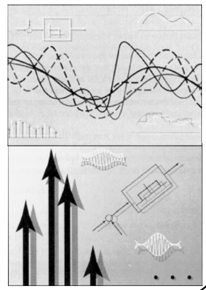
# EECE233 Signals and Systems

Lecture Note #21
Prof. Joon Ho Cho
(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

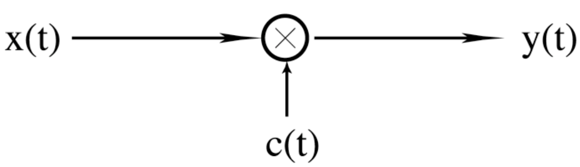
- Covers O & W pp. 601-604, 611-613, 619-623, 545-555
- AM with an Arbitrary Periodic Carrier
- Pulse Train Carrier and Time-Division Multiplexing
- Sinusoidal Frequency Modulation (**FM**)
- DT Sinusoidal AM
- DT Sampling, Decimation, and Interpolation



* These lecture notes are based on the lecture notes used in 6.003 taught b Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.



## **AM with an Arbitrary *Periodic* Carrier**




$c(t)$ – periodic with period $T$, carrier frequency $\omega_c = 2\pi/T$

Remember: periodic in $t \longleftrightarrow$ discrete in $\omega$

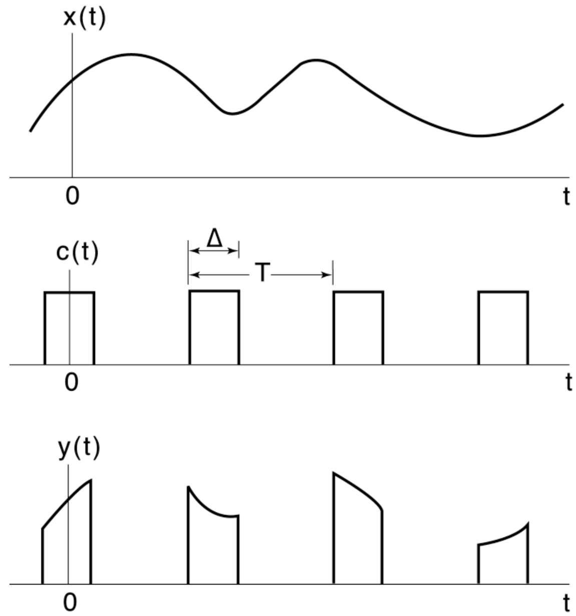
$$C(j\omega) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_c) \quad (a_k = \frac{1}{T} \text{ for impulse train})$$

$$\Downarrow$$

$$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * C(j\omega) = X(j\omega) * \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_c)$$

$$= \sum_{k=-\infty}^{+\infty} a_k X\big(j(\omega - k\omega_c)\big)$$

## Modulating a Carrier of Square Pulse Train




$$y(t) = x(t) \cdot c(t)$$

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스필버그의 영화 'ET'를 보면 ET가 나뭇가지에 줄을 달아 바람이 붊에 따라 주기적으로 변하는 신호를 만들어 낸다.

설령 이 주기 함수의 주기가 굉장히 길어도, sinusoid가 아닌 이상 이 주기함수에는 주기의 정수분의 일에 해당하는 성분들이 여기에는 들어있다. 다른말로 fundamental frequency의 harmonic성분들이 들어있다.

harmonic중에 충분히 높은 주파수에서 충분한 크기의 power를 가진다면 구조신호를 안테나에 실어 보낼 수 있을 것이다.(물론 현실적으로는 신호의 크기가 너무 작아 불가능하겠지만...)

