EECE233 Signals and Systems

Lecture Note #6

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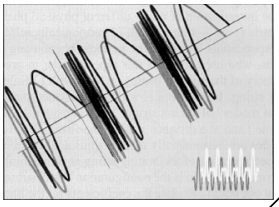
(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

- Covers O & W pp. 186-210
- Fourier Series representation of CT periodic signals
- How do we calculate the Fourier coefficients?
- Convergence and Gibbs' Phenomenon
- CT Fourier series examples



* These lecture notes are based on the lecture notes used in 6.003 taught by Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.

LN 5에서
DT LTI의 공통된 eigenfunciton이 complex exponential z^n,
CT LTI의 공통된 eigenfunciton이 complex exponential e^{st}임을 보았다.
이어서 focus를 narrow down하여
complex exponential중에서 특히 주기함수인
e^{j\omega t} for various \omega
그리고
e^{j\omega n} for various \omega
을 먼저 고려하여
이들의 linear combination으로 나타나는 LTI systems의 admissible input들이 무엇이 있는지를 공부하기 시작하였다.

LN 5에서는 DT signals 중 특히 주기 함수들이 어떻게 complex sinusoids로 만들어지는 지에 대해 공부한다.
LN 6에서는 CT signals 중 특히 주기 함수들이 어떻게 complex sinusoids로 만들어지는 지에 대해 공부한다.

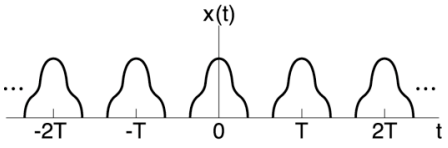
이러한 주기 함수가 input 으로 LTI system에 인가될때의 output에 대해서는 LN 7에서 다루게 된다. LN 7에서는 당연히
앞서 LN 5에서 배운대로 complex exponential 한 term마다
CT에서는 그 주파수에서의 Laplace transform값이
DT에서는 그 주파수에서의 z-transform 값이 곱해져서
linear combine된 것이 출력으로 나온다는 것을 확인하고
filter라는 개념을 도입하여 LTI system의 의미에 대해 생각한다.

Def. Fourier Series Representation of a CT Periodic Signal

Given a periodic signal

$x(t) = x(t+T)$ for all t

- smallest such T is the *fundamental period*
- $\omega_o = \frac{2\pi}{T}$ is the *fundamental frequency*



$e^{j\omega t}$ periodic with period T $\Leftrightarrow$ $\omega = k\omega_o$

if we have $\Downarrow$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk2\pi t/T} = x(t+T)$$

-periodic with period T
- $\{a_k\}$ are the *Fourier (series) coefficients*
- $k = 0$ DC
- $k = \pm 1$ first harmonic
- $k = \pm 2$ second harmonic

where the equality in some sense, then the RHS is called the .... in the sense.

Q. There are only N distinct complex exponential functions with period N. However, there are countably infinite periodic complex exponential functions with period T. Why?

Q. 앞서의 note에서는 any DT periodic signal에 대해
(i) DTFS가 존재하고
(ii) DTFS가 original signal과 모든 n값에서 일치함을 보였다.
같은 말을 CT periodic signal에 대해 할 수 있을까?

A. No. Actually, "limited yes".

유사하나 다른점이 많이 있으므로 이에 유의하면서 공부하자.

Lemma. For *real* periodic signals, there are two other commonly used forms for CT Fourier series:

$$x(t) = a_o + \sum_{k=1}^{\infty} \left[\alpha_k \cos k\omega_o t + \beta_k \sin k\omega_o t\right]$$

or

$$x(t) = a_o + \sum_{k=1}^{\infty} \left[\gamma_k \cos(k\omega_o t + \theta_k)\right]$$

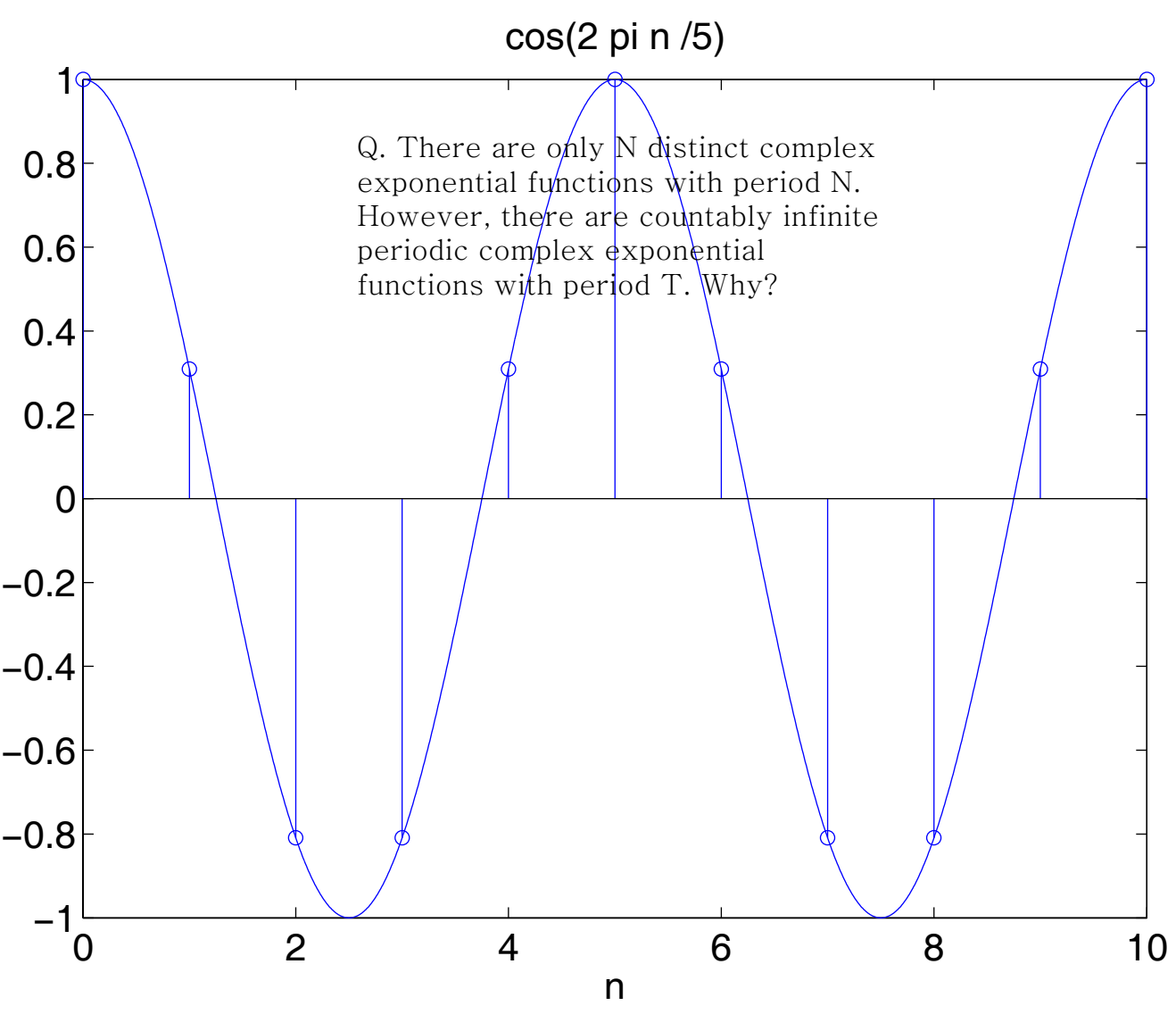
Remarks. Because of the eigenfunction property of $e^{j\omega t}$, we will only use the complex exponential form in EECE233.

