EECE233 Signals and Systems

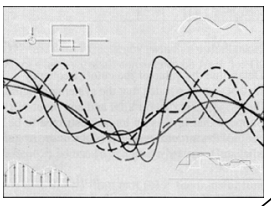
Lecture Note #19

Prof. Joon Ho Cho

(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

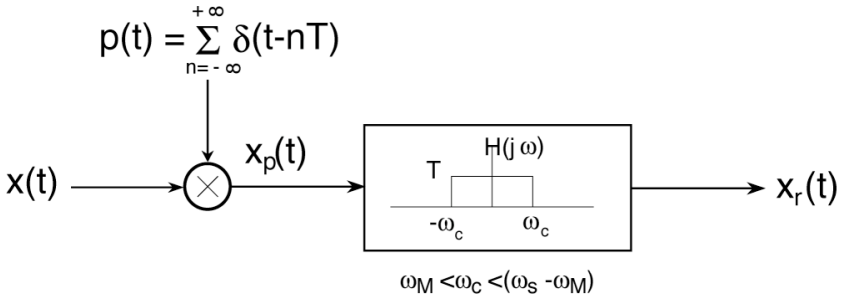
Covers O & W pp. 527-543

1. Review/Examples of Sampling/Aliasing
2. DT Processing of CT Signals



* These lecture notes are based on the lecture notes used in 6.003 taught b Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.

Sampling Review



Def. If $X(j\omega) = 0, \quad |\omega| > \omega_M \Rightarrow x(t)$ is ***band - limited***

and $\omega_s = \dfrac{2\pi}{T} > 2\omega_M$

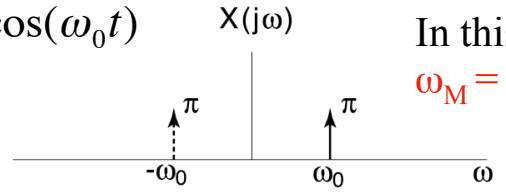
then, assuming we choose $\omega_M < \omega_c < \omega_s - \omega_M, \; x_r(t) = x(t)$

Def. Aliasing — if the sampling criterion is not satisfied, that is:

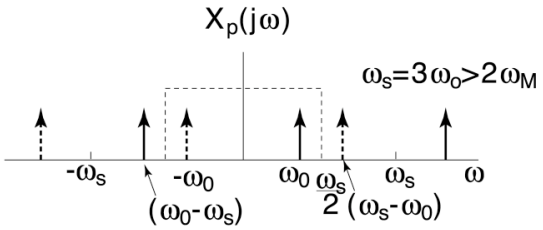
$\omega_s < 2\omega_M$.

A Simple Example of aliasing

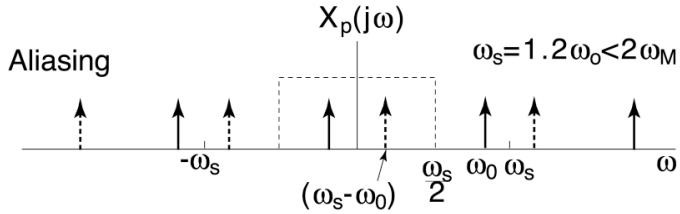
$x(t) = \cos(\omega_0 t)$



In this case,
$\omega_M = \omega_o$



After a LPF, get our original signal back. No problem.



Q: What happens to the signals within the LPF when the signal frequency ω_o increases?

Q. sin(\omega_0 t)의 경우 최대주파수가 \omega_0이다.
이를 2배인 2\omega_0로 sampling하면 어떻게 되는가?
후에 perfect reconstruction이 가능한가?
A. No.
sampling하면 언제나 0이다.

