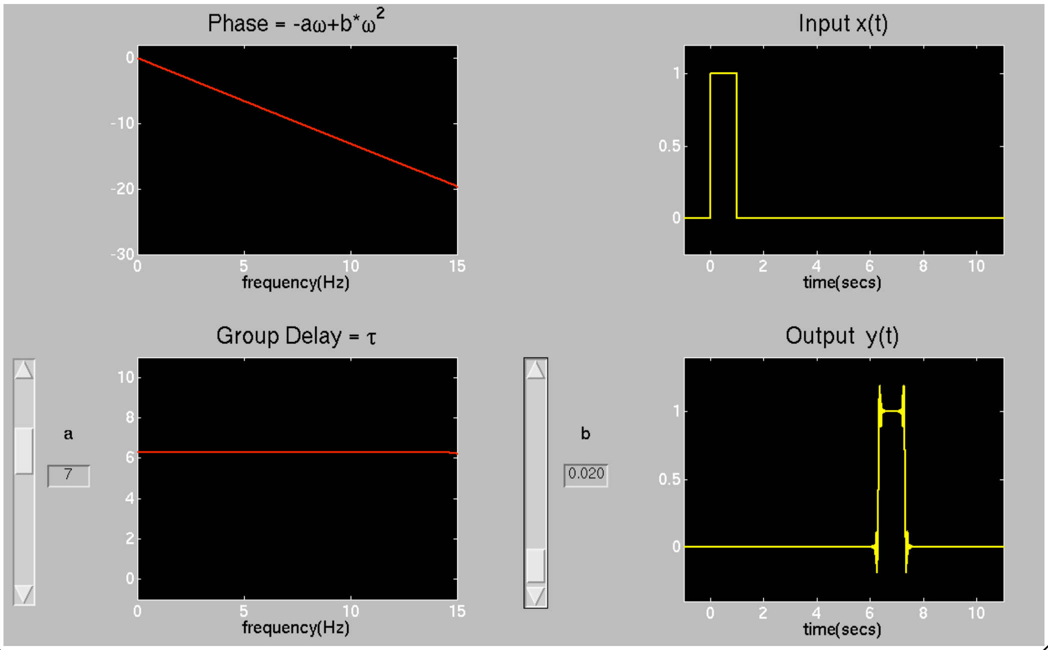
$$\tau(\omega_o) = -\frac{d}{d\omega}\left[\angle H(j\omega)\right]\Big|_{\omega=\omega_o} = \textbf{Group Delay}$$

$$\Downarrow$$

$$H(j\omega) = \underbrace{\left(|H(j\omega)|\, e^{j\phi}\right)}_{\text{Distortion of the signal}} \underbrace{e^{-j\tau(\omega_o)\omega}}_{\text{Delay by } \tau(\omega_o)}$$

where phi = angle H(j\omega_0 + \tau(\omega_0) \omega_0 이다.

Demo: All-pass system $H(j\omega) = e^{j\angle H(j\omega)}$,
and $\angle H(j\omega) = -a\omega + b\omega^2$.



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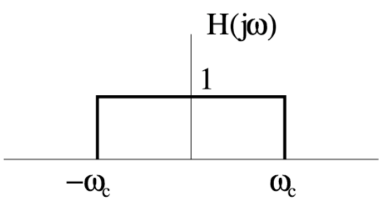
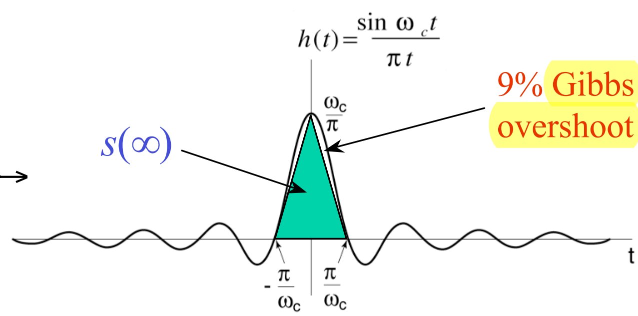
Ideal Lowpass Filter

