EECE233 Signals and Systems

Lecture Note #5

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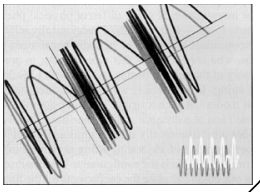
(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

- Covers O & W pp. 177-185,211-225
- Complex Exponentials as Eigenfunctions of LTI Systems
- DT Fourier series
- DT Fourier series examples



* These lecture notes are based on the lecture notes used in 6.003 taught by Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.

지금까지 우리는 DT LTI system과 CT LTI system에 대해 time domain에서 그 해석법을 배웠다.

가장 중요한 식은, convolution sum과 integral이었고
이로부터 LTI system은 impulse response로 characterize됨을 알 수 있다.

1. Analysis
LTI system의 출력은 입력의 delayed weighted sum/integral또는 등가적으로 impulse response의 delayed weighted sum임을 보았다.

2. Synthesis
echo mic과제에서 보았듯, 원하는 시스템을 만든다는 것은 원하는 기능을 하도록 impulse response를 만들어 내는 것이다. 또한 commutativity, distributivaty, associativity등을 이용해 synthesize할 수 있음도 보았다.

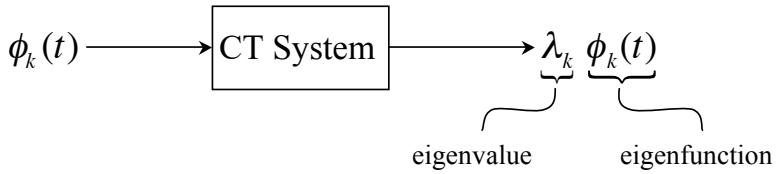
하지만, 과제에서처럼 주먹 구구로 만들어서는 engineering이라 할 수 없다. 게다가 원하는 impulse response가 무엇인지 알더라도 예를 들어 어떤 미분/차분 방정식으로 어떻게 근사해야 하는가?

Q. 어떤 보다 나은 해석/디자인 방법이 없을까?
A. 힌트는, 우리가 이미 배운 선형대수학에 있다. 행렬의 곱, row space, column space만 배우면 선형대수가 끝났던가? eigenvalue, eigenvector는 왜 필요한가?

The eigenfunctions $\phi_k(t)$ and their properties

(Focus on CT systems now, but results apply to DT systems as well.)

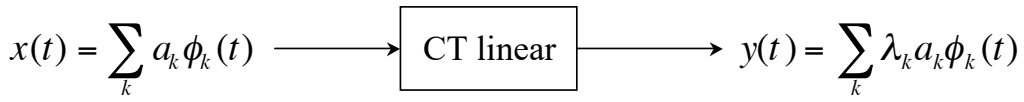
Def.



Eigenfunction in → same function out with a "gain"

Lemma. If a CT system is linear, then ....

From the superposition property of linear systems:



Now the task of finding response of linear systems is to determine λ_k. The solution is simple, general, and insightful.

Theorem. If a CT system is linear, then an eigenfunction and the corresponding eigenvalue can be found by solving the following Fredholm integral equation given by

\int h(t,\tau)x(\tau)d\tau =
\lambda\x(t)

for a non-zero function x(t), where h(t,\tau) is called the Kernel of the integral equation,x(t) is called an eigenfunction,
and lamba is called the corresponding eigenvalue.

Remarks.
(i) Existence of an eigenfunction
(ii)Uniqueness of an eigenfunction
(iii) Countably infinite vs. uncoutably infinite

Q1. Given a two-port RLC circuit, how can you compute the output to an arbitrary input?
A1. We can(?) find the output by solving a linear constant coefficient differential equation (LCCDE).

Q2. What if the input is a pure sinusoid? What is the most important property of the output?
A2. The output is a pure sinusoid with the same frequency as the input.

Q3. Do we still need to solve a differential equation when the input is a pure sinusoid?
A3. No. We just need to find the (complex valued) impedance and scale and phase shift the input to obtain the output.

Q4. What is the name of the above mentioned technique to find the output to a pure sinusoid input?
A4. the phasor analysis

Q5. Given an RLC circuit, what is the impedance? How can you find it?
A5. Use KCL and KVL to find the impedance with the mapping of
C -> 1/(j\omega C),
L -> j\omega L,
R - > R

Q6. Why the phasor analysis works?
A6. This question will be answered in this note.