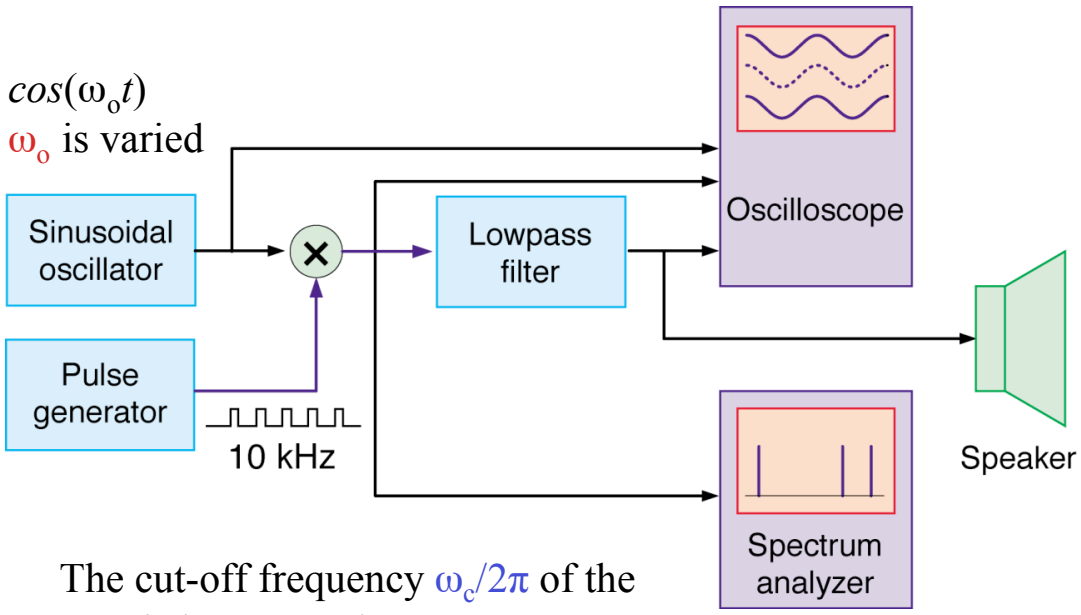
Q. 왜 이런일이 생기나?
A. 사실 엄밀히 말해 sampling theorem 은 finite energy signal에서 유도된다. finite energy signal이면 최대주파수의 2배 '이상'으로 샘플링하면 문제 없다. 그러나 sin함수는 fiinite energy signal이 아니다.

최대주파수의 2배 초과로 하면 된다?

Demo:

The effect of sampling on sinusoidal audio signals

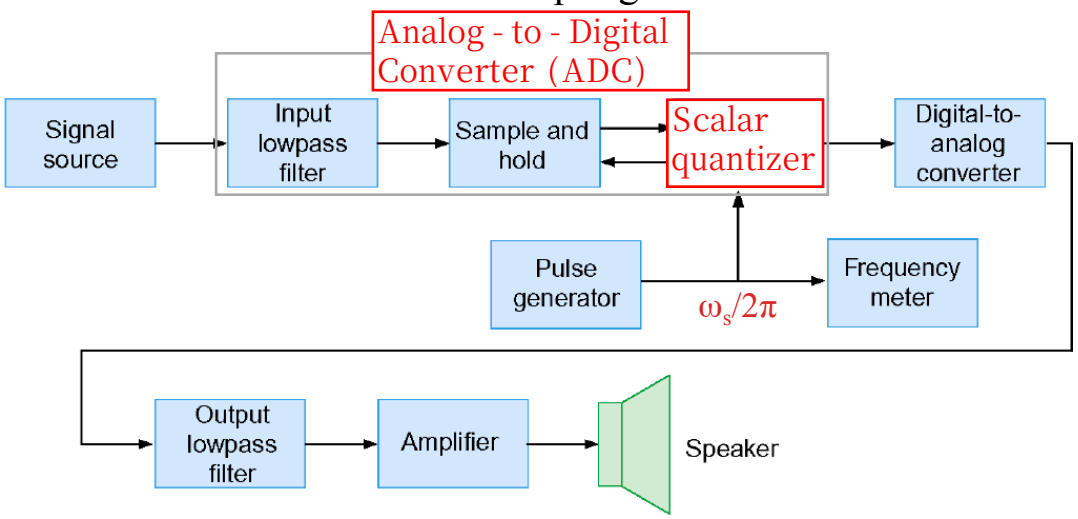
$cos(\omega_o t)$
ω_o is varied



The cut-off frequency $\omega_c/2\pi$ of the LPF is kept at ~4 kHz.

