EECE233 Signals and Systems

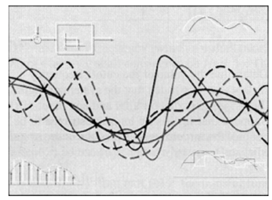
Lecture Note #18

Prof. Joon Ho Cho

(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

Covers O & W pp. 514-527

- Sampling of a CT Signal
- Analysis of Sampling in the Frequency Domain
- The Sampling Theorem — the Nyquist rate
- In the Time Domain: Interpolation
- Undersampling and Aliasing



* These lecture notes are based on the lecture notes used in 6.003 taught b Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.

Digital 시대가 도래하여 눈부신 발전(가격과 성능– processing speed, power consumption, size)을 이루고 있는데,
CT signal의 processing을 위해서는 계속 CT system을 만들어야만 할까?

CT signal을 이의 sample들로 나타내어 DT system으로 처리하면 어떨까?

sample들이 CT signal의 모든 정보를 손실없이 가질 수 있을까?

music CD가 나왔을때,

LP판과 CD의 '음질 차이를 알 수 있다'고 주장하는 사람들이 있었는데 이 주장은 옳은 주장일까?

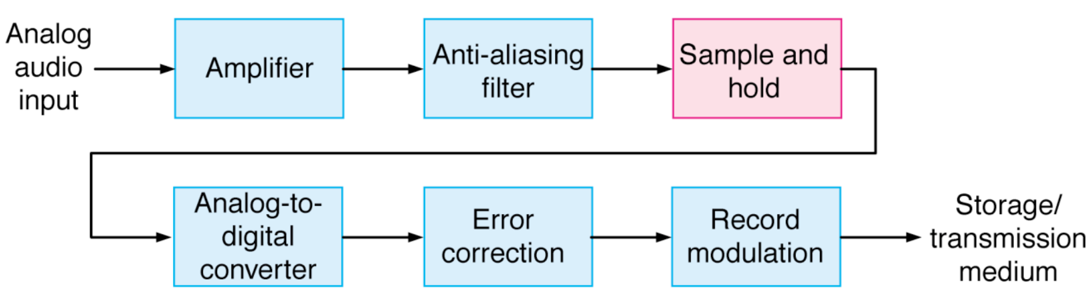
예를 들어 CD quality인 44 kHz sampling rate란 무슨 소리일까?

TD processing 인 sampling을 FD에서 이해해 보자.
핵심은
sampling전의 CT 신호의 CTFT와
sampling후의 DT 신호의 DTFT와의 관계를 규명하는 것이다.

SAMPLING

— A crucial step in converting CT signals to DT, so that we can use versatile digital computers or DSPs to process them.

Example: Digital recording of sounds



ADC output에 error가 있는 것이 아니고, 앞으로의 processing에서 error가 발생하면 correction할 수 있도록 redundancy를 추가하는 과정이다.

이 AAF가 왜 필요한지는 차차 설명한다.

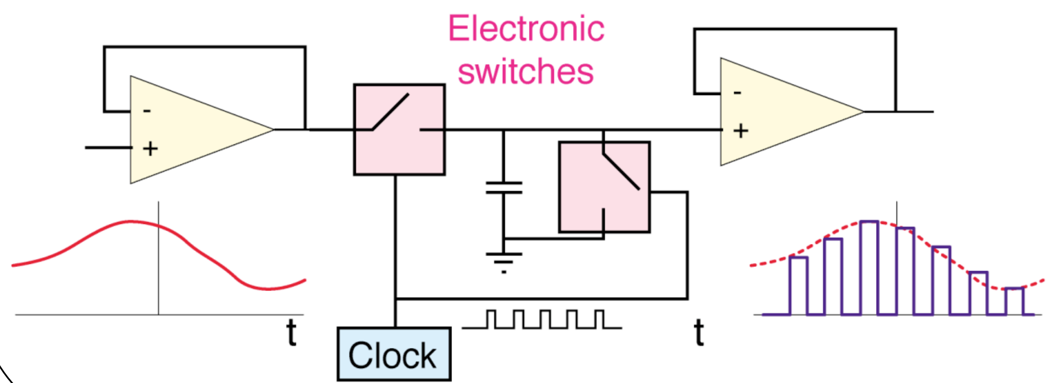
sample and hold는
CT signal을 piecewise constant function으로 approximate하는 CT-input CT-output 시스템이다.

.cda format
sampling rate 44.1.kHz
number of bits 16

How do we perform sampling?

Taking snap shots of $x(t)$ every T second. That is, taking $x(nT)$, and later convert $x(nT)$ to $x[n]$.

Example: A sample-and-hold circuit that produces such periodic snap shots.



