Lecture Note #3에서는 DT LTI system에 대해 배웠고, 여기서는 CT LTI system에 대해 배운다.

거의 대부분의 내용이 parallel하므로 특히 차이점에 대해 관심을 갖도록 하자.

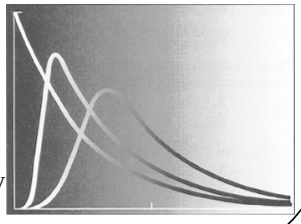
EECE233 Signals and Systems

Lecture Note #4

Prof. Joon Ho Cho

(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

- Covers O & W pp. 90-102, 127-136
- Representation of CT Signals in terms of shifted unit impulses
- Introduction of the unit impulse $\delta(t)$
- Convolution integral representation of CT LTI systems
- Properties and Examples

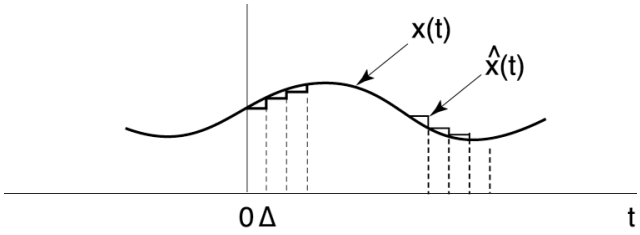


* These lecture notes are based on the lecture notes used in 6.003 taught by Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.

The main ideas in the previous note are summarized as follows.
1. Find an alternate representation of the input signal by using weighted and shifted delta functions.
2. Use linearity and time invariance to derive the notions of impulse response and convolution sum.
3. Examine the properties of the convolution sum and find the physical meanings.
Do the same for CT systems!!!

Representation of CT Signals

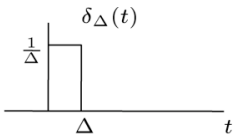
- Approximate any input $x(t)$ as a sum of shifted, scaled square pulses (in fact, that is how we do integration)



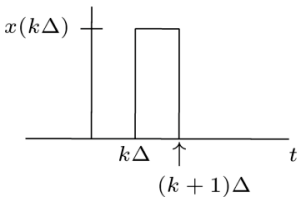
piecewise constant approximation of x(t)

$$\hat{x}(t) = x(k\Delta), \quad k\Delta < t < (k+1)\Delta$$

Representation of CT Signals (cont.)



$\delta_\Delta(t)$ has a unit area



$$= \quad x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\Downarrow$$

$$\hat{x}(t) = \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

Def. The sifting property of the unit impulse is given by

$$\downarrow \qquad \text{limit as } \Delta \to 0$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

높이 1, 폭 Delta, 면적 Delta

고등학교 구분구적법에서 했던 것과 같은 방식을 쓴다. 다만 적분구간이 이제는 무한대이고 delta_Delta(t)의 극한을 delta(t)로 정의하여 사용한다.

x(t)를 delta(t)의 weighted shifted 'sum'으로 나타낸 것으로 이해할 수 있다.

Q. What was the parallel result for DT signals?

앞서 배웠던 DT delta function의 sifting property를 CT delta function으로 확장한다.

The Unit-impulse function $\delta(t)$

Paul Dirac first introduced this function in 1920's to describe the charge density of an electron (diameter < 10^{-13} cm), extremely useful, but made many mathematicians uncomfortable. Definition of $\delta(t)$:

확률 및 랜덤 프로세스 과목에서 discrete random variable 의 pdf를 나타내기 위해 도입한다.

Def. delta(t) is defined as the limit of delta_Delta(t) as Delta tends to zero.

$$\left.\begin{aligned}\delta(t) &= 0 \quad for\ t \neq 0\\ \delta(t) &= \infty \quad for\ t = 0\end{aligned}\right\} \qquad (1)$$

$$\int_{0^-}^{0^+} \delta(t)dt = 1 \qquad (2)$$

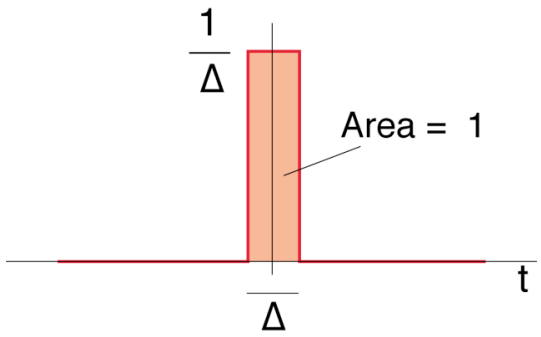
그후 수학자들이 distribution또는 generalized function이라는 이름으로 이를 체계화하였다.