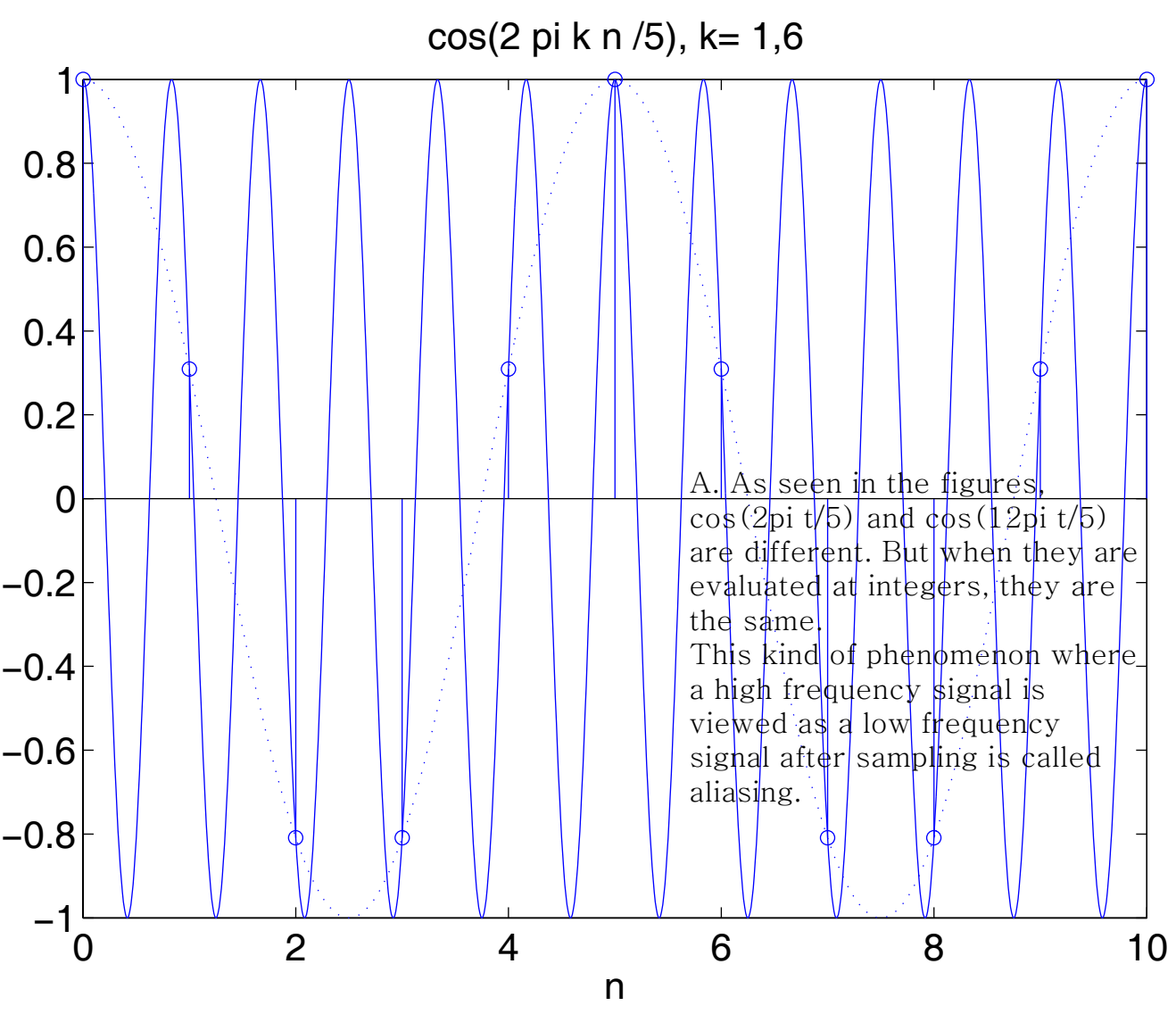
— A nontrivial consequence of this is that we need to include terms for *both* positive and negative frequencies:

$$e^{jk\omega_0 t} \quad , \quad e^{-jk\omega_0 t}$$

Q. 여기서 equality는 무슨 의미 인가?

A.
equals everywhere
vs.
equals almost everywhere
vs.
equals in L^2 (the mean square) sense
...


cos(2 pi n /5)
Q. There are only N distinct complex exponential functions with period N. However, there are countably infinite periodic complex exponential functions with period T. Why?
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
2
4
6
8
10
n


cos(2 pi k n /5), k= 1,6
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
2
4
6
8
10
n
A. As seen in the figures, cos(2pi t/5) and cos(12pi t/5) are different. But when they are evaluated at integers, they are the same.
This kind of phenomenon where a high frequency signal is viewed as a low frequency signal after sampling is called aliasing.

Question: How do we find the Fourier coefficients a_k?

Let's first take a detour by studying a three-dimensional vector:

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z},$$

$\hat{x}, \hat{y}, and\ \hat{z}$ *are unit vectors.*

How do we find the coefficients A_x, A_y, and A_z?