Demo:

The effect of sampling on music



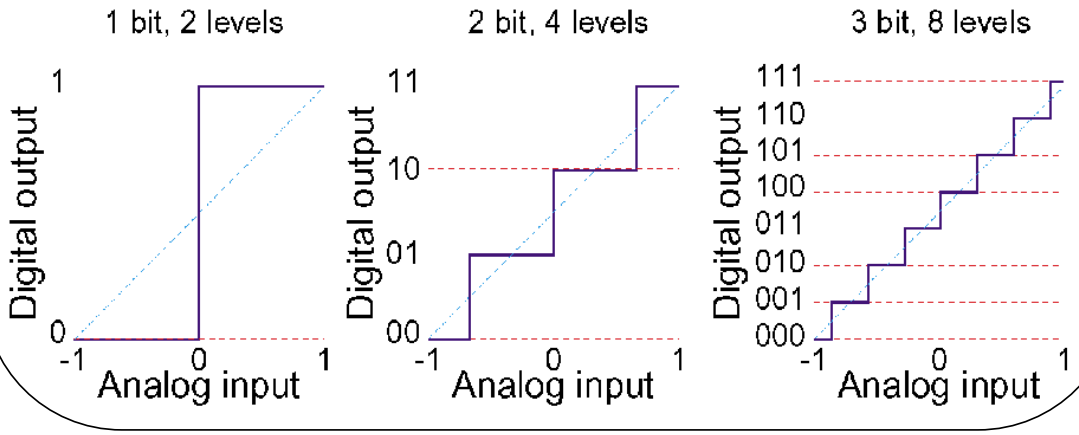
The sampling frequency $\omega_s/2\pi$ is varied and displayed on the frequency meter.

ADC (Analog-to-Digital Converter) consists of Sampler + Quantizer, where

a sampler consists of an AAF and a CDC (Continuous-time to Discrete-time Converter),
a quantizer also encodes to a bit sequence.

Demo: The effect of quantization on music.

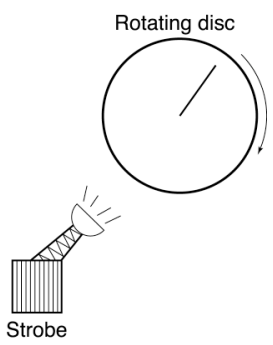
In addition to sampling to convert CT signals to DT signals, the DT signals are further *quantized* in their amplitudes. That is, $x \rightarrow$ (10110101001010), convert x into a binary number with a limited precision and maximum value, so that they can be processed by DSPs.

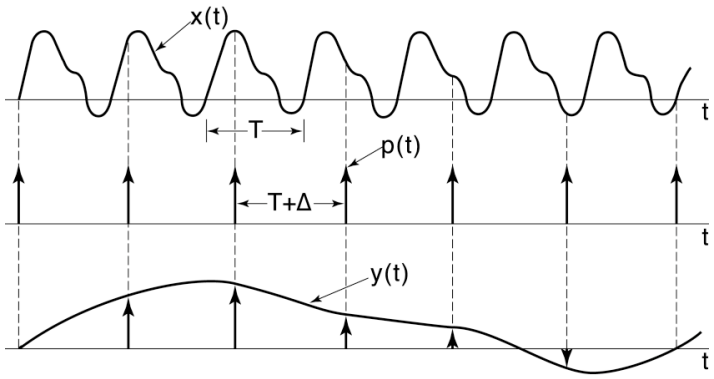


quantization noise가 작으면 귀로는 quantization effect를 알아채지 못한다.

scalar quantizer not only has
1. quantization noise
but also has
2. overflow error or
3. clipping error

Strobe Demo





$\Delta = 0$, strobed image still

$\Delta > 0$, strobed image moves forward, but at a slower pace

$\Delta < 0$, strobed image moves *backward*.

Applications of the strobe effect (*aliasing* can be useful sometimes):

— Sampling oscilloscope (will be discussed in recitation)

⋮

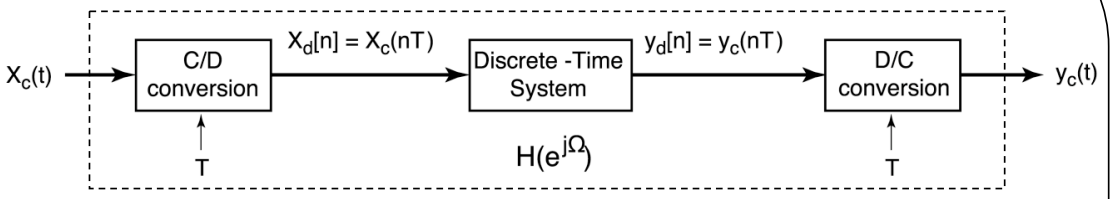
수성의 공전 주기는 87.969일

1880년대 이탈리아의 스키아파렐리는 수성은 '자전 주기와 공전 주기가 같은 동주기 자전을 하고 있다.' 라고 주장하였다.