## Analysis of AM of a Square Pulse Train Carrier

A square pulse train =
a square $*$ impulse train

$$C(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_c)$$

$a_k$ – a discrete *sinc*

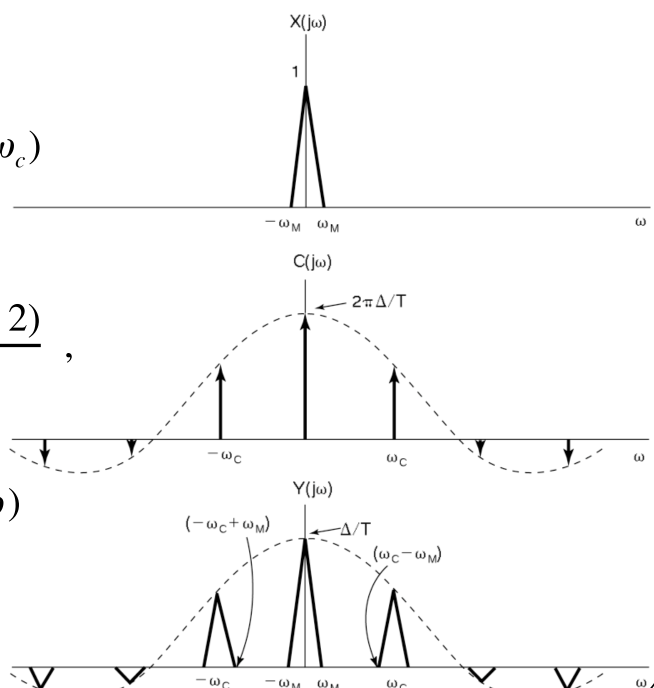
$$a_o = \frac{\Delta}{T}, \qquad a_k = \frac{\sin(k\omega_c \Delta / 2)}{\pi k},$$

$$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * C(j\omega)$$

Drawn assume:

$\omega_c > 2\omega_M$

Nyquist rate is met

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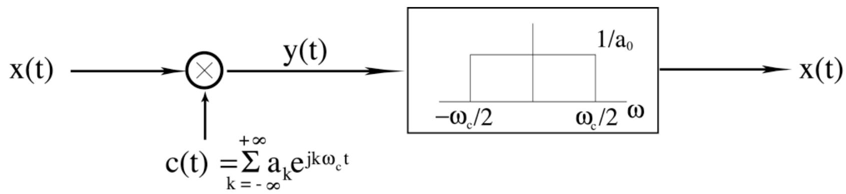
carrier로 사용되는 주기함수의 fundamental frequency \omega_c가 message signal의 bandwidth \omega_M의 2배 이상이어야 no aliasing!

# Observation

1) We get a similar picture with any $c(t)$ that is periodic with period $T$

2) As long as $\omega_c = 2\pi/T > 2\omega_M$, there is no overlap in the shifted and scaled replicas of $X(j\omega)$. Consequently, assuming $a_o \neq 0$:




*x*(*t*) can be recovered by passing *y*(*t*) through a LPF

3) Pulse Train Modulation is the basis for Time-Division Multiplexing
— Assign *time* slots instead of *frequency* slots to different channels,

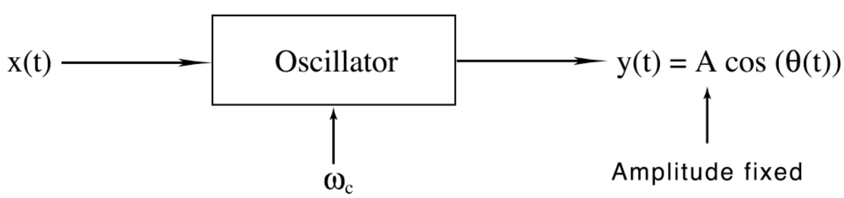
4) Really only need *samples* $\{x(nT)\}$ when $\omega_c > 2\omega_M$
$\Rightarrow$ Pulse Amplitude Modulation



AM의 문제점:
거리가 멀어지거나 교각 및 등에서 shadowing이 생기거나 산과 빌딩등에 의해 multi-path가 있으면 (참고. 아날로그 TV의 고스트현상. 물론 AM에서는 TV에서 처럼 delay가 길지는 않아 신호크기만 바뀐다.) 신호의 크기가 줄어들고 이에 따라 demod된 메시지의 크기가 변한다. 버스에서 많이 경험해 봤을 듯.

## Sinusoidal Frequency Modulation (FM)

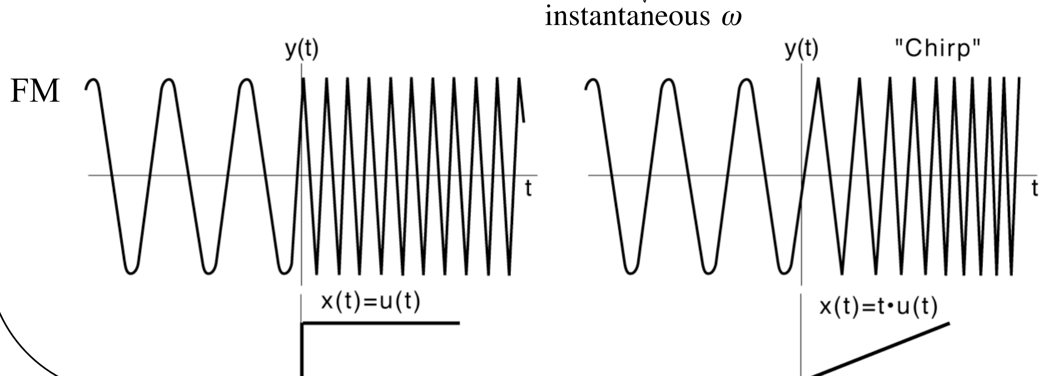
(A nonlinear operation in general, will only discuss the principle here)