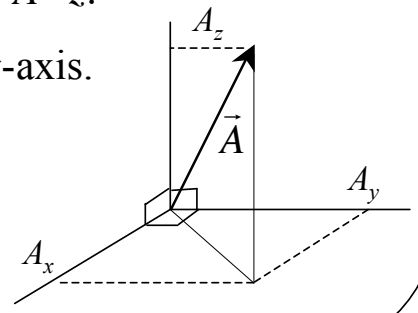
Easy, $A_x = \vec{A} \bullet \hat{x}, A_y = \vec{A} \bullet \hat{y}, and\ A_z = \vec{A} \bullet \hat{z}.$

Project the vector onto the *x*-, *y*-, and *z*-axis.

Why does it work this way?

Orthogonality:

$$\hat{y} \bullet \hat{x} = \hat{z} \bullet \hat{y} = \hat{x} \bullet \hat{z} \equiv 0$$



한 주기 내에서 finite energy를 갖는 주기신호 x(t)가 주어졌을때,

a finite-power periodic signal, or a square integrable (during a period) periodic signal

Back to Fourier series

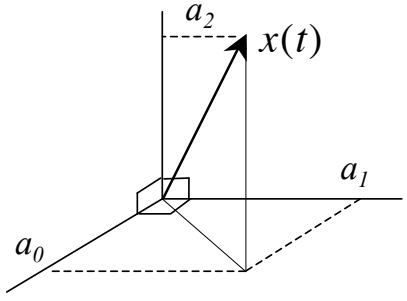
Now let's imagine that $x(t)$ is a vector in a space of ∞-dimensions and $e^{jk\omega_o t}$ $(k = 0, \pm 1, \pm 2, ...)$ is a unit vector in this space. (Such a space does exist, it is called Hilbert space, just nobody lives there.) Then,

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t}, \quad with\ the\ analogy$$

$$x(t) \leftrightarrow \vec{A}$$

$$e^{jk\omega_o t} \leftrightarrow \hat{x}, \hat{y}, \hat{z}$$

$$a_k \leftrightarrow A_x, A_y, A_z$$



Now all we need to do is to "project" $x(t)$ onto the $e^{jk\omega_o t} - axis$, then we will obtain the coefficients a_k. How to do that? Again, need **orthogonality.**

space: a set and an accompanying operator (e.g. norm, scalar multiplication, addition, metric, inner product, etc.)

vector/point in a space: an element of the set of the space

dimension: linear space에서 basis의 원소수

basis: a subset of the space로 임의의 한 point는 이들 basis vector 들의 linear combination으로 나타난다.

Hilbert space = a complete normed linear space with inner product

a complete space: every Cauchy sequence converges
(Cauchy sequence란 인접하는 term들간의 간격이 0으로 수렴하면 수열 자체도 수렴하는 수열)

a normed space: a set of elements with a norm defined

norm of an element: size of an element.

metric space: 거리(metric)를 정의한 space

metric: 거리를 정의하는 함수로서
positivity
symmetry
triangle inequality가 성립해야 함.

Note: Hilbert space 는 3 dimensional Euclidean space의 무한차원 확장으로 생각하면 된다.

직교좌표계상에서 점들간의 거리, 두 점들간의 내적이 잘 정의되는 무한 차원 공간이다.

-> 학부에서 수학과 real analysis를 공부한 후 수학과/통계학과 대학원에서 measure theory, probability theory를 들을 것.

Obtaining the Fourier series coefficients

Orthogonality in the Hilbert space:

$$\frac{1}{T}\int_T e^{jk\omega_o t}\cdot e^{-jn\omega_o t}dt = \frac{1}{T}\int_T e^{j(k-n)\omega_o t}dt = \begin{cases}1, & k=n\\ 0, & k\neq n\end{cases}$$
$$= \delta[k-n].$$

($\int_T$ = integral over *any* interval of length T, and the operation of $\frac{1}{T}\int_T e^{-jn\omega_o t}dt\cdot$ is to take an "inner product" with $e^{jn\omega_o t}$)

Now if we "project" $x(t)$ onto $e^{jn\omega_o t}$ by taking the operation :

$$\frac{1}{T}\int_T x(t)\cdot e^{-jn\omega_o t}dt = \frac{1}{T}\int_T \sum_{k=-\infty}^{+\infty} a_k e^{j(k-n)\omega_o t}\,dt,$$

then only one term (a_n) will be nonzero. That is :

$$\frac{1}{T}\int_T x(t)\cdot e^{-jn\omega_o t}dt = a_n.$$

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비슷한 것이 있으면
언제나 표를 만들어서
공통점과 차이점을 명확히 파악하라!

Finally

$$\frac{1}{T}\int_T x(t)\cdot e^{-jn\omega_o t}dt = a_n.$$

$$\Downarrow$$

CT Fourier Series Pair $\qquad (\omega_o = 2\pi/T)$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_o t}dt \qquad \text{(Analysis equation)}$$

DTFS의 synthesis equation과의 공통점은 countable sum이라는 것.

DTFS의 synthesis equation과의 차이는 term의 수, 즉 finite sum이 infinite sum으로 바뀌었다는 것.

따라서 각 a_k들이 모두 존재해도 이 infinite sum의 수렴여부가 문제됨.

DTFS의 analysis equation과의 차이는 finite sum이 integration (sum of uncountably infinite terms)으로 바뀌었다는 것.

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DTFS와의 결정적인 차이는, DTFS의 경우 주기 N인 신호의 DTFS term의 수가 at most N이었으나, CTFS의 경우 countably infinite개의 term수가 가능하다는 것이다.
infinity가 나오므로 CTFS의 convergence에 주의하여야 한다.