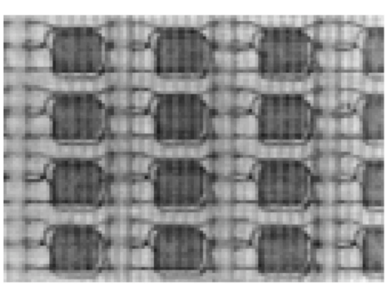
두 스위치는 하나가 닫히면 하나는 열린다.
충전과 방전을 반복한다.
첫번째 사이클에서 왼쪽의 스위치는 충전하고 오른쪽의 스위치는 open되어 있다. ADC 출력으로 신호가 적분되어 나온다.

두번째 사이클에서는 오른쪽의 스위치가 close되고 왼쪽이 open된다. 이에 따라 충전된 전하가 방전되고 출력에는 0이 나온다.

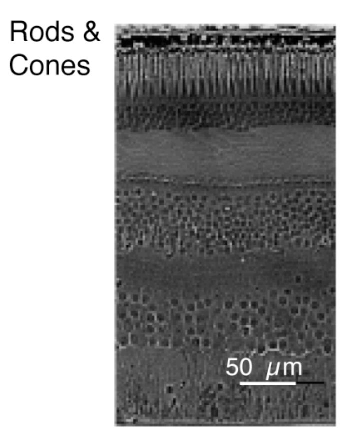
앞서 학기초에 잠시 설명했듯,
discrete-time signal processing이라고 하면, amplitude의 quantization과 finite resolution을 생각지 않는 경우가 많고
digital signal processsing이라하면 quatization과 finite resolution을 포함하여 말하는 경우가 많다.
이 과목에서는 다루지 않지만, quantization effect와 finite resolution에서 오는 한계는 매우 중요한 토픽이다.

The issue of sampling also applies to spatially varying signals

Examples: Light-sensing part of a digital camera — a CCD (Charge-Coupled-Device) device. The spacing between the adjacent elements determines the *Megapixel*. Also, human's (and all animals') light sensing is discrete.



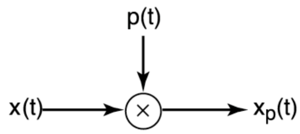
CCD (charge-coupled-device) at the focal plane of a digital camera



Cross section view of a human retina 망막

Mathematical Model of *Idealized* Sampling

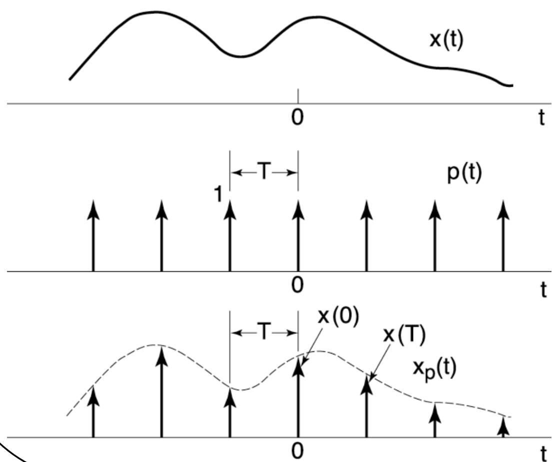
Impulse Train Approach — Multiplying $x(t)$ by the sampling function



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \;-\; \text{sampling function}$$

$$x_p(t) = x(t) \cdot p(t) = \sum_{n=-\infty}^{+\infty} x(t)\delta(t - nT)$$

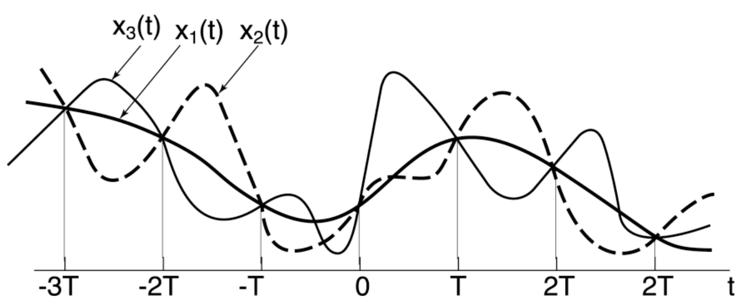
$$= \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$



여기서 sampling은 CT signal을 CT modulated impulse train으로 바꾸는 것을 의미한다.

Why/When Would a Set of Samples Be Adequate?

- Observation: $x_1(t)$, $x_2(t)$, $x_3(t)$ and many other signals have the same samples



- Obviously, by doing sampling we throw out some (a lot) information (all values of $x(t)$ between sampling points are lost).