CT



$\longleftrightarrow$

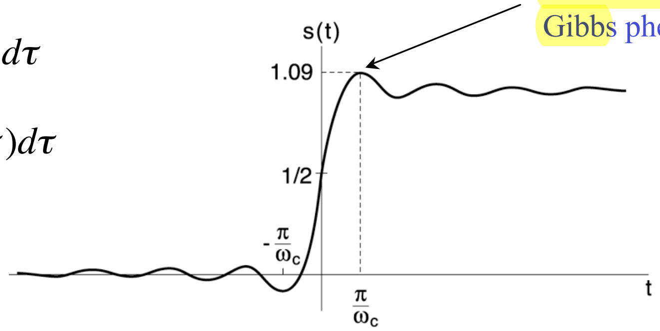


- Noncausal $h(t<0) \neq 0$
- Oscillatory Response — e.g. step response

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$s(\infty) = \int_{-\infty}^{\infty} h(\tau)d\tau$$
$$= H(j0)$$
$$= 1$$

Overshoot by 9%,
Gibbs phenomenon



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여기서 주의할 것은,
\omega_c가 달라져도
t=− \pi/\omega_c에서
t=+ \pi/\omega_c 까지의
적분값은 변하지 않는다는
것이다.
x축 방향으로 줄어드는 만큼 y축 방향으로 늘어나서
면적변화를 보상하게 된다.

step response가
t=−\infty에서는 0,
t=\infty에서는 1이고,
h(t)는 t=0을 중심으로
symmetrical 하므로,
step response는 t=0에서 1/2의 값을 갖는다.

\omega_c가 커지면
maximum overshoot되는 위치는 0에 가까와 지지만,
maximum overshoot의 양은 변하지 않는다.

Q. Gibbs phenomenon과의 관계는???

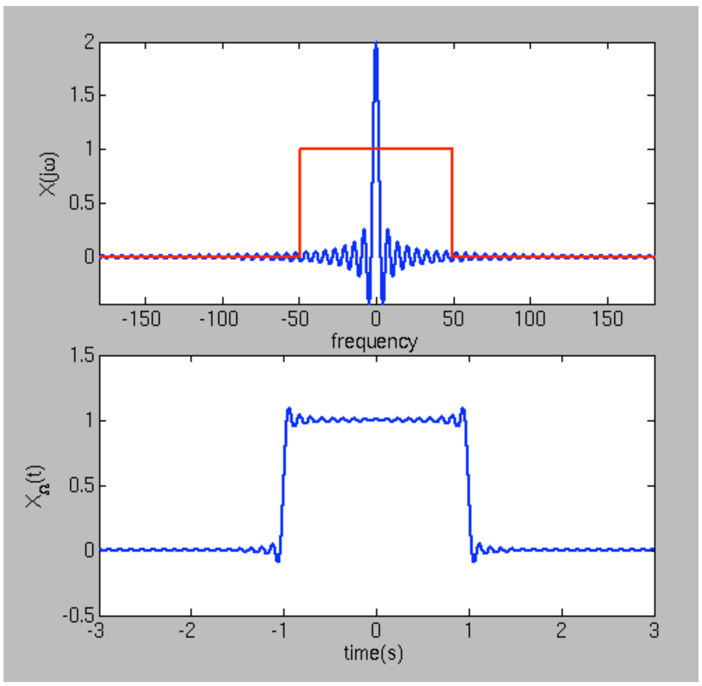
ex/ periodic square wave의 Fourier series를 truncate하는 것과 무슨 관계가???

Q. 그렇다면 jump discontinuity가 있는 신호의 Gibbs phenomenon은 어떻게 설명하나?

만약 $x(t)$가 t_0에서 alpha크기의 jump discontinuity가 있으면
$x(t) = x_0(t) + \text{alpha}\ u(t-t_0)$로 나타내어...