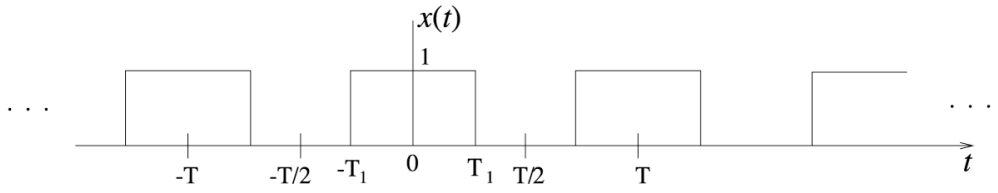
Q1. What is the definition of the convergence of a sequnece of numbers?
Q2. What is the definition of the convergence of a sequence of functions?
Q3. What is the definition of point-wise convergence of a sequence of functions?
Q4. What is the definition of almost everywhere convergence of a sequence of functions?
Q5. Following two expressions are the same? :
f(x) = 0 and
\int_{-\infty}^{\infty}|f(x)|^p ds = 0, for some 1\leq p.
Q6. What is the definition of uniform convergence of a sequence of functions?
Q7. What is the definition of the convergence in the p-th mean of a sequence of functions?
Q8. What is the definition of the convergence in measure of a sequence of functions?

Converge할 조건은 더더욱 sufficient conditions뿐이다. If S, then N (converges).

필요충분 조건 대신 많은 sufficient conditions가 발견되었다. 즉,
CTFS의 original function으로의 convergence in some sense인
iff set의 subset들이 발견되었다는 것이다.

가장 간단하고 쉬운 a sufficient condition: Wiki
"In mathematics, the Dirichlet - Jordan test gives sufficient conditions
for a real-valued, periodic function f
to be equal to the sum of its Fourier series
at a point of continuity.
Moreover, the behavior of the Fourier series at points of discontinuity is
determined as well (it is the midpoint of the values of the discontinuity).
It is one of many conditions for the convergence of Fourier series."

Example: Periodic Square Wave



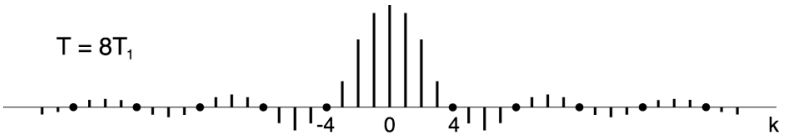
$$a_o = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{2T_1}{T}$$

$k \neq 0$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt$$

$(\omega_o = \frac{2\pi}{T})$:

$$= \frac{1}{-jk\omega_o} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} = \frac{\sin(k\omega_o T_1)}{k\pi}$$



오른쪽 그래프(?)는 윗 그림과 같은 주기함수의 각 k값에서의 Fourier series coefficients를 나타낸다.

Q. T가 그래로 있으면서 T1이 작아지면 오른쪽 그래프는 어떤 게 변하는가?

Precisely speaking, the CTFS \hat{x}(t) of $_9$ x(t) is defined as

\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{j 2\pi kt/T} where a_k = ...

If \hat{x}(t)=x(t) in some sense, then x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j 2\pi kt/T}
where a_k = ... is called the CTFS representation of x(t).

Convergence of CT Fourier Series

- How can the Fourier series (composed of continuous *sine* and *cos* functions) for the square wave (with many *discontinuities*) possibly make sense?
- The key is: What do we *mean* by

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad ?$$

- One useful notion for engineers: there is no *energy* in the difference

$$e(t) = x(t) - \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$\int_T |e(t)|^2 dt = 0$$

(just need $x(t)$ to have finite energy per period)

One set over which the CTFS representation exists in a.e. sense is

the set of every signal that is a.e. equal to a finite power periodic signal.

어떤 주기 함수 x(t)의 CTFS가 \sum a_k e^()로 나타날때,
Q1. What is the definition of this infinite sum?
Q2. What is the definition of =?

A.

equivalence relation =는

1) reflexive: x=x

2) symmetric: x=y이면 y=x

3) transitive: x=y이고 y=z이면 x=z

를 만족한다.

Q3. What is an equivalence class?

참고로

e(t)=0 almost everywhere와

\int |e(t)|^p dt =0 (for some p\geq 1)

은 같은 말이다.

Carleson's Theorem

If a periodic signal x(t) has a finite power, then x(t) and its CTFS converges almost everywhere.

주의.

a periodic signal x(t) has a finite power란 말은

x(t)가 주기 T상에서 finite energy를 갖는다는 말이고

또 다른 말로 x(t)가 주기상에서 square integrable하다는 말이다.

아래: 이 conditions들을 모두 만족하는 CT주기함수들의 집합이 CTFS와 origianl signal이 converges almost everywhere 되는 집합의 하나의 유용한 부분집합이라는 말씀!

앞서는 finite power signals의 집합이었고 여기서는 Dirichlet가 발견한 다른 집합을 제시.

그런데 공통점은 converges almost everywhere이고 차이점은 Dirichlet의 집합의 경우는 converges almost everywhere 가 성립할 뿐아니라 더 강력하게 converges everywhere except discontinuity points 이고 게다가 이들 discontinuity points에서는 좌극한과 우극한의 평균값으로 수렴한다는 것이다.

Under a different, but reasonable set of conditions
(the Dirichlet conditions)

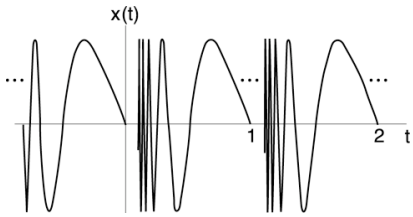
Condition 1. $x(t)$ is *absolutely integrable* over one period, i. e.

$$\int_T |x(t)|\,dt < \infty$$

Condition 2. In a finite time interval, $x(t)$ has a *finite* number of maxima and minima.

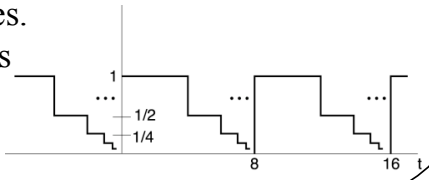
Ex. An example that violates Condition 2.

$$x(t) = \sin(2\pi/t) \quad 0 < t \le 1$$



Condition 3. In a finite time interval, $x(t)$ has only a *finite* number of discontinuities.

Ex. An example that violates Condition 3.



앞서본 a sufficient condition for almost everywhere convergence of CTFS와는 달리

여기서는 또 다른 another sufficient condition for almost everywhere convergence of CTFS를 선보인다.

둘의 큰 차이의 하나로, 먼저것은 square integrable, 이번 것은 absolutely integrable조건이 있다.

The Dirichlet Conditions (cont.)