Phase modulation: $\theta(t) = \omega_c t + \theta_0 + kx_1(t)$

Frequency modulation: $\dfrac{d\theta}{dt} = \underbrace{\omega_c + kx_2(t)}_{\text{instantaneous } \omega}$

Signal to be transmitted





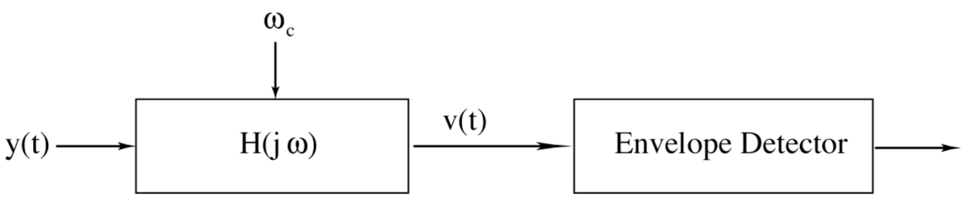

반면 FM은, 신호의 크기에 정보가 실리지 않고 phase에 실리기 때문에 신호의 크기가 변해도 demod된 message의 크기는 변하지 않는다. 물론 지나치게 작아지면 noise가 dominant해져서 소리가 일그러 지지만 그 일그러지는 패턴이 다르다!!!

Amplitude Modulation에 상대되는 말은

Angle Modulation이고
그 밑에
Phase Modulation (PM)과
Frequency Modulation (FM)방식이 있다.

# Sinusoidal FM (Continued)

- Transmitted power does not depend on $x(t)$ (Average power = $A^2/2$)
- Bandwidth of $y(t)$ *does* depend on *amplitude* of $x(t)$
- Demodulation of **FM** signals
  - a) Direct tracking of the phase $\theta(t)$ by using PLL
  - b) Use of an LTI system that acts like a differentiator




$H(j\omega)$ — Tunable band-limited differentiator, over the bandwidth of $y(t)$

$$H(j\omega) \cong j\omega$$

$$\Downarrow$$

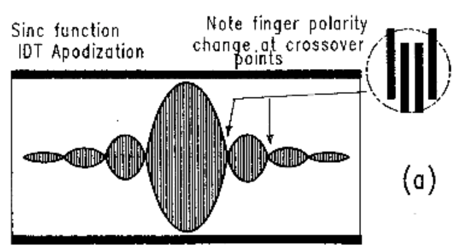
$$v(t) \cong \frac{dy(t)}{dt} = -\underbrace{\left(\omega_c + kx(t)\right)}_{d\theta/dt} A\sin\theta(t)$$ — AM, then envelope detection

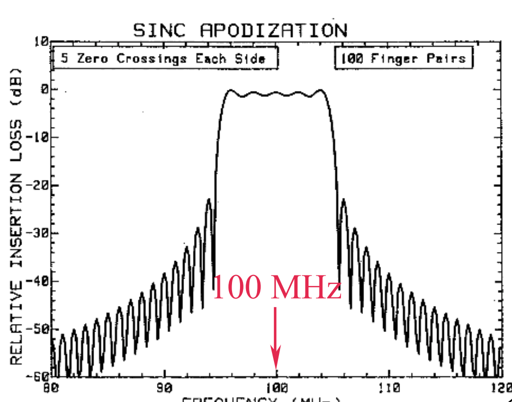
phase locked loop: 정보통신공학개론, 전자회로 시간에 배울 것!



# DT Sinusoidal AM — *E.g. SAW* BPF

- From Lecture #10
- A *sinc* $h[n]$ is a low-pass filter. Here we use a modulated *sinc* (multiplied with $\cos\omega_o n$) to construct a bandpass filter centered at $\omega_o$.