매리너 10호가 수성을 지나가면서 측정한 수성의 자전 주기는 58.66일로 밝혀졌다.

그런데 이 값은 정확히 수성의 공전 주기(87.969일)의 2/3가 되는 값이다. 스키아파렐리는 왜 잘못된 주장을 하였을까?

DT Processing of *Band-limited* CT Signals



$H_c(j\omega)$

Why do this?

— Inexpensive, versatile, and higher noise margin.

How do we analyze this system?

— We will need to do it in the frequency domain in both CT *and* DT

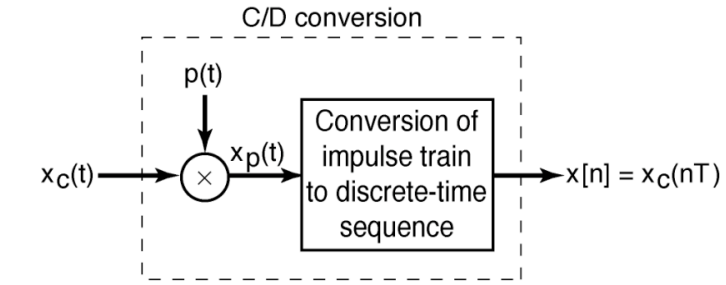
— In order to avoid confusion about notations, specify

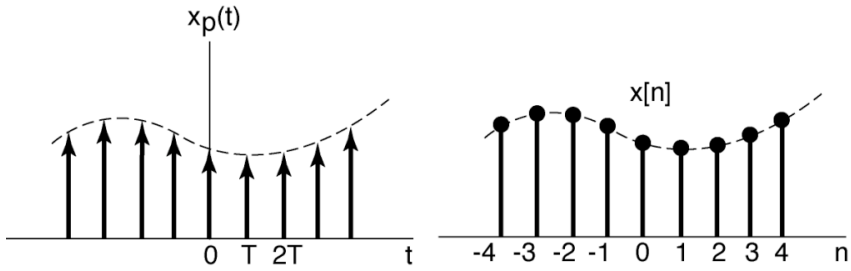
ω — CT frequency variable

Ω — DT frequency variable ($\Omega = \omega T$)

Step 1: Find the relation between $x_c(t)$ and $x_d[n]$, or $X_c(j\omega)$ and $X_d(e^{j\Omega})$

Time-Domain Interpretation of C/D Conversion





Discontinuous CT signal

DT Sequence $x_d[n] = x_c(nT)$

impulse train을 곱하고 amplitude modulated impulse train을 DT로 바꾸는 이 그림은, 개념을 나타낸 것일 뿐이지 실제로는 이렇게 하지 않는다.
임펄스 열은 무한학 대역폭을 갖는데, 실제로는 유한한 대역폭의 AAF후 CT in and DT out 변환이 이루어진다.

Frequency-Domain Interpretation of C/D Conversion

$$x_p(t) = x_c(t) \cdot \sum_{n=-\infty}^{+\infty} \delta(t - nT) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t - nT)$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$X_p(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} X_c\left(j(\omega - k\omega_s)\right) = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\omega nT} \quad \text{— CT (1)}$$

– (periodic with period $\omega_s = \frac{2\pi}{T}$)

Note: $x_d[n] = x_c(nT)$

$$X_d(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x_d[n]\, e^{-j\Omega n} = \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\Omega n} \quad \text{— DT (2)}$$

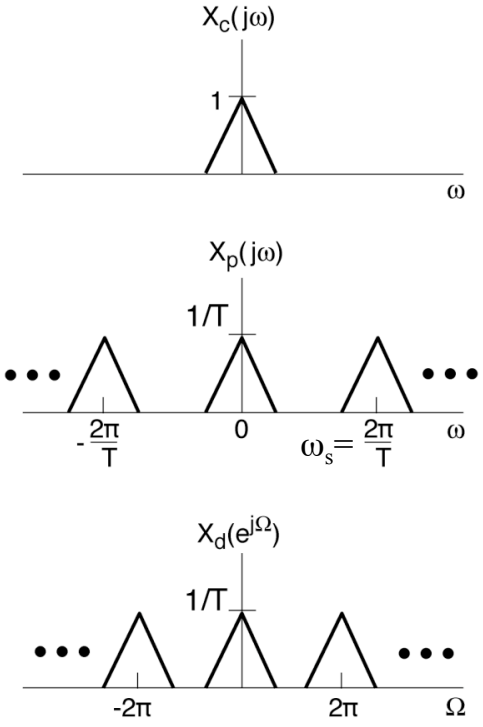
– (periodic with period 2π)