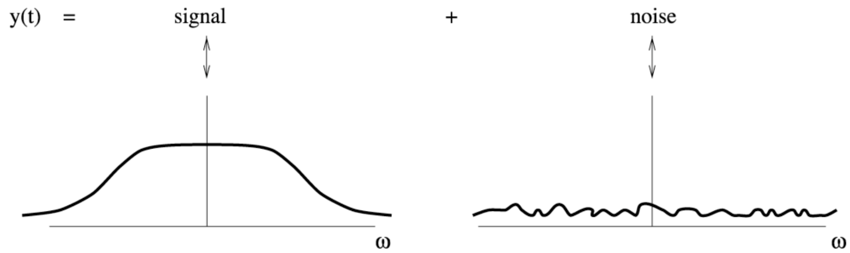
Demo: Gibbs phenomenon in *FT*

Passing a square pulse through an ideal LPF

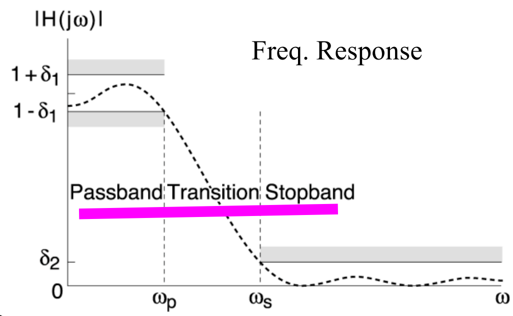


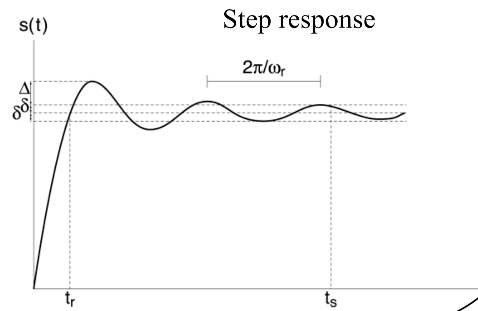
Nonideal Lowpass Filter

- Sometimes we don't *want* a sharp cutoff, e.g.



- Often have specifications in time and frequency domain ⇒ Trade-offs





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이제 qualitative하게 LCCDE로 나타나는 LTI system을 설계하는 법의 기초를 배운다. 보다 자세한 것은 Laplace transform과 함께 배운다.

우리는 LCCDE로 나타나는 시스템이 임의의 real-valued signal을 real-valued signal로 mapping하기를 원하는 경우가 많다.
이렇게 하기위한 sufficient한 방법으로는 H(j\omwga)가 real-valued coefficient를 갖는 rational function들의 product이면 된다.

CT Rational Frequency Responses

If the CT system is described by LCCDEs,

$$\sum a_k \frac{d^k y(t)}{dt^k} = \sum b_k \frac{d^k x(t)}{dt^k}$$

then $\dfrac{d^k}{dt^k} \longleftrightarrow (j\omega)^k$, and $H(j\omega) = \dfrac{\sum b_k (j\omega)^k}{\sum a_k (j\omega)^k} = \prod_i H_i(j\omega)$

$H_i(j\omega)$ = First - or Second – order factors

Prototypical Systems $H_1(j\omega) = \dfrac{1}{j\omega\tau + 1}$ — First - order system, has only *one* energy storing element, *i.e. L or C*

$$H_2(j\omega) = \frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\xi\left(j\frac{\omega}{\omega_n}\right) + 1} = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2}$$

— Second - order system, has *two* energy storing elements, *i.e. L and C*

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\Pi
대문자 \pi이다.
곱(product)을 나타낸다.
\Pi뒤에 나오는 것을 multiplicand라고 한다.
cf) summand, integrand

이렇게 하면 system을 구성할때, h_i(t)를 갖는 작은 system들을 구성하여 serially connect하면 된다. parallel connection으로 구성하려면 PFE한다.

DT Rational Frequency Responses

If the DT system is described by LCCDE's (Linear-Constant-Coefficient Difference Equations,

$$\sum a_k y[n-k] = \sum b_k x[n-k]$$

then from $y[n-k] \longleftrightarrow Y(e^{j\omega})e^{-jk\omega}$, $x[n-k] \longleftrightarrow X(e^{j\omega})e^{-jk\omega}$

We have:

$$H(e^{j\omega}) = \frac{\sum b_k e^{-jk\omega}}{\sum a_k e^{-jk\omega}} = \frac{\sum b_k (e^{-j\omega})^k}{\sum a_k (e^{-j\omega})^k}$$

$$= \prod_i H_i(e^{j\omega})$$

$$H_i(e^{j\omega}) = \text{First} - \text{ or second} - \text{order}$$

CT에서와 마찬가지로 factoring하면 serial connection으로 구성하는데 유용하고 PFE하면 parallel connection으로 구성하는데 유용하다.