Two Key Questions

1. If they exist, what are the eigenfunctions of a general LTI system?

linear algebra를 기억해보면, matrix마다 eigenvector들의 집합이 다 달랐다. LTI system에서도 일단, 주어진 impulse response마다 해당하는 eigenfunction들의 집합이 다를 것이라 예상할 수 있다. 여기서 질문은, 혹시 모든 LTI system에 대해서 eigenfunction이 되는 그런 signal이 존재할까? 하는 것이다.

2. What kinds of signals can be expressed as superpositions of these eigenfunctions?

Q1. It will turn out that the complex sinusoids (actually not only the sinusoids but also all complex exponential funtions) are all eigenfunctions of ANY LTI system.

Now, the question is: What kinds of input signals can be represented by a linear combination of these complex sinusoids?

Q2. True of False: A summation of two periodic signals (e.g. sinusoids) is periodic.

A specific LTI system can have more than one type of eigenfunction

Ex. #1: Identity system

$$x(t) \longrightarrow \boxed{\delta(t)} \longrightarrow x(t) * \delta(t) = x(t)$$

Any function is an eigenfunction for this LTI system.

Linear algebra에서 보았듯, an identity matrix는 any non-zero vector is an eigenvector.

Ex. #2: A delay

$$x(t) \longrightarrow \boxed{\delta(t-T)} \longrightarrow x(t-T)$$

Any periodic function $x(t) = x(t+T)$ is an eigenfunction.

Linear algebra에서 MNxMN matrix로서 N만큼 down shift하는 matrix 를 생각해보라. 그러면 any downshift invariant vector가 eigenvector가 된다.

Ex. #3: $h(t)$ even

$$\cos \omega t \longrightarrow \boxed{h(t) = h(-t)} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\cos[\omega(t-\tau)]d\tau = \int_{-\infty}^{\infty} h(\tau)[\cos\omega t \cdot \cos\omega\tau + \sin\omega t \cdot \sin\omega\tau)]d\tau$$

$$= \cos\omega t \underbrace{\int_{-\infty}^{\infty} h(\tau)\cos\omega\tau d\tau}_{H(j\omega)} \quad \text{— } \cos\omega t \text{ is an eigenfunction}$$

Q. 만약 h(t)가 odd이면, sin(wt)가 eigenfunction인가?

많은 경우, complex exponentials라고 하면 s가 pure imaginary일 경우를 가리키나, 그런 경우는 특별히complex sinusoids라고 하기도 한다.

Complex Exponentials are the *only* Eigenfunctions of *any* LTI Systems

$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau$

$= \left[\int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau \right] e^{st}$

$= \underbrace{H(s)}_{\text{eigenvalue}} \underbrace{e^{st}}_{\text{eigenfunction}}$

where s is any complex number that makes the convolution integral well defined.

Def. H(s) as a function of s is called the Laplace transform of h(t).

$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{m=-\infty}^{\infty} h[m] z^{n-m}$

$= \left[\sum_{m=-\infty}^{\infty} h[m] z^{-m} \right] z^n$

$= \underbrace{H(z)}_{\text{eigenvalue}} \underbrace{z^n}_{\text{eigenfunction}}$

where z is any complex number that makes the convolution sum well defined.

Def. H(z) as a function of z is called the z-transform of h(t).

impulse response가 time domain에서 LTI system을 이해하고 해석하고 디자인하는 데 이용된다면,
system function은 s또는 z domain에서 LTI system을 이해하고 해석하고 디자인하는 데 이용된다.

System Functions *H(s)* or *H(z)* – Eigenvalues for now, and more about them later

CT: $x(t) \xrightarrow{e^{st}} \boxed{h(t)} \xrightarrow{H(s)e^{st}} y(t)$

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

$$x(t) = \sum a_k e^{s_k t} \longrightarrow y(t) = \sum a_k H(s_k) e^{s_k t}$$

DT: $x[n] \xrightarrow{z^n} \boxed{h[n]} \xrightarrow{H(z)z^n} y[n]$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

$$x[n] = \sum a_k z_k^n \longrightarrow y[n] = \sum a_k H(z_k) z_k^n$$

Question 2. What kinds of signals can we represent as "sums" of complex exponentials?