Dirichlet conditions are met for most of the signals we will encounter in the real world. Then

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t} = \frac{1}{2}[x(t+0) + x(t-0)]$$

- The Fourier series = $x(t)$ at points where $x(t)$ is continuous
- The Fourier series = "midpoint" at points of discontinuity

Here, we have a stronger result than simple a.e. convergence.

- Still, convergence has some interesting characteristics:

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

At discontinuous points, it converges to the mid-point.

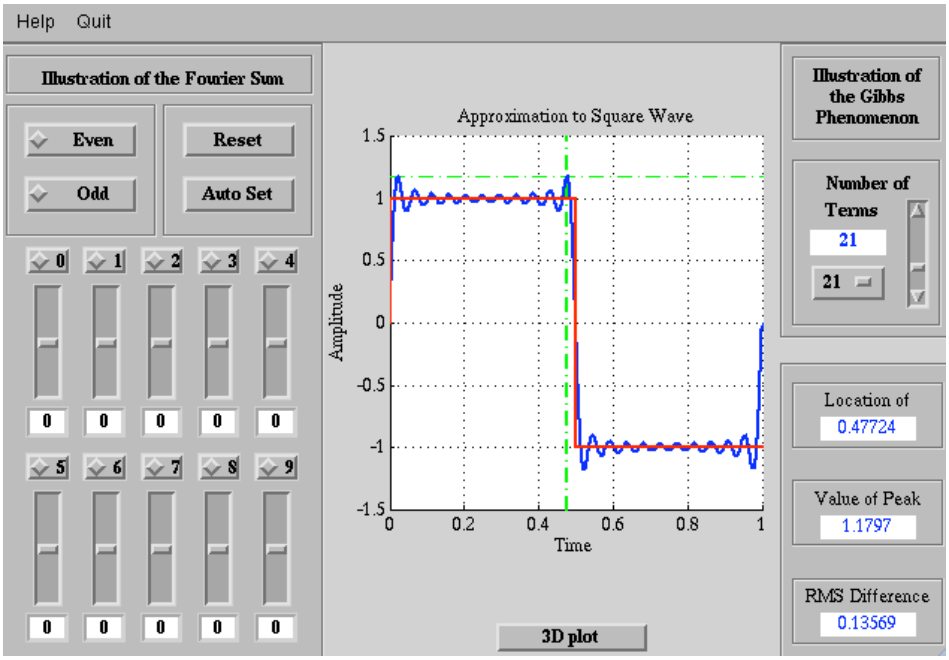
  - As $N \to \infty$, $x_N(t)$ exhibits *Gibbs'* phenomenon at points of discontinuity

디리클레 조건들을 만족하면,

1) x(t)가 연속인 points에서는 CTFS는 수렴하여 x(t)와 같고

2) x(t)가 불연속인 (finite number of) points에서는 CTFS가 x(t)의 좌/우극한의 평균값에 수렴한다.

•**Demo:** Fourier Series for CT square wave (Gibbs phenomenon).



Gibbs' phenomenon:

the oscillatory behavior of the Fourier series of a piecewise continuously differentiable periodic function around a jump discontinuity.
주기함수 x(t)와
이의 CTFS의 부분합
을 비교했을때,
x(t)의
불연속적인 점 근처에서의 squared error량은 partial sum의 term수를 늘이면 줄어드나
최대오차의 크기는 전혀 줄지 않는 현상.

그러니까 다른말로
uniform convergence는 guarantee되지 못한다는 말씀...