# DT Sinusoidal AM

Multiplication Property

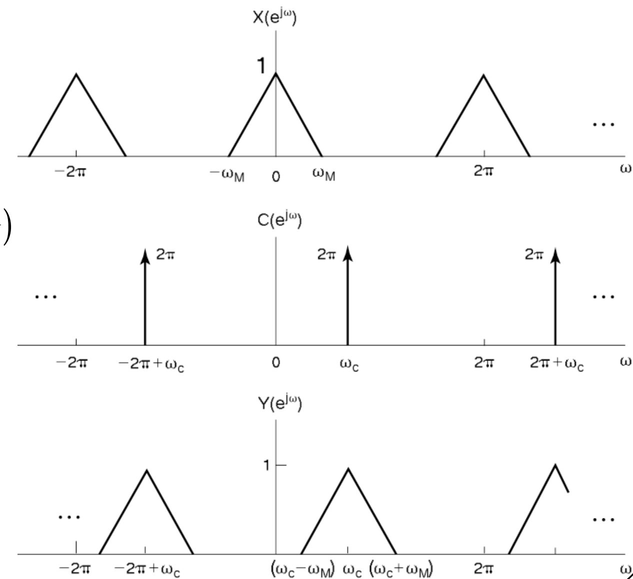
$$y[n] = x[n]\cdot c[n] \longleftrightarrow Y\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\theta}\right)C\left(e^{j(\omega-\theta)}\right)d\theta$$ — periodic convolution

**Example #1:**

$$c[n] = e^{j\omega_c n}$$

$$\Rightarrow \quad C\left(e^{j\omega}\right) = 2\pi \sum_{k=-\infty}^{+\infty} \delta(\omega - \omega_c + 2\pi k)$$

$$Y\left(e^{j\omega}\right) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \otimes C\left(e^{j\omega}\right)$$




CT에서와 마찬가지로 complex sinusoid는 전체 DTFT를 shift할 뿐이다.

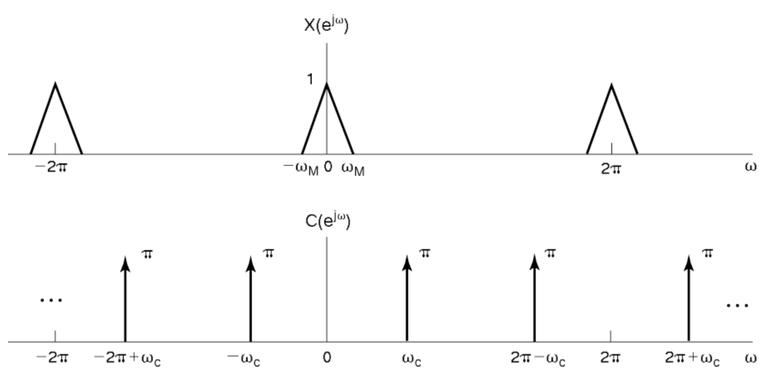


**Example #2:** Sinusoidal AM

$$c[n] = \cos(\omega_c n)$$

$$C(e^{j\omega}) = \pi\left\{\sum_{k=-\infty}^{+\infty} \delta(\omega - \omega_c + 2\pi k) + \delta(\omega + \omega_c + 2\pi k)\right\}$$

역시 Euler's identity가 중요하다.




Drawn assume:

$\omega_c - \omega_M > 0$ &

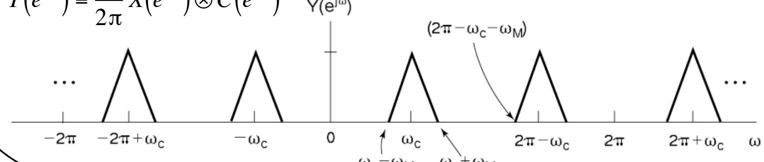
$2\pi - \omega_c - \omega_M > \omega_c + \omega_M$

*i.e.*

$\omega_M < \omega_c < \pi - \omega_M$

$\boxed{\Rightarrow \omega_M < \pi/2}$

$$Y\left(e^{j\omega}\right) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \otimes C\left(e^{j\omega}\right)$$