- **Key Question for Sampling**:
  Under what conditions can we reconstruct the original CT signal $x(t)$ from the sampled signal $x_p(t)$? — $x(t)$ cannot change too fast.

Q.
rate 1/T인 sampler를 사용할때,
임의의 CT신호에 대해 sampling후의 DT신호가 sampling전의 CT신호의 모든 정보를 그대로 유지하고 있을 필요충분 조건 on sampling rate?
H.
uncountably infinite개의 정보를 요구하는 CT signal을
countably infinite개의 정보로 바꾸는 sampling은 일반적으로 가역적인 처리가 아니다.
왼쪽 그림에서처럼, 서로다른 CT signals가 sampling후에 같은 신호로 바뀐다.

Analysis of Sampling in the Frequency Domain

$$x_p(t) = x(t) \cdot p(t)$$

Multiplication Property $\Rightarrow X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$

But $P(j\omega)$ is also a "sampling function" in the frequency domain

$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

where $\omega_s = \dfrac{2\pi}{T}$ = Sampling Frequnecy

Important to note: $\omega_s \propto 1/T$

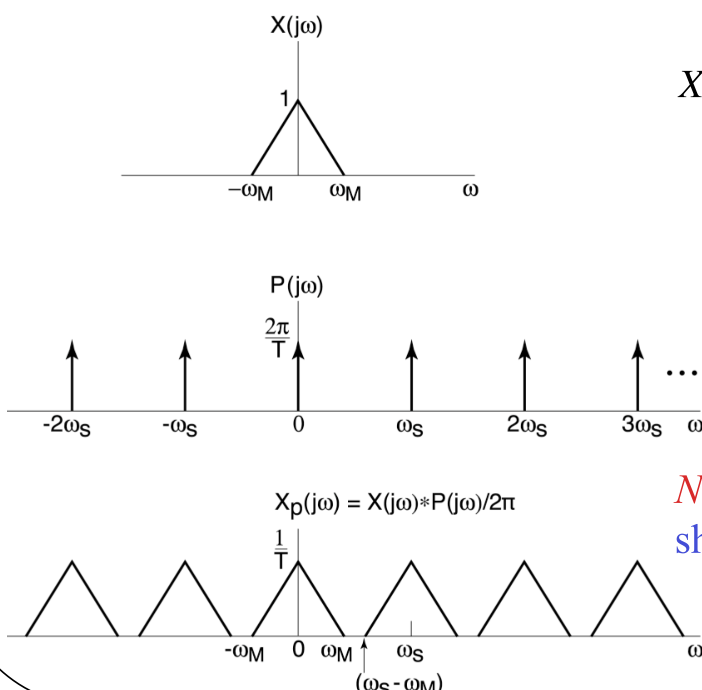
then $\Downarrow$

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(j\omega) * \delta(\omega - k\omega_s)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} X\big(j(\omega - k\omega_s)\big)$$

Superposition of shifted spectra

Illustration of sampling in the frequency-domain for a ***band-limited*** ($X(j\omega)$=0 for $|\omega| > \omega_M$) signal



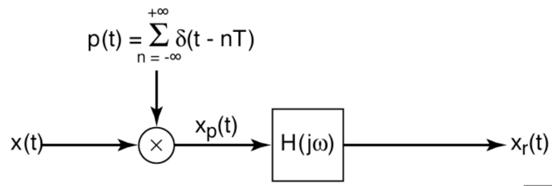
$X_p(j\omega)$ drawn assuming

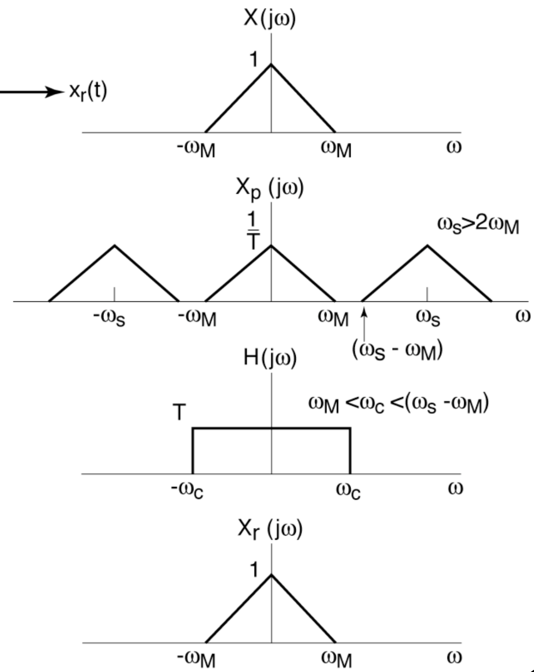
$$\omega_s - \omega_M > \omega_M$$

i.e. $\boxed{\omega_s > 2\omega_M}$

No overlap between shifted spectra

Reconstruction of $x(t)$ from sampled signals





Clearly in this case,

$$X_r(j\omega) = X(j\omega)$$

$$\Rightarrow \quad x_r(t) = x(t)$$

If there is no overlap between shifted spectra, a LPF can reproduce $x(t)$ from $x_p(t)$

즉, 이 경우는 적절한 cutoff frequency를 갖는 ideal LPF를 쓰면 perfect reconstruction이 가능하다.