오른쪽 정의는 수학적이지 못한 (엉터리같지만 유용한) 정의이다.

— an infinitesimally sharp pulse with an unity area.

Construction of the Unit-impulse function $\delta(t)$

One of the simplest way — rectangular pulse, taking the limit $\Delta \to 0$.



1사분면에만 있는 rectangular pulse의 극한으로 봐도 되고

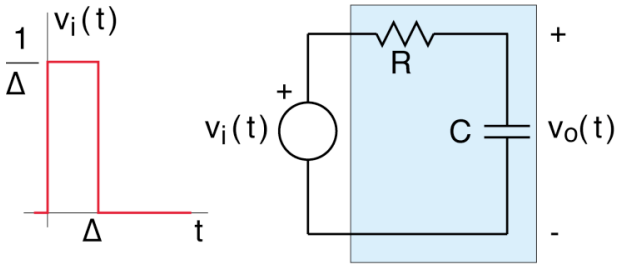
2사분면에만 있는 rectangular pulse의 극한으로 봐도 된다.

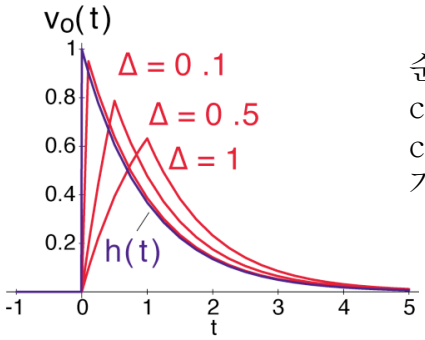
But this is by no means the only way. One can construct a $\delta(t)$ function out of many other functions, *Eg.* Gaussian pulses, triangular pulses, sinc functions, *etc.*, as long as the pulses are short enough — much shorter than the characteristic time scale of the system.

Def. sinc(x) = sin (\pi x) / (\pi x)

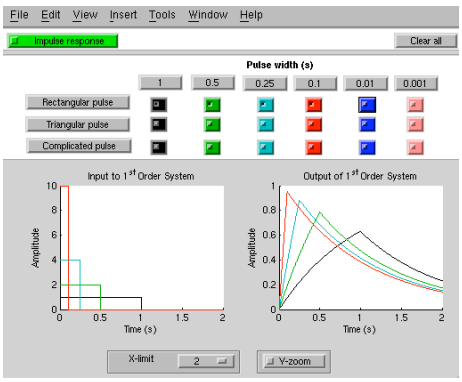
혹자는 sinc(x) = sin (x) /x로 정의하기도 하니, sinc를 사용할때는 어느 정의를 따르는지 확실히 할 것.

Demo: Response of an RC circuit to short pulses





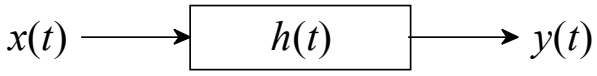
순간적으로 capacitor가 charge되었다가 방전됨.



Response from initial rest to pulses of different shapes and durations, but with unit area. As the duration decreases, the response becomes similar for different pulse shapes.

Q. What was the definition of initial rest?

Response of a CT LTI System



Def. Given an **LTI** system, the *unit impulse response* $h(t)$ is defined as

$$\delta(t) \longrightarrow h(t)$$

Lemma.

From Time-Invariance:

$$\Downarrow$$

$$\delta(t-\tau) \longrightarrow h(t-\tau)$$

Theorem.

From Linearity and Time-Invariance:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow y(t) = \underbrace{\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau}_{\text{Convolution Integration}} = x(t) * h(t)$$

Interpretation.
(i) The output is a weighted integral of delayed impulse responses.
(ii) The output is the sliding correlation between the input and the time-reversed impulse response.