For Now: Focus on restricted sets of complex exponentials

CT: $s = j\omega$ - purely imaginary ,

i.e. signals of the form $e^{j\omega t}$

DT: $z = e^{j\omega}$, i.e. signals of the form $e^{j\omega n}$

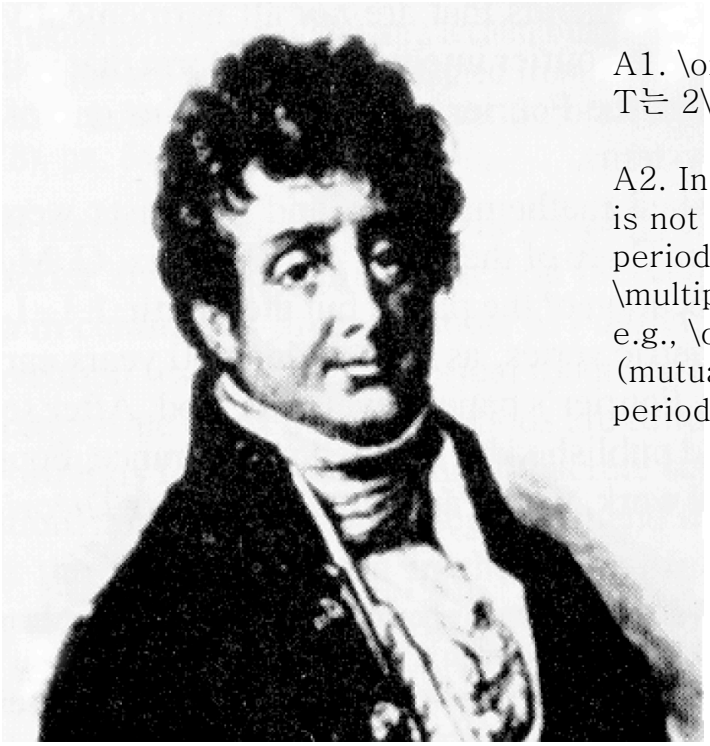
⇓

CT & DT Fourier Series and Transforms

Q1. What is the period of e^{j \omega t}?

Q2. What is the period of e^{j \omega n}?

Q1. It turned out that the complex exponential funtions are all eigenfunctions of ANY LTI system.
Now, the question is: What kind of input signals can be represented by a linear combination of these complex exponentials?
Q2. True of False: A summation of two periodic signals (e.g. sinusoids) is periodic.



Joseph Fourier (1768-1830)

A1. \omega T = 2 \pi에서 주기 T는 2\pi/=omega

A2. In general, e^{j\omega n} is not a periodic function. It is periodic iff \omega is a 2\pi \multiple of a rational number, e.g., \omega = 2\pi M/N (mutually prime). Then, the period is N.

Q. What happens to H(j\omega) or H(e^{j\omega}) if h(t) or h[n] is real?

A. H(j\omega)=
H(−j\omega)^*.

H(e^{j\omega))=
H(e^{−j\omega))^*=

Q. What is the output of a CT LTI system when x(t) = cos(\omega_0t)?

A. r cos(\omega_0t + theta)

where

r = mag(H(j\omega_0))
theta = phase(H(j\omega_0))

Q. Find all DT complex exponential functions with period N.

A. There are only N distinct DT complex exponential functions with period N.

Fourier Series Representation of DT Periodic Signals

- $x[n]$ - periodic with fundamental period N, fundamental frequency

$$x[n+N] = x[n] \text{ for all n} \qquad \textit{and} \qquad \omega_o = \frac{2\pi}{N}$$

- Only $e^{j\omega n}$ which is periodic with period N will appear in the *FS*

$$\omega N = k2\pi \leftrightarrow \omega = k\omega_o, \;\; k = 0, \pm 1, \pm 2, \cdots$$

- There are only N distinct signals of this form

$$e^{j(k+N)\omega_0 n} = e^{jk\omega_0 n} e^{\overbrace{jN\omega_0 n}^{2\pi n}} = e^{jk\omega_0 n}$$

- So we *could* just use $e^{j0\cdot\omega_0 n}$, $e^{j1\cdot\omega_0 t}$, $e^{j2\cdot\omega_0 t}$,…, $e^{j(N-1)\cdot\omega_0 t}$

- However, it is often useful to allow the choice of N consecutive values of k to be *arbitrary* (*E.g.* choose {-(N-1)/2, (N-1)/2} if N is odd and $x[n]$ has definite parity).

⇓

Q1. When x[n] is a periodic signal with period N, is it always possible to represent x[n] in the form of

x[n] = \sum_{k=<N>} a_k exp(j 2\pi kn/N) ?

A1. Yes.

Q2. How?

A2.
[
x[0]
x[1]
x[2]
...
x[N-1]
] = [exp(j*2*\pi (i-1)(j-1)/N)]
[
a_0
a_1
...
a_{N-1}
]

Then, we have
x = U \sqrt{N} a
where U is unitary.
Thus, we obtain a = inv(U)x/sqrt{N}.