The Sampling Theorem

Suppose $x(t)$ is *bandlimited*, so that

$$X(j\omega) = 0 \quad \text{for } |\omega| > \omega_M.$$

Then $x(t)$ is uniquely determined by its samples $\{x(nT)\}$ if

$$\boxed{\omega_s > 2\omega_M = \text{The Nyquist rate}}$$

where $\omega_s = 2\pi/T$.

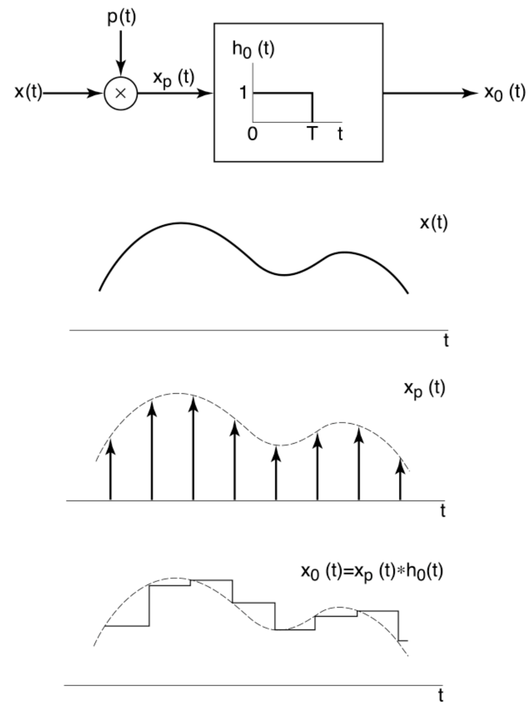
Intuitively, this makes sense since a *bandlimited* signal has a limited amount of information, which can be *fully* captured in the snap shots $\{x(nT)\}$.

주의! if and only if 가 아니다.

Observations on Sampling

(1) In practice, we obviously don't sample with impulses or implement ideal lowpass filters.

One practical example:
— The Zero-Order Hold

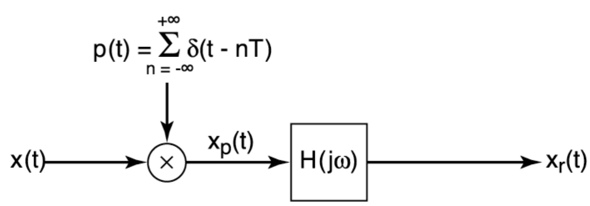


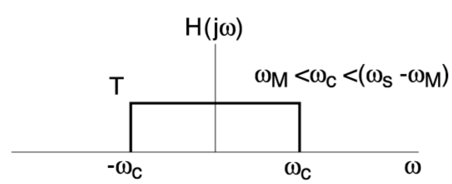
11

Q. Sampling은 linear operation인가?
A. linear 이다.
Q. time-varying 인가 invariant인가?
A. Time-varying이다. 일단 sampling을 CT2CT인 경우로 생각하면 ouput에서 input에서는 없던 spectrum 성분이 생겨난다. 따라서 LTI가 아니다.

Observations (Continued)

(2) Sampling is fundamentally a *time-varying* operation, since we multiply $x(t)$ with a time-varying function $p(t)$. However,





is the identity system (which is *TI*) for bandlimited $x(t)$ satisfying the sampling theorem ($\omega_s > 2\omega_M$).

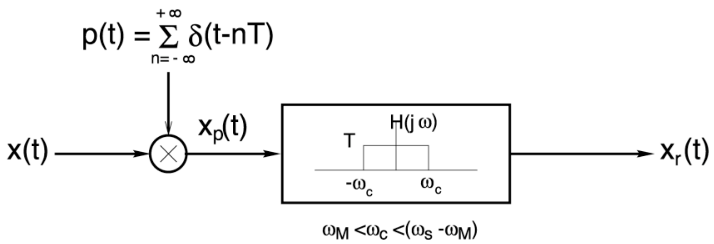
(3) What if $\omega_s \leq 2\omega_M$? Something different: more later.

12

여기서 \omega_s-\omega_M은 오른쪽에 생긴 이미지의 lowest frequency component 이다.