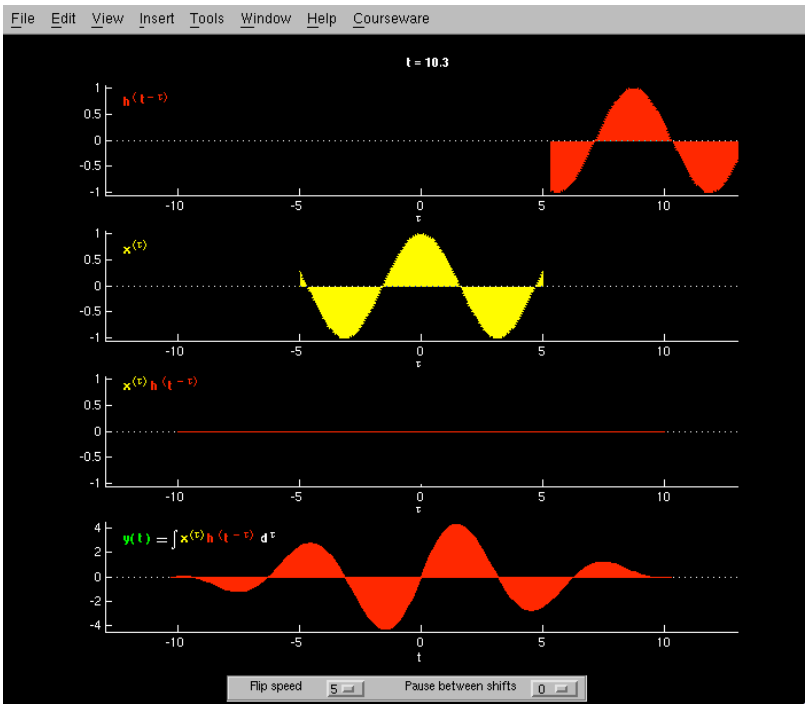
Operation of CT Convolution

$$y(t) = x(t) * h(t) \equiv \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$h(\tau) \xrightarrow{Flip} h(-\tau) \xrightarrow{Slide} h(t-\tau) \xrightarrow{Multiply}$$

$$x(\tau)h(t-\tau) \xrightarrow{Integrate} \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

Demo: CT convolution



Properties and Examples

LTI의 output은 weighted integral of delayed input 으로 보는 interpretation 을 얻는다.

1) Commutativity: $x(t) * h(t) = h(t) * x(t)$

2) $x(t) * \delta(t - t_o) = x(t - t_o)$ (Sifting property: $x(t) * \delta(t) = x(t)$)

3) An integrator:

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

$$\Downarrow$$

$$h(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$$

The impulse repose of the identity system is the impulse itself.
delta(t) in
delta(t) out

Q1. The integrator는 LTI system인가?

Q2. LTI system이라면 impulse response는 어떻게 되는가?

That is

$$\boxed{x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau}$$

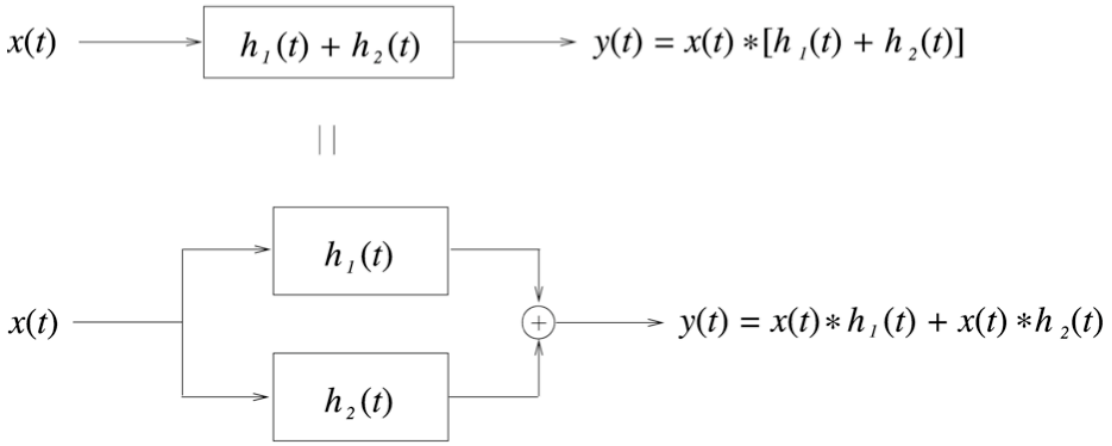
4) Step response

Def. A CT LTI system에 unit step input을 가했을때의 출력을 its unit step response 라 한다.