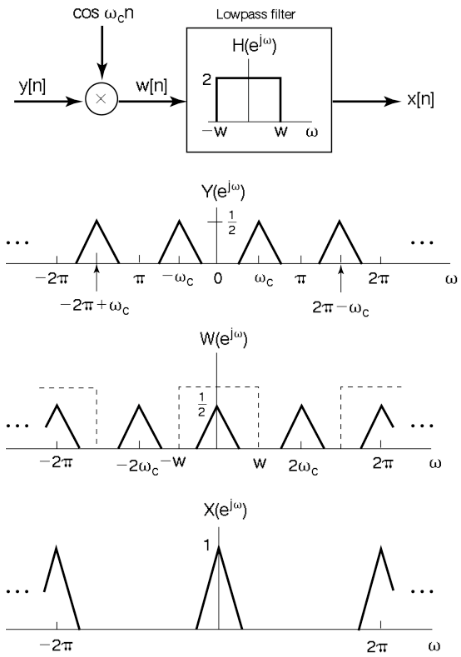

No overlap of shifted spectra

## Example #2 (continued): Demodulation

Possible as long as there is no overlap of shifted replicas of $X(e^{j\omega})$, i. e. as long as $\omega_c - \omega_M > 0$ and $\omega_c + \omega_M < 2\pi - \omega_c - \omega_M$ , *i.e.*

$$\omega_M < \omega_c < \pi - \omega_M$$

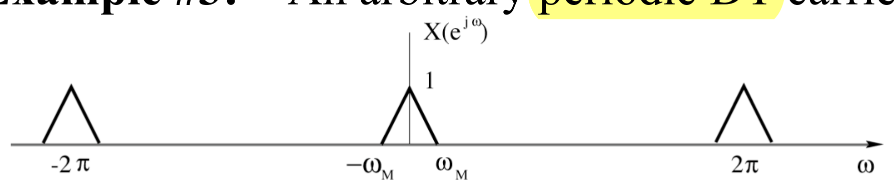
$$W\left(e^{j\omega}\right) = \frac{1}{2\pi} Y\left(e^{j\omega}\right) \otimes C\left(e^{j\omega}\right)$$



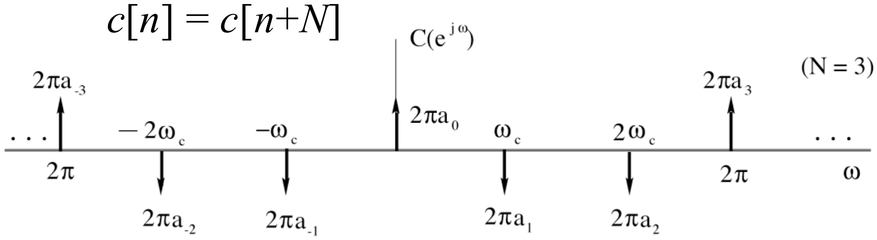



## Example #3: An arbitrary periodic DT carrier

CT에서와 마찬가지...

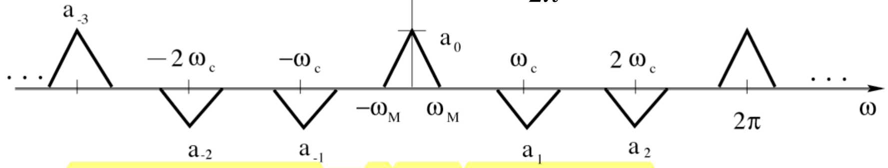



$c[n] = c[n+N]$




\omega_c의 어떤 정수배는 반드시 2\pi가 됨을 잊지말자.

$$Y(e^{j\omega}) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \otimes C\left(e^{j\omega}\right)$$




No overlap when: $\omega_c > 2\omega_M$ (Nyquist rate) $\Rightarrow \omega_M < \pi/N$

## Example #3 (cont.)

$$c[n] = \sum_{k=<N>} a_k e^{j2\pi kn/N} = c[n+N] \ , \quad \omega_c = \frac{2\pi}{N}$$

$$C\left(e^{j\omega}\right) = 2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

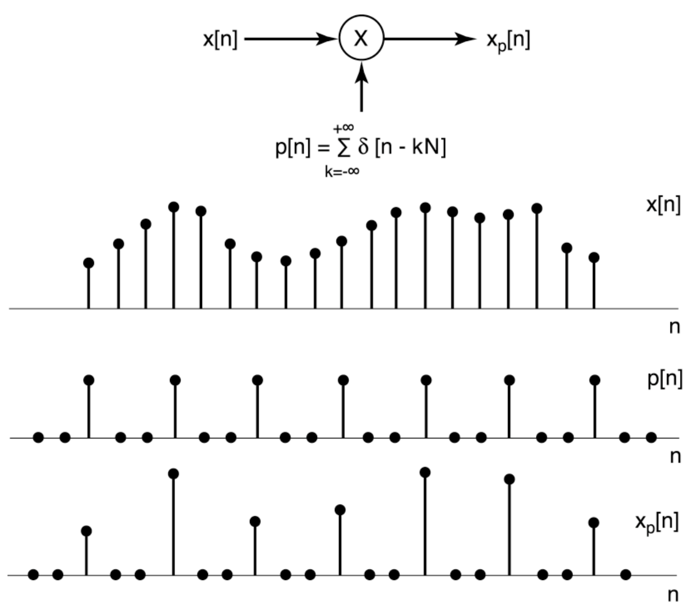
$$Y\left(e^{j\omega}\right) = \underbrace{\frac{1}{2\pi} X\left(e^{j\omega}\right) \otimes C\left(e^{j\omega}\right)}_{\uparrow \text{ periodic convolution}} = \underbrace{\frac{1}{2\pi} X\left(e^{j\omega}\right) * \sum_{k=0}^{N-1} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)}_{\uparrow \text{ regular convolution}}$$