\omega_M<\omega_c<\omega_S-\omega_M 에서 오른쪽과 왼쪽항을 모으면 \omega_x>2 \omega_M이라는 답이 나온다.

Time-domain Interpretation of Reconstruction of Sampled Signals — Band-limited Interpolation



$$x_r(t) = x_p(t) * h(t) \ , \ h(t) = \frac{T\sin(\omega_c t)}{\pi t}$$

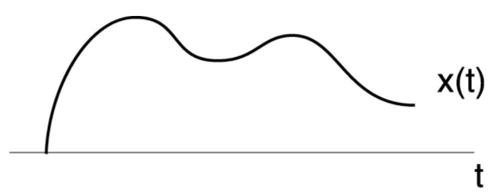
$$= \left(\sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT) \right) * h(t)$$

$$= \sum_{n=-\infty}^{+\infty} x(nT)h(t - nT) = \sum_{n=-\infty}^{+\infty} x(nT)\frac{T\sin[\omega_c(t - nT)]}{\pi(t - nT)}$$

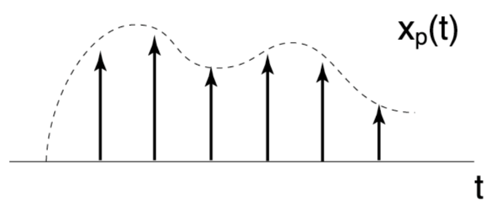
The lowpass filter interpolates the samples *assuming* $x(t)$ contains no energy at frequencies $\geq \omega_c$

Graphic Illustration of Time-domain Interpolation

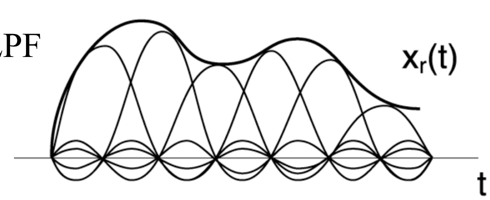
Original CT signal



After sampling



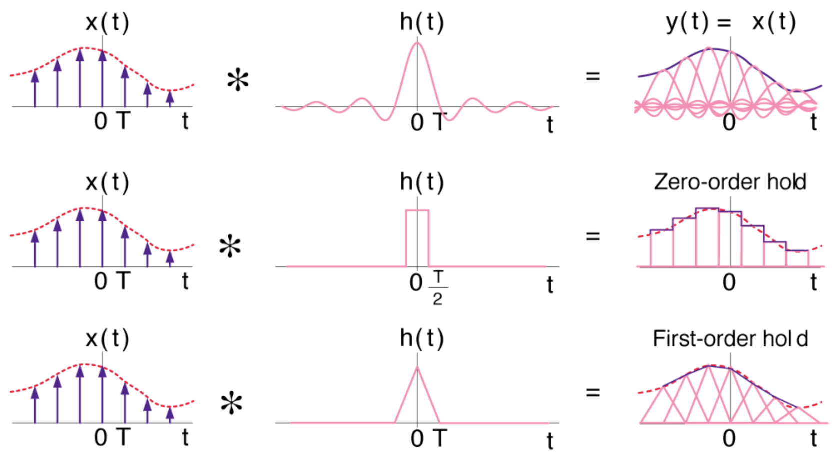
After passing the LPF



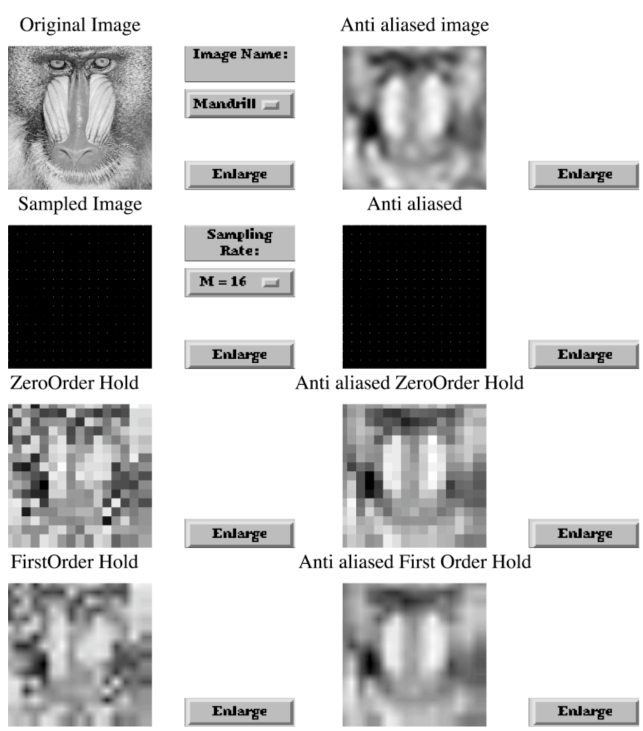
The LPF smoothes out sharp edges and fill in the gaps.