DT Fourier Series Representation

If x[n] is a DT signal with period N, then...

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

This summation is a finite sum, which is always well defined.

$\sum_{k=<N>}$ = Sum over *any* N consecutive values of k

— This is a *finite* series

$\{a_k\}$ - Fourier (series) coefficients

Questions:

1) What DT periodic signals have such a representation?

2) How do we find a_k?

A. All.
Draw a Venn diagram.

A. Both of the questions are related to what you learned in linear algebra.

Convert the questions to equivalent linear algebra problems.

Answer to Question #1:

Any DT periodic signal has a Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$\Downarrow$$

$$x[0] = \sum_{k=<N>} a_k$$

$$x[1] = \sum_{k=<N>} a_k e^{jk\omega_0}$$

$$x[2] = \sum_{k=<N>} a_k e^{j2k\omega_0}$$

$$\vdots$$

$$x[N-1] = \sum_{k=<N>} a_k e^{j(N-1)k\omega_0}$$

Put these equations in an equation in terms of a matrix and vectors.

N equations for N unknowns, a_0, a_1, ... a_{N-1}

A More Direct Way to Solve for a_k

Finite geometric series

$$\sum_{n=0}^{N-1}\alpha^n = \begin{cases} N & , \ \alpha=1 \\ \frac{1-\alpha^N}{1-\alpha} & , \ \alpha\neq 1 \end{cases}$$

$$\Downarrow \ \alpha = e^{jk\omega_0}$$

$$\sum_{n=<N>} e^{jk\omega_0 n} = \sum_{n=0}^{N-1}(e^{jk\omega_0})^n = \sum_{n=0}^{N-1}(e^{jk2\pi/N})^n$$

$$= \begin{cases} N & , \ k=0,\pm N,\pm 2N \\ \frac{1-e^{j\overbrace{k\omega_o}^{k2\pi}N}}{1-e^{jk\omega_o}} = 0 & , \ \text{otherwise} \end{cases}$$

$$= N\delta[k-mN]$$

omega_0= 2\pi/N

So, from

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$\Downarrow \ \sum_{n=<N>} e^{-jm\omega_0 n} \ \textit{to both sides}$$

— Taking an "inner product" of $x[n]$ and $e^{jm\omega_0 n}$

$$\sum_{n=<N>} x[n]e^{-jm\omega_0 n} = \sum_{n=<N>}\left(\sum_{k=<N>} a_k e^{jk\omega_0 n}\right)e^{-jm\omega_0 n}$$

$$= \sum_{k=<N>} a_k \underbrace{\left(\sum_{n=<N>} e^{j(k-m)\omega_0 n}\right)}$$

$N\delta[k-m]$ — orthogonality

$$= Na_m$$

$$\Downarrow$$

DT Fourier Series Pair $\left(\omega_o = \frac{2\pi}{N}\right)$

Theorem. If x[n] is periodic with period N, then

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk\omega_0 n} \qquad \text{(Analysis equation)}$$

This is a Fourier series representation of a DT periodic signal x[n].

This shows how to find the Fourier series coefficients.

Note: It is convenient to think a_k as being defined for *all* integers k. So:

1) $a_{k+N} = a_k$ — *Unique* for DT. Since $x[n] = x[n+N]$, there are only N pieces of information, whether in the time-domain or in the frequency-domain.

2) We only use N consecutive values of a_k in the synthesis equation

일반화하면, k를 N으로 나누었을때 서로다른 나머지를 갖는 a_k들을 쓰면 언제나 성립한다. 굳이 연속적인 k들을 쓸 필요는 없다.

Q. Given x[n], find its Fourier series representation.

Example #1: Sum of a pair of sinusoids

$$x[n] = \cos(\pi n / 8) + \cos(\pi n / 4 + \pi / 4)$$

— periodic with period $N = 16 \Rightarrow \omega_o = \pi / 8$

$$x[n] = \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right] + \frac{1}{2}\left[e^{j\pi/4} e^{j2\omega_0 n} + e^{-j\pi/4} e^{-j2\omega_0 n}\right]$$

더 이상 sine cosine을 쓰지 말라!