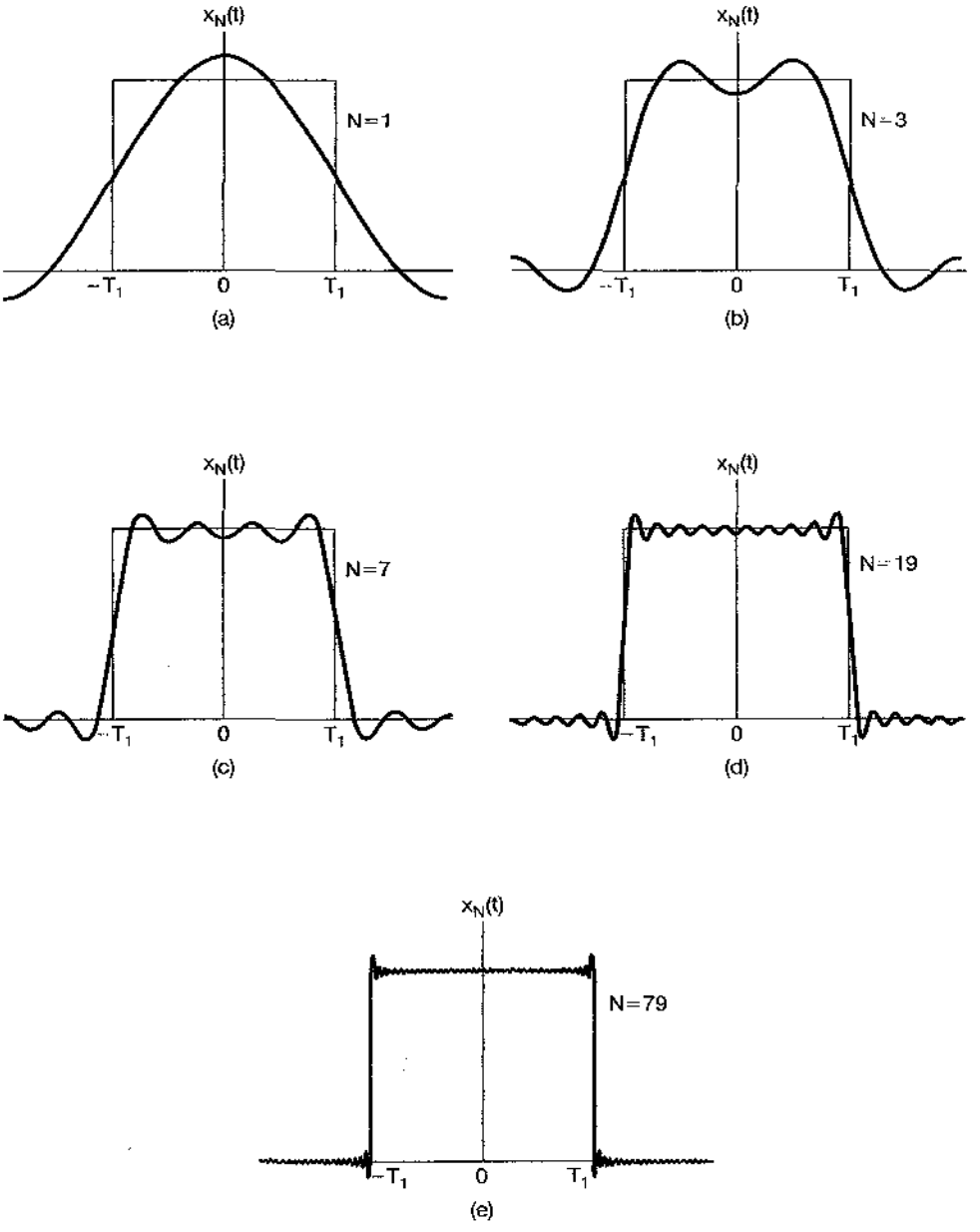


Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

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CT Fourier Series Pairs

$$(\omega_o = \frac{2\pi}{T})$$

Review:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{j2\pi kt/T}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

Skip it in future for shorthand

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

16

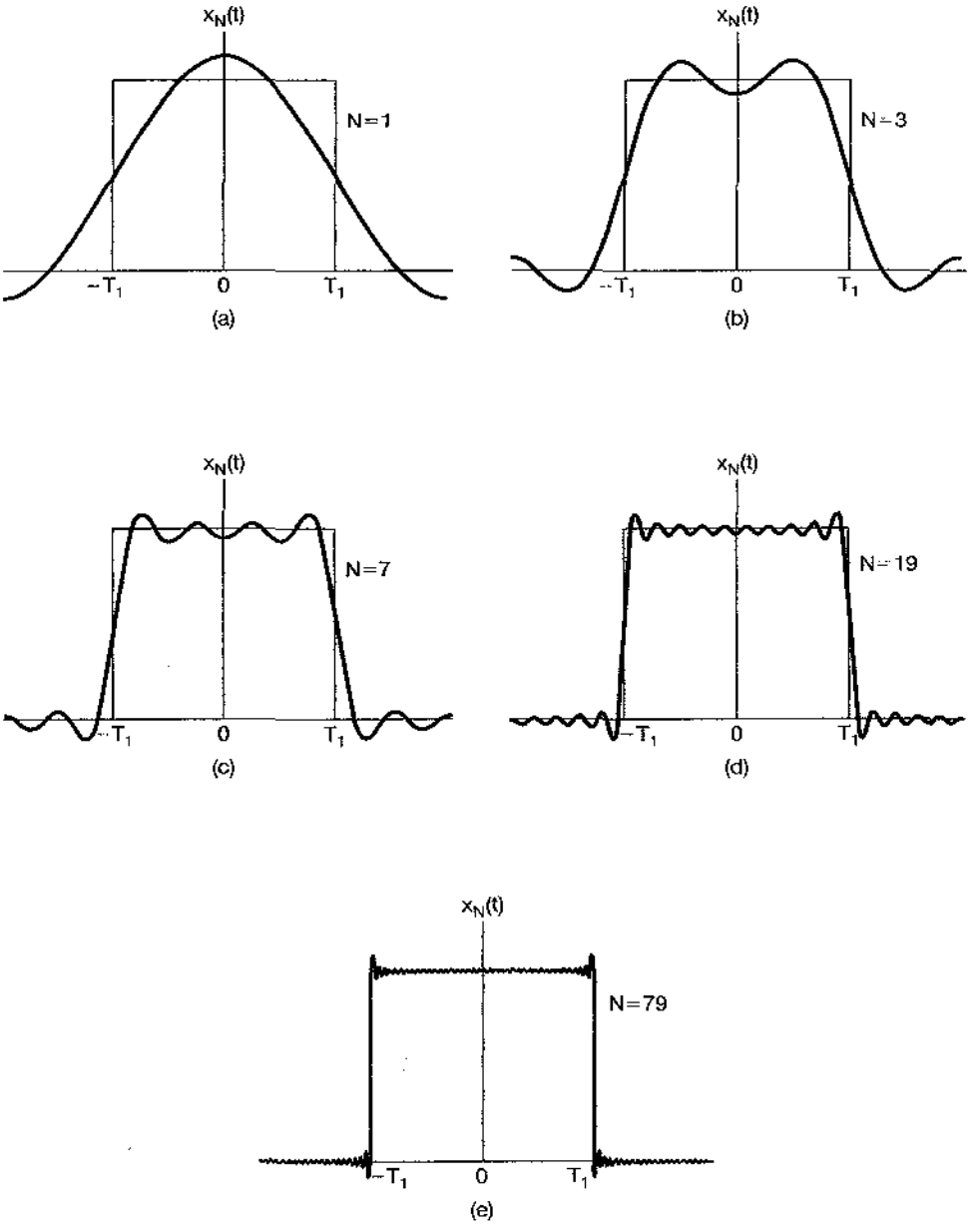
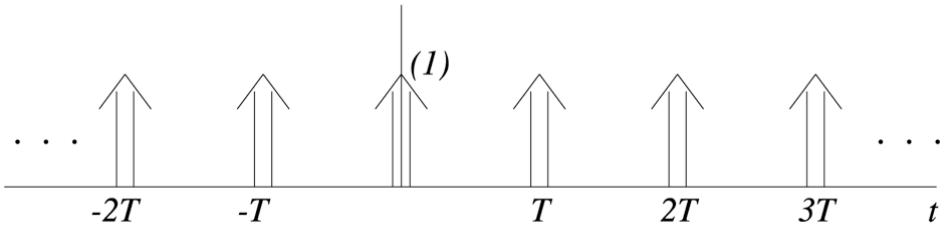


Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

Another (important!) example: Periodic Impulse Train

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$ — Sampling function, important for sampling later



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_o t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_o t}dt$$

$$= \frac{1}{T} \quad \text{for } all \ \ k !$$

$$\Downarrow$$

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} e^{jk\omega_0 t}$$

— All components have:
(1) the same amplitude,
&
(2) the same initial phase.

유변을 한주기 동안 적분(?)하면 1이 된다. 당연하다!

여기서 \omega0는 2\pi/T이다. 우변을 한 주기 동안 적분하면 1이 된다. 당연하다?