Commonly Used Interpolation Methods

- Bandlimited Interpolation
- Zero-Order Hold (*E.g.* movie projection)
- First-Order Hold — Linear interpolation, commonly used in plotting.



Demo: Sampled Images

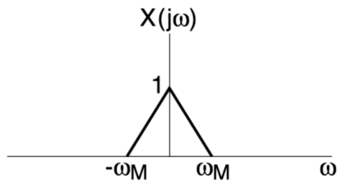


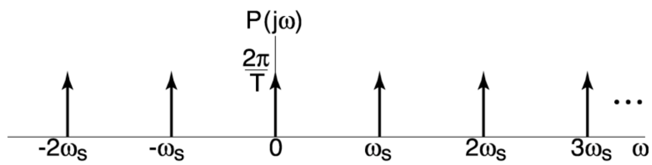
AAF (Anti-Aliasing Filter) — A presampling LPF to make sure that the signals are bandlimited before going through sampling.

예를 들어,
audio signal에서
우리가 들을 수 있는 범위가 22kHz라고해서
마이크로부터의 신호를
그냥 44.1kHz로 샘플링하면
마이크 자체의 가청(?)주파수가 22kHz보다 크므로
가청 주파수를 벗어나는 신호가 aliasing되어 가청 주파수 대역에 나타나게 된다.
따라서 AAF는 반드시 필요하다.
그런데 문제는 ideal LPF로 AAF를 만들기가 어렵다. 따라서 실제로는
oversampling을 하여
digital filtering하여 AAF효과를 낸다. 자세한 내용은 학기말에 배운다.

Undersampling and Aliasing

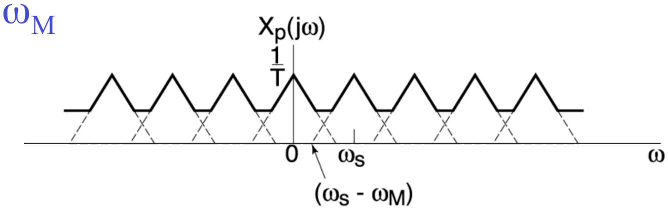
When $\omega_s \le 2\,\omega_M \Rightarrow$ Undersampling





Note $\omega_s - \omega_M \le \omega_M$

$\Rightarrow \omega_s \le 2\,\omega_M$

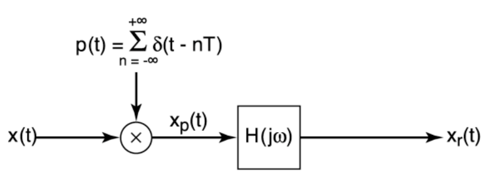


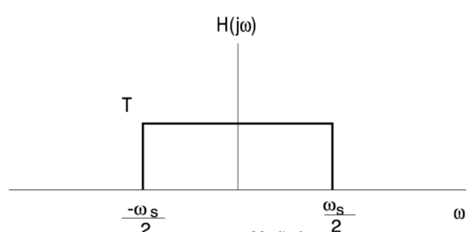
영화는 초기에는 초당 16프레임이었다가 현재는 초당 24프레임이다.
NTSC방식은 초당 30프레임이다.

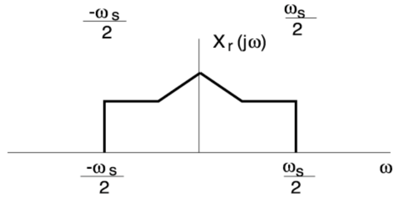
Aliasing때문에 수성의 자전주기를 관측했던 초기의 천문학자들이 잘못 계산된 주기를 믿었다.

왼쪽 그림처럼 aliasing이 일어나고 나면
그 이후에는 무슨 수를 써도 (일반적인 신호에서는) perfect reconstruction이 불가능하다.

Undersampling and Aliasing (continued)