즉, LCCDE로 시스템을 구성하려면 orders M and N과 coefficients a_k, b_k 들을 선정해야 하는데, Fourier transform 만으로는 어떻게 해야 하는지 확실히 보이지 않는다. -> 뭔가 더 필요한다. Laplace transform...

모든 H_i(j\omega)가 1차 또는 2차의 j\omega의 rational function이고 real - valued coefficients를 갖는 경우를 공부하자.

DT First-Order Systems

$$y[n] - ay[n-1] = x[n], \qquad |a| < 1, \quad \text{inital rest}$$

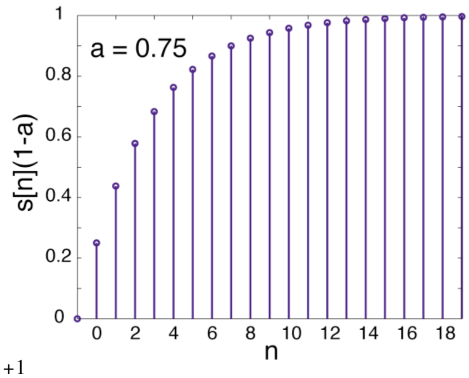
$$H(e^{j\omega}) = \frac{1}{1 - a\,e^{-j\omega}} = \frac{1}{(1 - a\cos\omega) + ja\sin\omega}$$

$$\Downarrow$$

Frequency Domain

$$\left|H(e^{j\omega})\right| = \frac{1}{\left(1 + a^2 - 2a\cos\omega\right)^{\frac{1}{2}}}$$

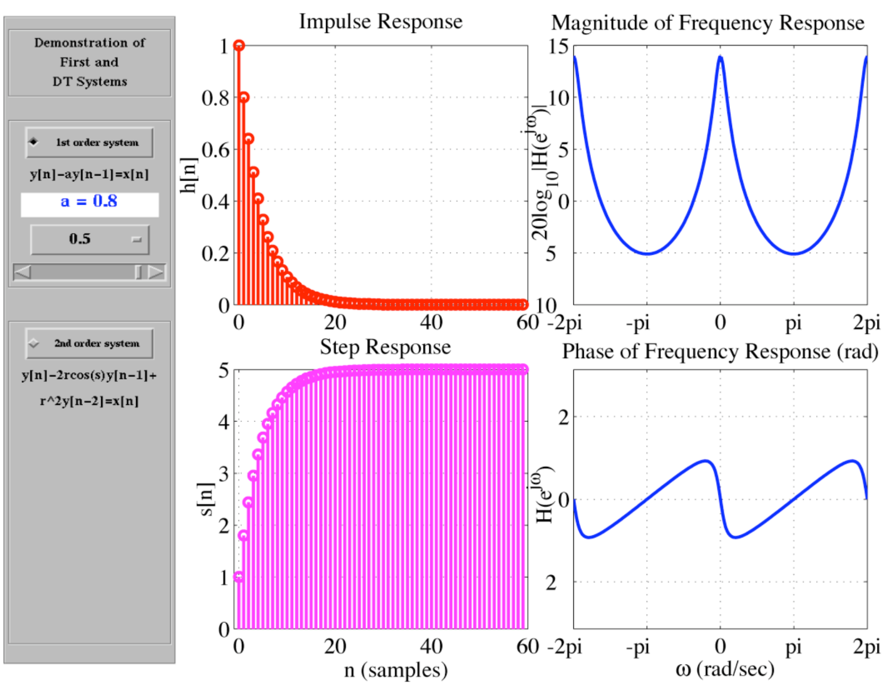
and $\quad \angle H(e^{j\omega}) = -\tan^{-1}\left[\dfrac{a\sin\omega}{1 - a\cos\omega}\right]$