Q. DTFS에서 이와 유사한 결과는?

(A few of the) Properties of CT Fourier Series

- Linearity $x(t) \longleftrightarrow a_k, y(t) \longleftrightarrow b_k, \Rightarrow \alpha x(t) + \beta y(t) \longleftrightarrow \alpha a_k + \beta b_k$
- Conjugate Symmetry

$$x(t) \text{ real} \quad \Rightarrow \quad a_{-k} = a_k^*$$

Proof: $$a_{-k} = \frac{1}{T}\int_T x(t)e^{jk\omega_o t}dt = \left[\frac{1}{T}\int_T x^*(t)e^{-jk\omega_o t}dt\right]^* = a_k^*$$

$$\Downarrow \quad a_k = \text{Re}\{a_k\} + j\,\text{Im}\{a_k\}$$

$$= |a_k| e^{j\angle a_k}$$

$\text{Re}\{a_k\}$ is even, $\text{Im}\{a_k\}$ is odd

or $|a_k|$ is even, $\angle a_k$ is odd

Q. DTFS에서 이와 유사한 property는?

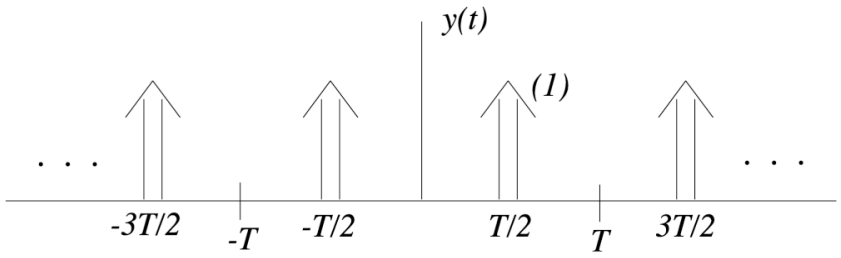
- Time shift $x(t) \longleftrightarrow a_k$

$$x(t - t_o) \longleftrightarrow a_k e^{-jk\omega_o t_o} = a_k e^{-jk2\pi t_o/T}$$

Introduce a linear phase shift $\propto t_o$

Example: Shift by half period

$$y(t) = x(t - T/2) \xleftrightarrow{e^{-jk\omega_o T/2} = e^{-jk\pi}} a_k e^{-jk\pi} = (-1)^k a_k$$



$$y(t) \longleftrightarrow (-1)^k a_k \qquad (a_k = \frac{1}{T} = \text{F.C. of } \sum_{n=-\infty}^{+\infty} \delta(t - nT))$$

$$\| \quad \frac{(-1)^k}{T}$$

- **Thm. Parseval Relation**

$$\underbrace{\frac{1}{T}\int_T |x(t)|^2 \, dt}_{\text{Average signal power}} = \sum_{k=-\infty}^{+\infty} \underbrace{|a_k|^2}_{\substack{\text{Power in the} \\ k_{th} \text{ harmonic}}}$$

Energy is the same whether measured in the time-domain or the frequency-domain

Q. DTFS에서 이와 유사한 결과는?

- **Thm. Multiplication Property**

$$x(t) \longleftrightarrow a_k, \; y(t) \longleftrightarrow b_k \text{ (Both } x(t) \text{ and } y(t) \text{ are}$$

$$\Downarrow \qquad \text{periodic with the same period } T)$$

$$x(t) \cdot y(t) \longleftrightarrow c_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l} = a_k * b_k$$

Proof: $\underbrace{\sum_l a_l e^{jl\omega_o t}}_{x(t)} \cdot \underbrace{\sum_m b_m e^{jm\omega_o t}}_{y(t)} = \sum_{l,m} a_l b_m e^{j(l+m)\omega_o t} \xrightarrow{l+m=k} \sum_k \underbrace{\left[\sum_l a_l b_{k-l}\right]}_{c_k} e^{jk\omega_o t}$

Q. 같은 주기를 갖는 두 주기함수의 곱은 주기함수인가? 그렇다면 주기는? 주기함수라면 이 주기함수의 CTFS는?

Q. DTFS에서 이와 유사한 결과는?

Q1. Given x(t) = \sum_k a_k e^{2pi kt/T} and y(t) = \sum_k b_k e^{2pi kt/T}, find c_k of

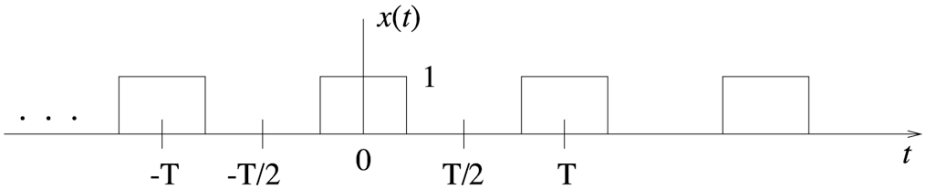
z(t) = x(t)y(t) = \sum_k c_k e^{2pi kt/T}.

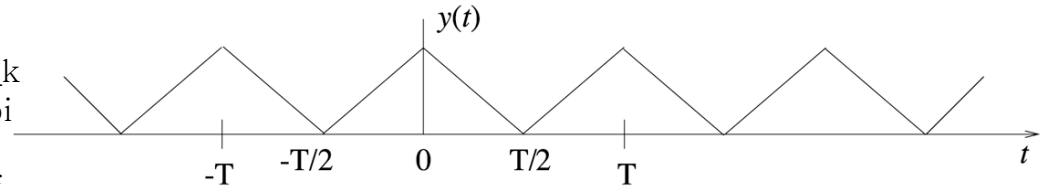
Q2. Given x(t) = \sum_k a_k e^{2pi kt/T} and y(t) = \sum_k b_k e^{2pi kt/T}, find the time-domain description of

z(t) = \sum_k (a_k b_k) e^{2pi kt/T} in terms of x(t) and y(t).