$$= \sum_{k=0}^{N-1} a_k X\left(e^{j(\omega - 2\pi k/N)}\right)$$



## DT Sampling

Motivation: Reducing the amount of signals to be stored or transmitted. *e.g.* CD music recording (Lecture #24).




N = 3

한개는 두고 (N−1)개를 zero로 바꾼다.

Q. DT sampling 은 LTI operation인가?
A. No, it is an LTV processing.

스펙트럼을 보면 없던 성분이 생긴다.

# DT Sampling (continued)

$$p[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN] \longleftrightarrow P(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s), \quad \omega_s = \frac{2\pi}{N}$$

$$x_p[n] = x[n] \cdot p[n] = \sum_{k=-\infty}^{+\infty} x[kN]\delta[n - kN]$$

$$= \left\{ \begin{array}{ll} x[n] & , \text{ if } n \text{ is a multiple of } N \\ 0 & , \text{ otherwise} \end{array} \right\} \Rightarrow \text{Pick one out of } N$$

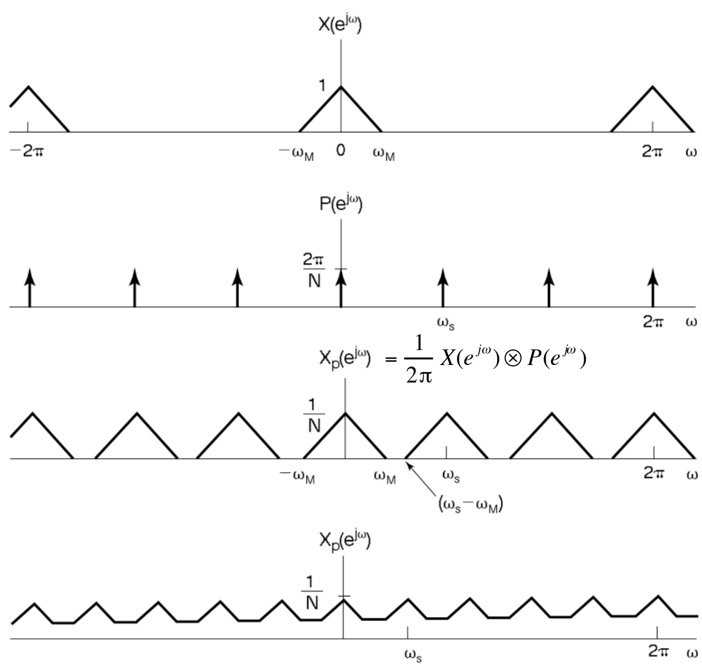
$$X_p\left(e^{j\omega}\right) = \frac{1}{2\pi} X\left(e^{j\omega}\right) \otimes P\left(e^{j\omega}\right) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(e^{j(\omega - k\omega_s)}\right)$$

— periodic with period $\omega_s = \dfrac{2\pi}{N}$



# DT Sampling Theorem

We can reconstruct $x[n]$ if $\omega_s = 2\pi/N > 2\omega_M$




Drawn assume
$\omega_s > 2\omega_M$
Nyquist rate is met
$\Rightarrow \quad \omega_M < \pi/N$

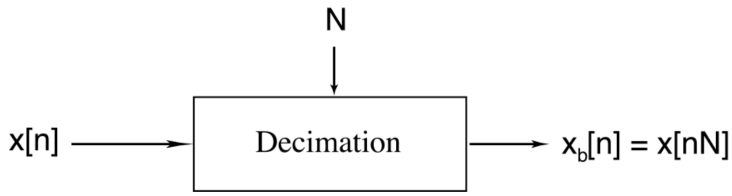
Drawn assume
$\omega_s < 2\omega_M$
Aliasing!