⇓ Compare Eqs. (1) & (2), and remember $\Omega = \omega T$

$$X_d(e^{j\Omega}) = X_p\left(j\left(\frac{\Omega}{T}\right)\right)$$

여기서 주의할 것은, 왼쪽 식은 ADC input 인 CT 신호와 output 인 DT 신호의 CTFT와 DTFT의 관계가 아니고 CT신호에 impulse train을 곱한것과 DT 신호의 관계이다.

Illustration of C/D Conversion in the Frequency-Domain



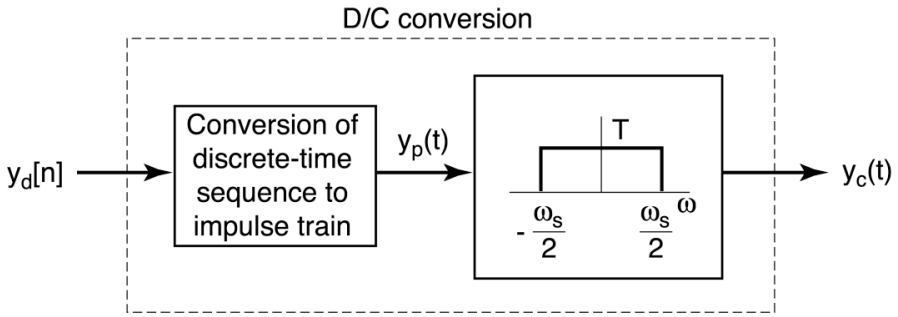
Note: $\Omega = \omega T$

$2\pi \Leftrightarrow \omega_s$

DT CT

D/C Conversion $y_d[n] \rightarrow y_c(t)$

Reverse of the process of C/D conversion



Again, $\Omega = \omega T$

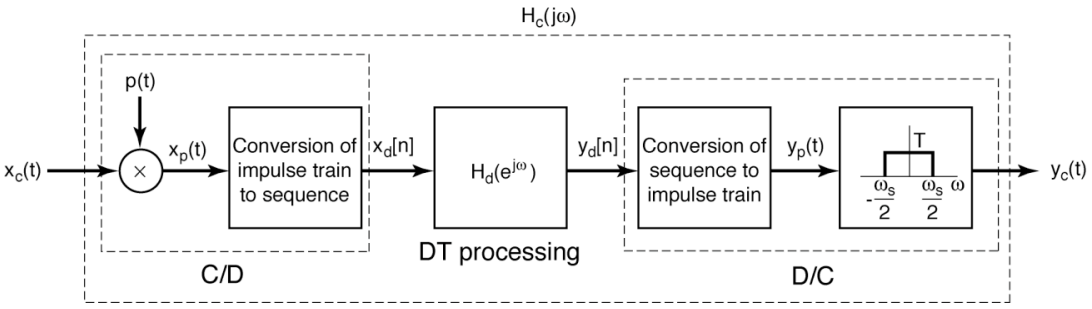
$Y_p(j\omega) = Y_d(e^{j\omega T})$ — Reverses frequency scaling

$$Y_c(j\omega) = \begin{cases} T Y_d(e^{j\omega T}) & |\omega| < \frac{\omega_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

— ***band - limited***

여기서 T는
sampling rate가 T여서 나타난 것이 아니고
LPF의 높이가 T여서 나타난 것이다. 높이가 1이면 1이 된다.