Time Domain $\qquad h[n] = a^n u[n]$

$$s[n] = h[n] * u[n] = \sum_{m=0}^{n} a^m = \frac{1 - a^{n+1}}{1 - a}\,u[n]$$

Demo: Unit-sample, unit-step, and frequency response of DT first-order systems



DT Second-Order System (O & W pp.465-466)

$$y[n] - 2r\cos\theta y[n-1] + r^2 y[n-2] = x[n], \qquad 0 < r < 1 \, and \, 0 \le \theta \le \pi$$

$$H(e^{j\omega}) = \frac{1}{1-(2r\cos\theta)e^{-j\omega} + r^2 e^{-j2\omega}},$$

$$= \frac{1}{1-(re^{j\theta})e^{-j\omega}} \cdot \frac{1}{1-(re^{-j\theta})e^{-j\omega}}$$

$$= \frac{A}{1-(re^{j\theta})e^{-j\omega}} + \frac{B}{1-(re^{-j\theta})e^{-j\omega}}$$

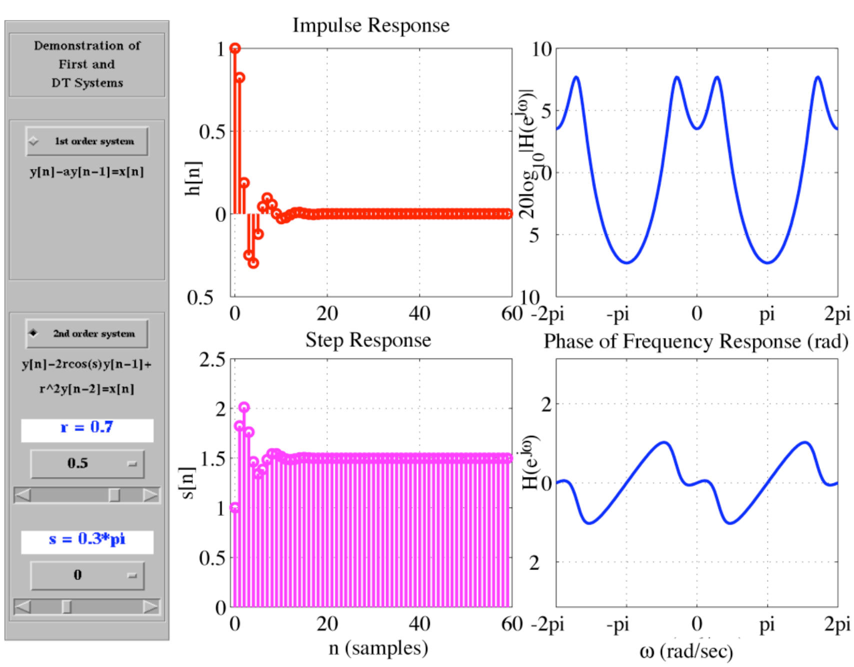
$$h[n] = \left[A r^n e^{jn\theta} + B r^n e^{-jn\theta} \right] u[n]$$

$$= \left[\frac{r^n \sin\left[(n+1)\theta\right]}{\sin\theta} \right] u[n]$$

$$s[n] = h[n] * u[n]$$

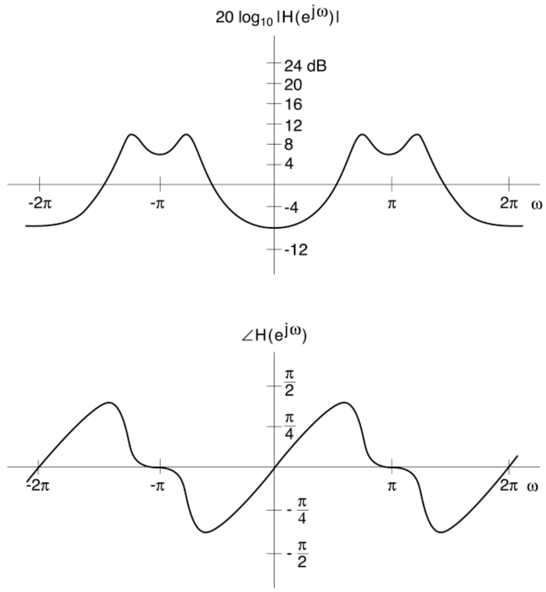
분모를 e^{−j\omega}로
풀면, 항상 켤레복소근을
갖는다.
켤레복소근의 특징은
mag는 갖고
phase는 반대.

Demo: Unit-sample, unit-step, and frequency response of DT second-order systems



A typical plot of the log-magnitude and phase of a second-order DT frequency response

$20\log|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$ vs. ω



Next lecture covers O & W pp. 514-527