TD에서 N개 마다
samping 하면
FT에서는
0초과 2pi 사이에 (N−1)
개의 impulse를 넣는다.
impulse들 간의 간격은
2pi/N이다.

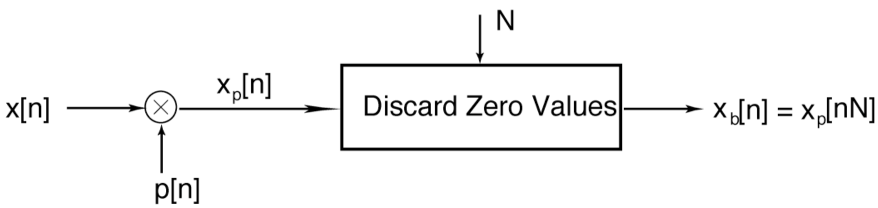
## *Downsampling*

$x_p[n]$ has $(n - 1)$ zero values between nonzero values:

Why keep them around?




Useful to think of this as sampling followed by discarding the zero values



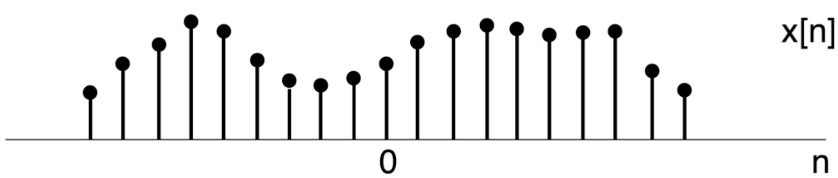

우리 교과서에서는 DT sampling은 일부를 zero로 바꾸어 새로운 DT signal을 얻는 것이고 downsampling은 그 zero들을 제거하여 새로운 DT signal을 얻는 것으로 정의한다.

구별할 것!

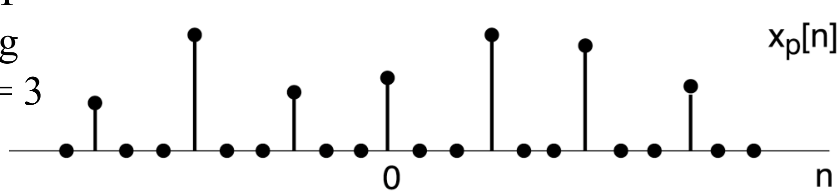
downsampling도 LTV processing이다. 따라서 스펙트럼에서 없던 성분이 생길 수 있다.



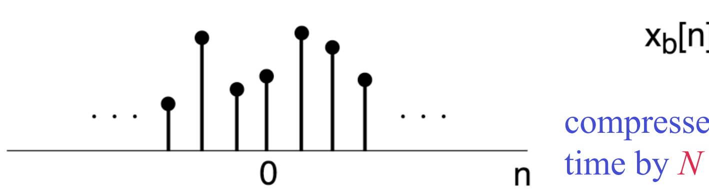
## **Illustration of Downsampling in the Time-Domain** (for $N = 3$)




After DT sampling with $N = 3$




After discarding zero values




compressed in time by $N$

## Downsampling in the Frequency Domain

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} x_b[k] e^{-j\omega k} \qquad (x_b[k] = x_p[kN])$$

$$= \sum_{k=-\infty}^{+\infty} x_p[kN] e^{-j\omega k} \qquad \text{Let } n = kN \text{ or } k = n/N$$