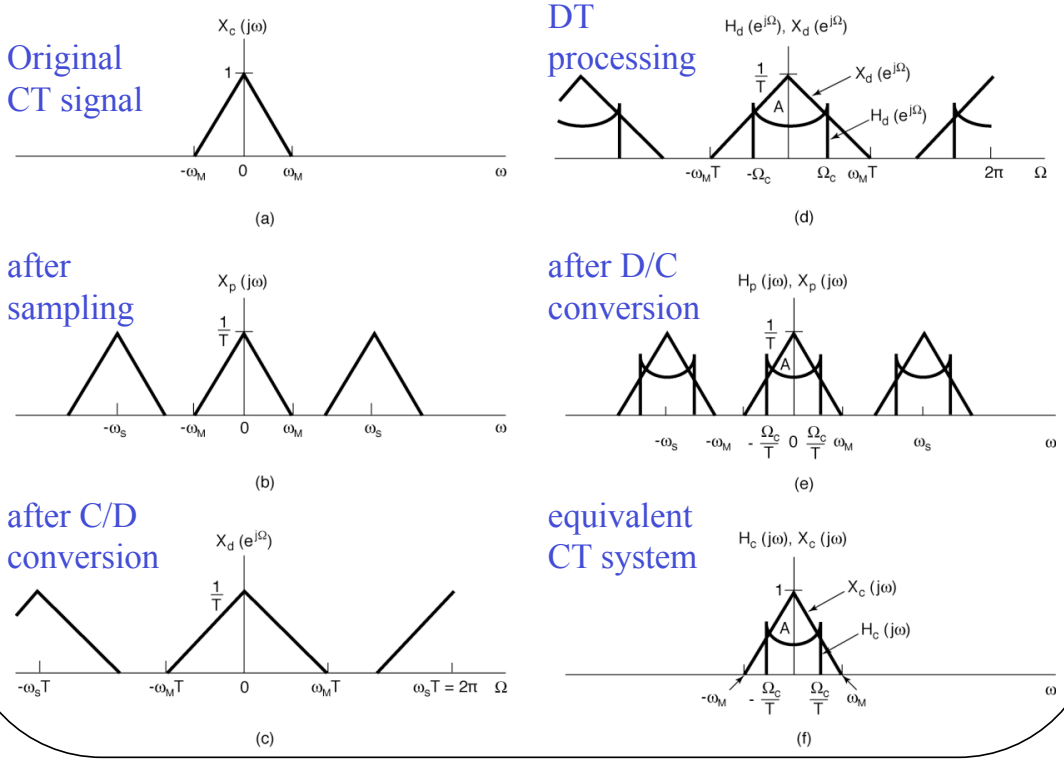
Now the whole picture



- Overall system is time-varying if sampling theorem is *not* satisfied
- It is LTI if the sampling theorem *is satisfied*, *i.e.* for *bandlimited* inputs $x_c(t)$, with $\omega_M < \frac{\omega_s}{2}$
- When the input $x_c(t)$ is band-limited ($X(j\omega) = 0$ at $|\omega| > \omega_M$) and the sampling theorem is satisfied ($\omega_s > 2\omega_M$), then

$$Y_c(j\omega) = H_c(j\omega)\, X_c(j\omega) \longleftrightarrow y_c(t) = h_c(t) * x_c(t) \qquad \text{LTI}$$

Frequency-domain Illustration of DT Processing of CT Signals



Assuming No Aliasing (using an AAF)

Step # 1 CT → DT: $X_d(e^{j\Omega}) = X_p(j\Omega / T)$ — periodic

$$= \frac{1}{T} X_c(j\Omega / T) \quad , \quad -\pi < \Omega < \pi \quad \text{if no aliasing}$$

Step # 2 DT system: $Y_d(e^{j\Omega}) = H_d(e^{j\Omega}) X_d(e^{j\Omega})$

$$= \frac{1}{T} H_d(e^{j\Omega}) X_c(j\Omega / T) \ , \ -\pi < \Omega < \pi$$

Step# 3 D / C: $Y_c(j\omega) = \begin{cases} TY_d(e^{j\omega T}) = H_d(e^{j\omega T}) X_c(j\omega) & , \quad -\frac{\omega_s}{2} < \omega < \frac{\omega_s}{2} \\ 0 & , \quad \text{otherwise} \end{cases}$

$$\Downarrow$$

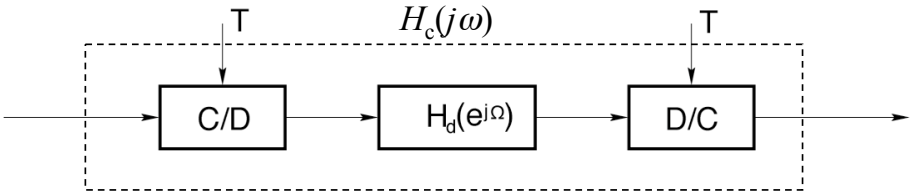
$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}) & , \quad |\omega| < \frac{\omega_s}{2} \\ 0 & , \quad \text{otherwise} \end{cases}$$

In practice, first specifies the desired $H_c(j\omega)$, then design $H_d(e^{j\Omega}) = H_c(j\Omega/T)$.