Thm. Periodic Convolution

$x(t), y(t)$ periodic with period T





$$x(t) * y(t) = \int_{-\infty}^{+\infty} x(\tau) y(t-\tau) d\tau$$ — not very meaningful

E.g. If both $x(t)$ and $y(t)$ are positive definite, then $x(t) * y(t) = \infty$

Q. What is the convolution of two periodic signals of the same period? Is it well defined?

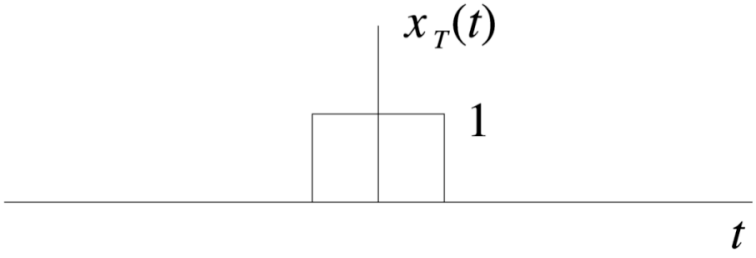
A. x(t)*y(t) diverges in general. So, it is not well defined.

Periodic Convolution (continued)

Periodic convolution: Integrate over *any* one period (**e.g.** -T/2 to T/2)

$$z(t) = \int_{-T/2}^{T/2} x(\tau) y(t-\tau) d\tau = \int_{-\infty}^{\infty} x_T(\tau) y(t-\tau) d\tau$$

where $$x_T(t) = \begin{cases} x(t) & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$$



Periodic convolution 의 두가지 해석법

1. convolution에서 한 주기 동안만 적분함.

2. 두 함수중 하나를 한 주기만 취한 후, 일반적 convolution을 함.

Periodic Convolution (continued) Facts

1) $z(t)$ is periodic with period T (why?)

From Lecture #2, $x(t)=x(t+T) \rightarrow y(t)=y(t+T)$ for LTI systems.
In this example, treat $y(t)$ as the input and $x_T(t)$ as $h(t)$.

2) Doesn't matter what period over which we choose to integrate:

$$z(t) = \int_T x(\tau) y(t-\tau) d\tau = x(t) \otimes y(t)$$

3) $x(t) \longleftrightarrow a_k$, $y(t) \longleftrightarrow b_k$, $z(t) \longleftrightarrow c_k$

$$c_k = \frac{1}{T}\int_T z(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_T \left(\int_T x(\tau) y(t-\tau) d\tau \right) e^{-jk\omega_0 t} dt$$

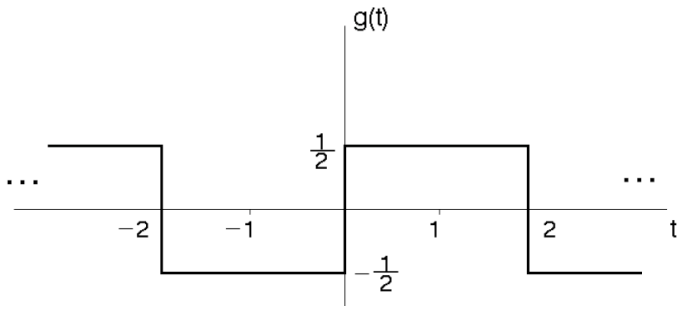
$$= \int_T \underbrace{\left(\frac{1}{T}\int_T y(t-\tau) e^{-jk\omega_0 \overbrace{(t-\tau)}^{t'}} dt \right)}_{b_k} x(\tau) e^{-jk\omega_0 \tau} d\tau$$

$$= \int_T b_k x(\tau) e^{-jk\omega_0 \tau} d\tau = T a_k b_k$$

엄밀히 말해 time invariant 이기만 하면
주기함수 input leads to 주기함수 output of the same period

Q. 세 주기함수 of the same period 의 CTFS들의 관계는?

Example: Periodic square wave with zero average



$T = 4$

$\omega_o = \frac{\pi}{2}$

이 신호가 여러 sine 함수들의 linear combination으로 분해되어 보이는가?

$g(t) \longleftrightarrow d_k$ $\qquad d_k = \int_{-2}^{2} g(t)\, e^{jk\omega_o t} dt$

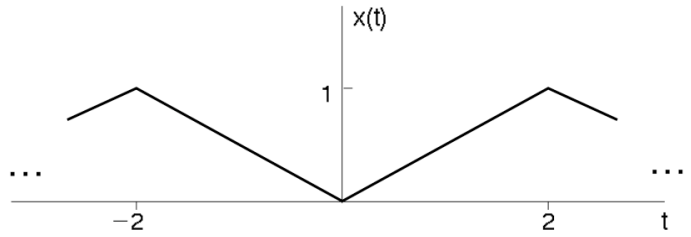
$$= \begin{cases} \dfrac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2} & for\ k \neq 0\ \ decays\ as\ \dfrac{1}{k} \\ 0 & for\ k = 0\ \ no\ dc\ component \end{cases}$$

d_k를 MATLAB으로 그려보라.

A practical example: Pentium 4 processor, clock frequency 3.2 GHz, third harmonic 9.6 GHz, fifth harmonic 16 GHz, ... challenging to avoid cross-talk.

여기서 모든 짝수 FS coefficients는 0이 된다.

Example: Periodic triangle wave



$T = 4$

$\omega_o = \frac{\pi}{2}$

Clearly, $x(t) = \int_{-\infty}^{t} g(\tau)\, d\tau$

Then $x(t) = \sum_k a_k e^{jk\omega_o t} = \int_{-\infty}^{t} \sum_k d_k e^{jk\omega_o \tau}\, d\tau$

$$= \sum_k \frac{d_k}{jk\omega_o} e^{jk\omega_o t}$$

$a_k = \frac{d_k}{jk\omega_o}$ *decays as* $\frac{1}{k^2}$, faster than the square wave, because the edges are smoother.

아래 신호가 여러 cosine함수들의 linear combination으로 분해되어 보이는가?

Q. square wave 둘을 periodic convolution하면?

a_k를 MATLAB으로 그려보라.

Thus, in general, *none* of the finite partial sums in eq. (3.52) yield the exact values of $x(t)$, and convergence issues, such as those considered in Section 3.4, arise as we consider the problem of evaluating the limit as the number of terms approaches infinity.

3.7 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

There are strong similarities between the properties of discrete-time and continuous-time Fourier series. This can be readily seen by comparing the discrete-time Fourier series properties summarized in Table 3.2 with their continuous-time counterparts in Table 3.1.

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$x[n]$, $y[n]$ } Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	a_k, b_k } Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k$ (viewed as periodic with period mN)
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \measuredangle a_k = -\measuredangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$x(t)$, $y(t)$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k, b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t}x(t) = e^{jM(2\pi/T)t}x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \measuredangle a_k = -\measuredangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$