$$\Downarrow$$

$a_0 = 0$

$a_1 = 1/2$

$a_{-1} = 1/2$

$a_2 = e^{j\pi/4}/2$

$a_{-2} = e^{-j\pi/4}/2$

$a_3 = 0$

$a_{-3} = 0$

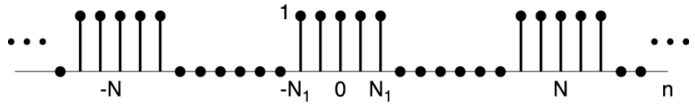
$\vdots$

$a_{15} = a_{-1+16} = a_{-1} = 1/2$

$a_{66} = a_{2+4\times16} = a_2 = e^{j\pi/4}/2$

Q. Find the Fourier series representation of x[n].

Example #2: DT Square wave



$$a_0 = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n] = \frac{(2N_1+1)}{N} = a_N = a_{-N} = a_{6N} = \cdots$$

For $k \neq$ multiple of N:

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} \overset{n=m-N_1}{=} \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\omega_0 (m-N_1)}$$

$$= \frac{1}{N} e^{jk\omega_0 N_1} \sum_{m=0}^{2N_1} (e^{-jk\omega_0})^m = \frac{1}{N} e^{jk\omega_0 N_1} \frac{1-e^{-jk\omega_0(2N_1+1)}}{1-e^{jk\omega_0}}$$

$$= \frac{1}{N} \frac{\sin\left[k(N_1+1/2)\omega_0\right]}{\sin(k\omega_0/2)} = \frac{1}{N} \frac{\sin\left[2\pi k(N_1+1/2)/N\right]}{\sin(\pi k/N)}$$

$\Downarrow$

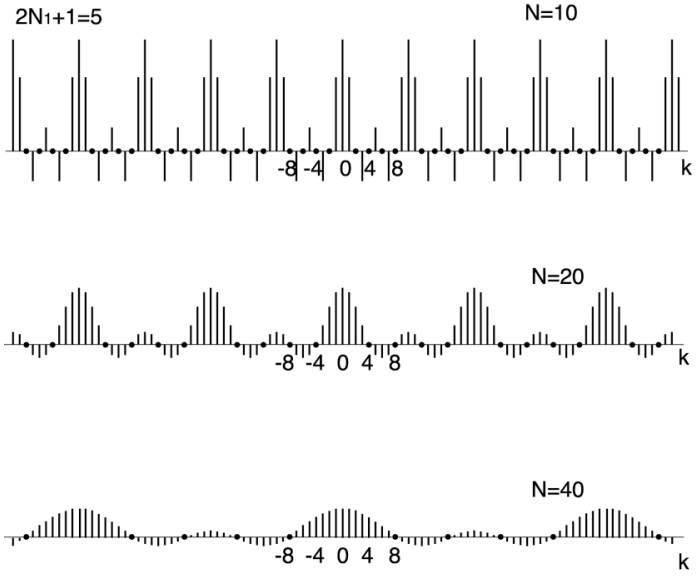
Q. Rewrite this answer using sinc functions, where
sinc(x) = 1 for x=0
sin(\pi x)/(\pi x) for x \ne 0

There are two conventions in defining a sinc function.
1. normalized sinc function
Sa(x) = sinc(x) = sin(pi x)/(pi x)
2. unnormalized sinc function
Sa(x) = sinc(x) = sin(x)/(x)

A related function: sine integral
Si(x) = int_{0}^{x} sin(y)/y dy

Example #2: DT Square wave (continued)

$$a_k = \frac{1}{N} \frac{\sin\left[2\pi k(N_1+1/2)/N\right]}{\sin(\pi k/N)}$$



Convergence Issues for DT Fourier Series:

Not an issue, since all series are finite sums.

Properties of DT Fourier Series: Lots, just as with CT Fourier Series

Example:

$$x[n] \longleftrightarrow a_k$$

x[n]이 N주기함수이고 DTFS coefficients가 a_k일때,

$$e^{jM\omega_0 n}x[n] \longleftrightarrow b_k = ?$$

$$x[n]e^{jM\omega_0 n} = \sum_{r=<N>} a_r e^{jr\omega_0 n} e^{jM\omega_0 n}$$

$$\overset{k=r+M}{=} \sum_{k=<N>} a_{k-M} e^{jk\omega_0 n}$$

$$\Downarrow$$

$$b_k = a_{k-M}$$

Frequency shift
$jk\omega_o \rightarrow j(k\text{-}M)\omega_o$

Q1. e^{j2\pi Mn/N}x[n]는 주기함수인가? 만약 그렇다면 주기는 얼마인가? 만약 그렇다면 이의 DTFS coefficients는?

Q2. 주기함수 x[n−n_0]의 DTFS coefficients는?

Q3. If x[n] is real and periodic, what is the property of the Fourier series coefficients?