Example: Digital Differentiator

Applications: Edge Enhancement

Construction of *Band-limited* Digital Differentiator



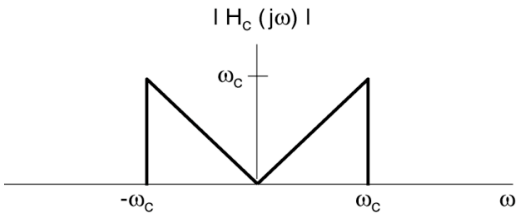
Desired: $H_c(j\omega) = \begin{cases} j\omega & , \ |\omega| < \omega_c \\ 0 & , \ |\omega| > \omega_c \end{cases}$

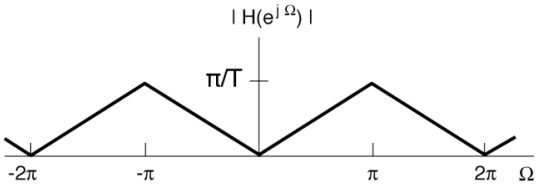
Set $\omega_s = 2\omega_c \Rightarrow T = \dfrac{2\pi}{\omega_s} = \dfrac{\pi}{\omega_c}$. Assume $\omega_M < \omega_c$ → Nyquist rate is met

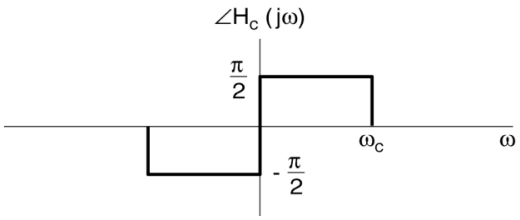
Choice for $H_d(e^{j\Omega})$: $H_d(e^{j\Omega}) = \begin{cases} H_c(j\Omega / T) \ , & |\Omega| < \pi \\ \text{periodic} & |\Omega| \geq \pi \end{cases}$

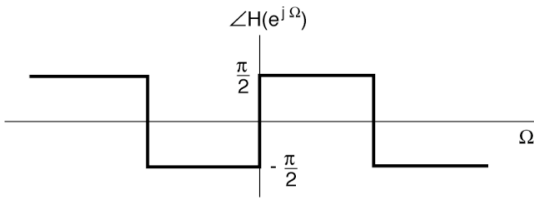
$= \begin{cases} j(\Omega / T) \ , & |\Omega| < \pi \\ \text{periodic} & |\Omega| \geq \pi \end{cases}$

Band-limited Digital Differentiator (continued)









Desired CT system

DT implementation

Digital Differentiator in the Time-Domain

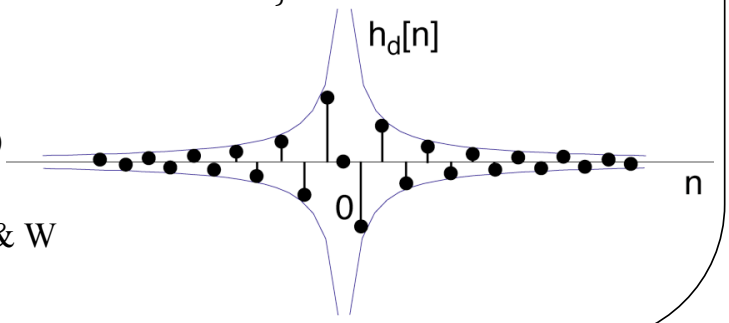
For: $$H_d(e^{j\Omega}) = \begin{cases} j(\Omega / T) & |\Omega| < \pi \\ periodic & |\Omega| \geq \pi \end{cases}$$

$h_d[n]$ is the corresponding "Fourier series" of the triangle wave

$$h_d[n] = \frac{1}{2\pi}\int_{2\pi} H_d(e^{j\Omega})e^{j\Omega n}d\Omega \overset{H_d\ odd}{=} \frac{2}{2\pi}\int_0^{\pi} j(\Omega / T)\cdot j\sin(\Omega n)\, d\Omega$$

$$= -\frac{1}{\pi T}\left\{\frac{1}{n}\left[-\Omega\cos(\Omega n)\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\cos(\Omega n)d\Omega\right\}$$

$$= \begin{cases} \frac{1}{nT}(-1)^n & n \neq 0 \\ 0 & n = 0 \end{cases}$$



Next lecture covers O & W
pp. 582-599