$X_r(j\omega) \neq X(j\omega)$
Distortion because of *aliasing*

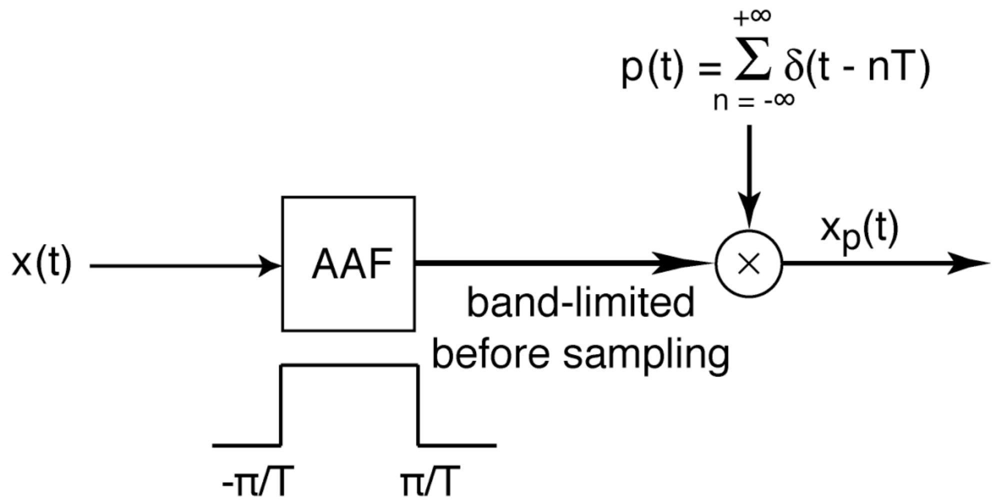
— Higher frequencies of $x(t)$ are "folded back" and take on the "aliases" of lower frequencies

— Note that at the sample times, $x_r(nT) = x(nT)$

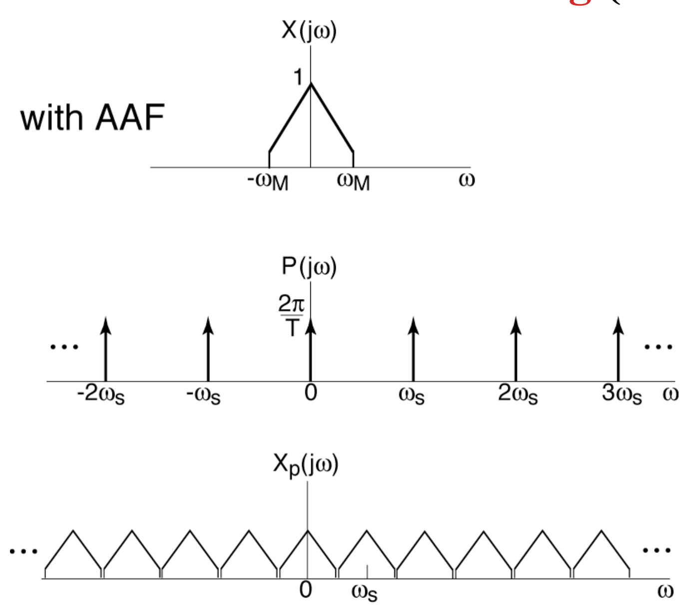
여전히 sample time에서는 perfect하다. 그러나 샘플 사이의 값에서 잘못된다.

Therefore, the first step in sampling is anti-alias filtering (AAF) — a LPF to assure that $\omega_s \geq 2\omega_M$

만약 sampling rate가 주어진다면, 다음 할 일은 sampling rate의 절반 이하를 갖는 LPF로 AAF하는 일이다.



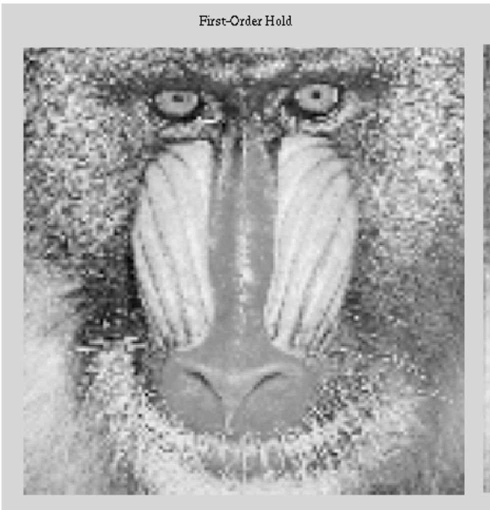
The effect of anti-alias filtering (AAF)

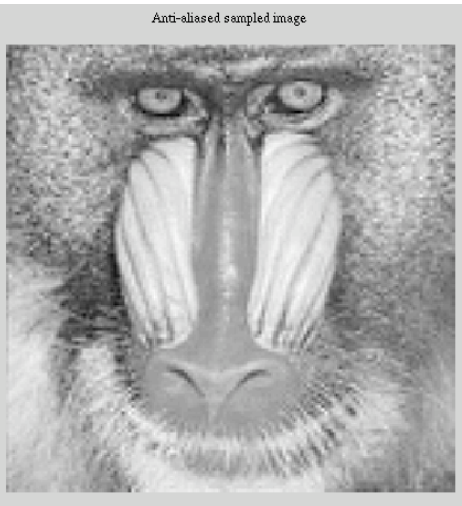


The AAF low-pass filter will rid of the information in $x(t)$ beyond $|\omega| > \omega_s/2$, but will avoid the much more serious aliasing problem. trade−off가 있다.