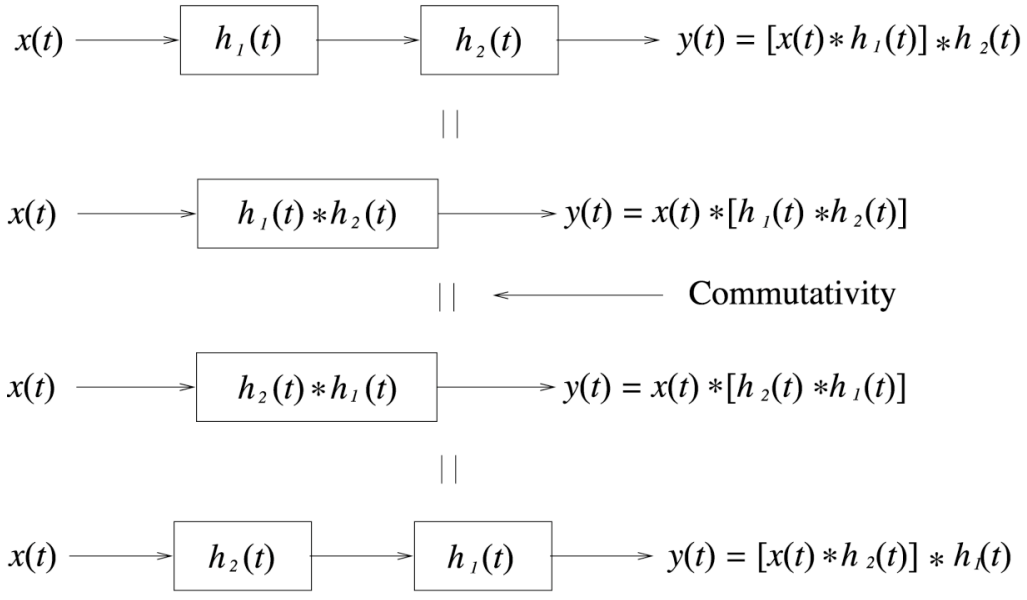
$$s(t) = u(t) * h(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

Q1. Find s(t) from h(t).
Q2. Find h(t) from s(t).

Distributivity



Associativity



More properties

Theorem. A CT LTI system is causal $\leftrightarrow h(t) = 0$, for all $t < 0$.

As a result:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

$y(t)$ only depends on $x(\tau < t)$.

Def. A CT signal is absolutely integrable if ...

Thm. A CT LTI system is BIBO stable $\leftrightarrow \int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < \infty$

Sufficient condition: For $|x(t)| \le x_{\max} < \infty$.

$$|y(t)| = \left|\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right| \le x_{\max}\left|\int_{-\infty}^{+\infty} h(t-\tau)d\tau\right| < \infty.$$

Necessary condition: Suppose $\int_{-\infty}^{+\infty} |h(\tau)|d\tau = \infty$

Let $x(t) = h^*(-t)/|h^*(-t)|$, then $|x(t)| \equiv 1$ bounded

But $y(0) = \int_{-\infty}^{+\infty} x(\tau)h(-\tau)d\tau = \int_{-\infty}^{+\infty} \frac{h^*(-\tau)h(-\tau)}{|h(-\tau)|}d\tau = \int_{-\infty}^{+\infty} |h(-\tau)|d\tau = \infty$

More About $\delta(t)$: The Operation Definition of the Unit Impulse

$\delta(t)$ — idealization of a unit-area pulse that is so short that, for any physical systems of interest to us, the system responds only to the area of the pulse and is insensitive to its duration

Operationally: The unit impulse is the signal which when applied to any LTI system results in an output equal to the impulse response of the system. That is,

$$\delta(t) * h(t) = h(t) \qquad \text{for all } h(t)$$

보다 정확히 이야기 하면 delta function은 sifting property에 의해 정의되는 mathematical entity이다

— $\delta(t)$ is defined by what it does under convolution.