$$= \sum_{n \text{ a multiple of } N}^{+\infty} x_p[n] e^{-j\omega(n/N)}$$

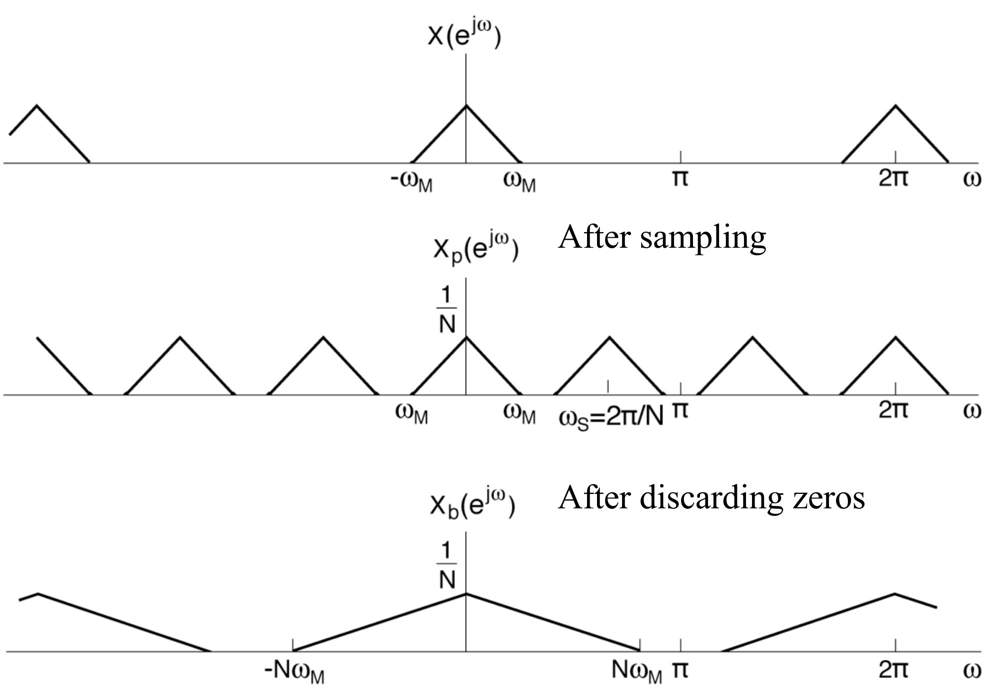
$$= \sum_{n=-\infty}^{+\infty} x_p[n] e^{-j(\omega/N)n} \qquad (\text{since } x_p[n \neq kN] = 0)$$

$$= X_p\left(e^{j(\omega/N)}\right)$$

stretched in frequency by $N$



## Illustration of Downsampling in the Frequency Domain



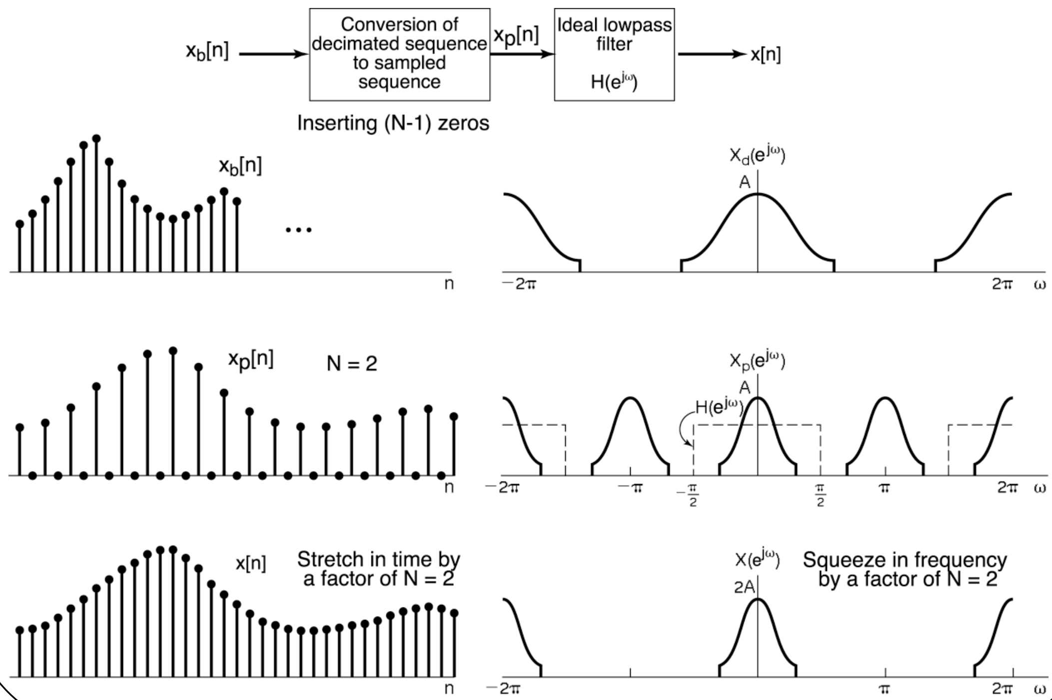

Q. DT processing으로 빠르게 플레이하는 기능을 만들고 싶다. 어떻게 하면될까?

A. AAF후 Downsampling!!!

= decimation

결과적으로 CT에서의 sampling과 동일한 모양이다.
단, sum이 finite sum이다.

## **The Reverse Operation: Upsampling** (*e.g.* CD playback)

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Q. DT processing 만으로 느리게 플레이 하고 싶다.

A. Upsamplilng과 적절한 filtering!!!

= interpolation

upsampling

interpolation filtering

Def.