Effect of sampling on images

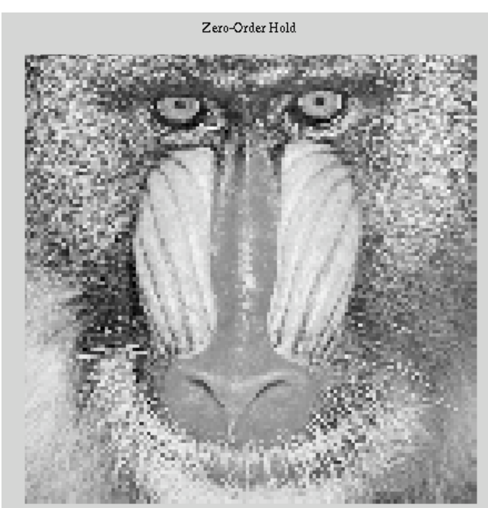
The effect of the anti-aliasing filter (AAF) is seen by comparing the reconstructed image without AAF (left) and with AAF (right). Both used a first-order hold.

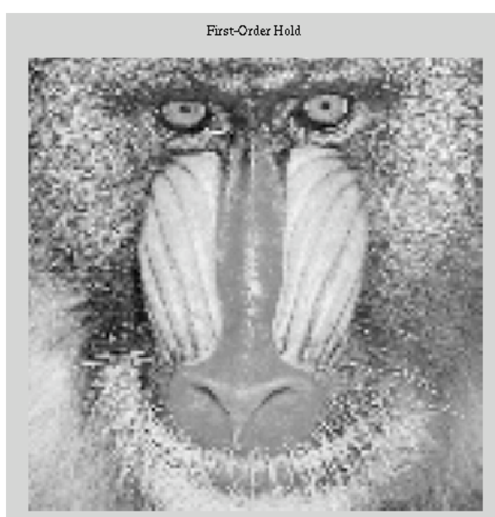




Effect of sampling on images (cont.)

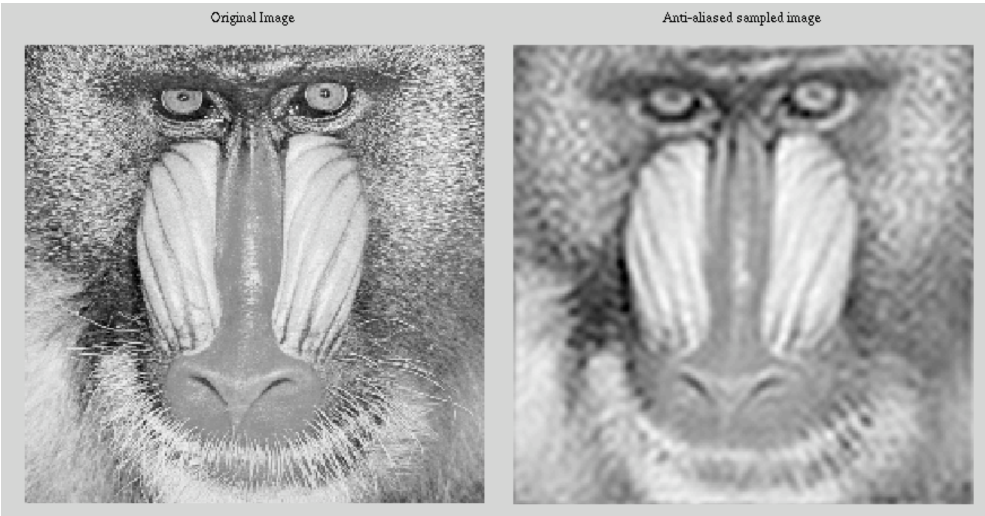
The sampled image (×2) has been reconstructed with a zero-order hold on the left, and a first-order hold (linear interpolation) on the right.





Effect of sampling on images (cont.)

The original image is shown on the left and the same image passed through an anti-aliasing filter appropriate to sampling every 4th point is shown on the right.



ADC란 1) sampler와 2)quantizer로 구성되어 있다.

엄밀히 sampler는 CT/DT (CDC: CT to DT converter)인데

x(t)라는 CT signal을
x[n]이라는 DT signal로 변환하는 시스템이고
그 input output relation은

$x[n] = x(nT+T_0)$, forall n

이 된다.
여기서 two parameters가 중요한데, 하나는 sampling rate $1/T$이고 다른 하나는 time offset T_0 이다.

보통은 time offset T_0를 0으로 놓고 생각한다.