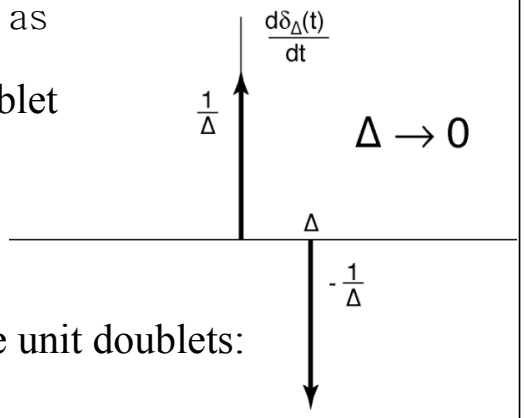
Q1. Is this an LTI system?
Q2. If so, what is the impulse response?

Differentiator and the Unit Doublet

$$x(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow y(t) = \frac{dx(t)}{dt}$$

Def. The unit doublet is defined as

Impulse response = unit doublet



Notation.

$$u_1(t) = \frac{d\delta(t)}{dt}$$

The operational definitions of the unit doublets:

$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$

1. x(t) -> d/dt -> d/dt -> ... -> d/dt -> y(t)
2. <=> x(t) -> d^n/dt^n ->y(t)
3. LTI?
4. If so, h(t) = ?

Triplets and beyond!

Def.

$$n > 0$$

$$u_n(t) = \underbrace{u_1(t) * \cdots * u_1(t)}_{n\ times}$$

Operational definitions:

$$x(t) * u_n(t) = \frac{d^n x(t)}{dt^n} \qquad (n > 0)$$

Integrators

$$x(t) \longrightarrow \boxed{\int} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Q. Is this system LTI? If yes, find the impulse response.

Def. $u_{-1}(t) \equiv u(t)$

Operational definition: $x(t) * u_{-1}(t) = \int_{-\infty}^{t} x(\tau)d\tau$

Cascade of *n* integrators:

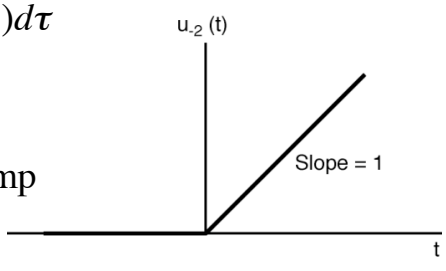
$$u_{-n}(t) = \underbrace{u_{-1}(t) * \cdots * u_{-1}(t)}_{n \text{ times}} \qquad (n > 0)$$

Q. u_{−1}(t)는 BIBO stable한가?
Q. What about u_{−2}(t)?

Integrators (continued)

$$\delta(t) \longrightarrow \boxed{\int} \xrightarrow{u_{-1}(t)} \boxed{\int} \longrightarrow u_{-2}(t)$$

$$u_{-2}(t) = \int_{-\infty}^{t} u_{-1}(\tau)d\tau = \int_{-\infty}^{t} u(\tau)d\tau$$
$$= u(t)\int_{0}^{t} d\tau$$
$$= t \cdot u(t) \qquad \text{the unit ramp}$$



More generally, for $n > 0$

$$u_{-n}(t) = \frac{t^{(n-1)}}{(n-1)!}u(t)$$

Mathematical induction의 3 steps:
1. Basis (step)
2. Induction/inductive step
3. Conclusion

주의! Conclusin은 오직 fininte한 parameter 값에 대해서만 성립함이 증명된 것임.

Q. 좌식을 mathematical induction을 이용하여 증명하시오.
(Mathematical induction vs. transfinite induction Wikipedia찾아봐요~)

Notation

Def. $u_0(t) = \delta(t)$

Lemma. $u_n(t) * u_m(t) = u_{n+m}(t)$

n and m can be $\pm$

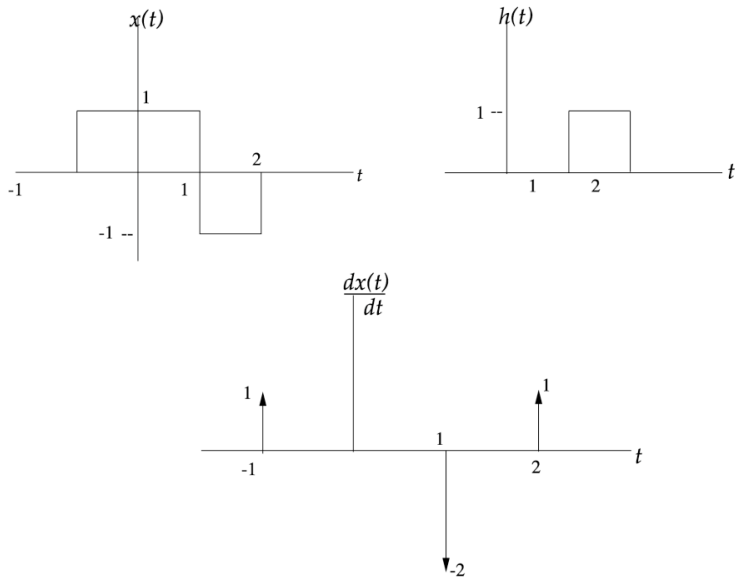
E.g. $u_1(t) * u_{-1}(t) = u_0(t)$

$||$

$$\left(\frac{d}{dt}u(t)\right) = \delta(t)$$

Note: The order of integration and differentiation may not be exchangeable.

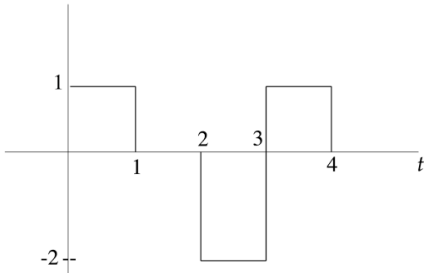
Example: Calculating $x(t) * h(t)$



$$\frac{dx(t)}{dt} = \delta(t+1) - 2\delta(t-1) + \delta(t-2)$$

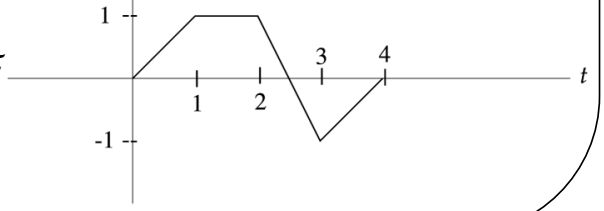
Example (continued)

$$\frac{dx(t)}{dt} * h(t) = h(t+1) - 2h(t-1) + h(t-2)$$



$$x(t) * h(t) = u_{-1}(t) * \{[u_1(t) * x(t)] * h(t)\}$$

$$= \int_{-\infty}^{t} \left[\frac{dx(\tau)}{d\tau} * h(\tau) \right] d\tau$$



Q. 폭이 같은 두 사각파의 convolution integral은 ?
A. 폭이 2배인 삼각파.
Q. 이를 이용하여 이 example 의 x(t)*h(t)를 구하시오.

Note: 여기서의 trick은,
rectangular pulse의 미분은 delta들로 나오고 delta입력에 의한 출력은 impulse response로 쉽게 구해지며 구형파의 적분도 쉽다는 것을 이용하여

y(t) = x(t)*h(t)
= x(t)* delta(t) *h(t)
= x(t)* u_0(t) *h(t)
= x(t)*[u_1(t)*u_{−1}(t)]*h(t)
= ([x(t)*u_1(t)]*h(t))*u_{−1}(t)
로 alternate way to compute y(t)를 찾아내는 것이다.

사실 이 문제는 u_1(t)*u_{−1}(t) = u_0(t) = delta (t)을 사용하는 예를 보여주기 위해 위와 같이 풀었으나, 이렇게 복잡하게 할 필요 없이, 폭이 같은 두 rectangular pulses의 convolution이 삼각파임에 주목하여 inearity와 time invariance를 이용하면 output y(t)를 쉽게 구할 수 있다.
input이 하나의 pulse의 weighted and uniformly time shifted versions의 합으로 되어 있으면 이 방법으로 이해하는 것이 아주 유용하다−> 디지털 통신에서 pulse amplitude modulation을 이해하기 위해 꼭 필요하다.