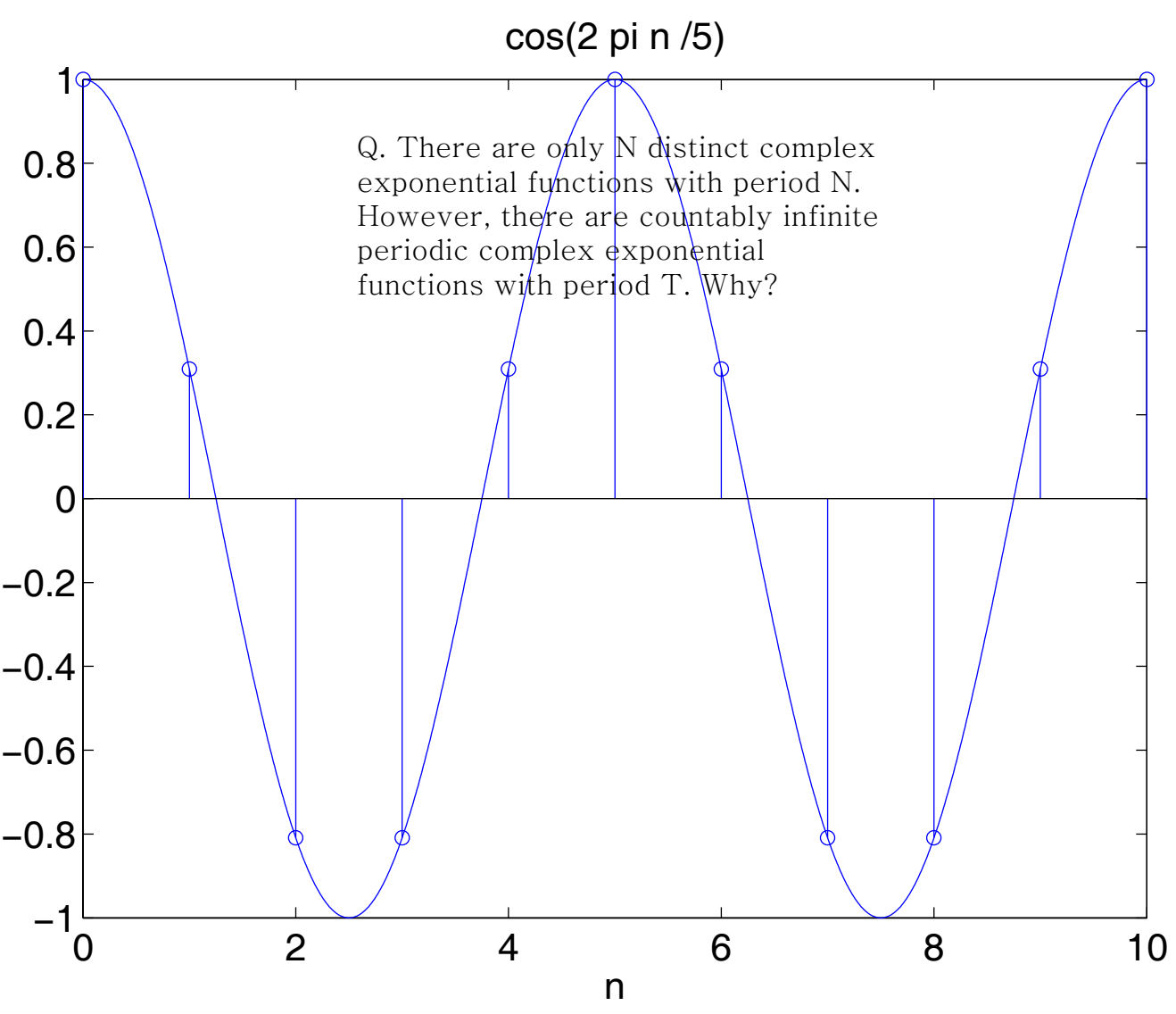
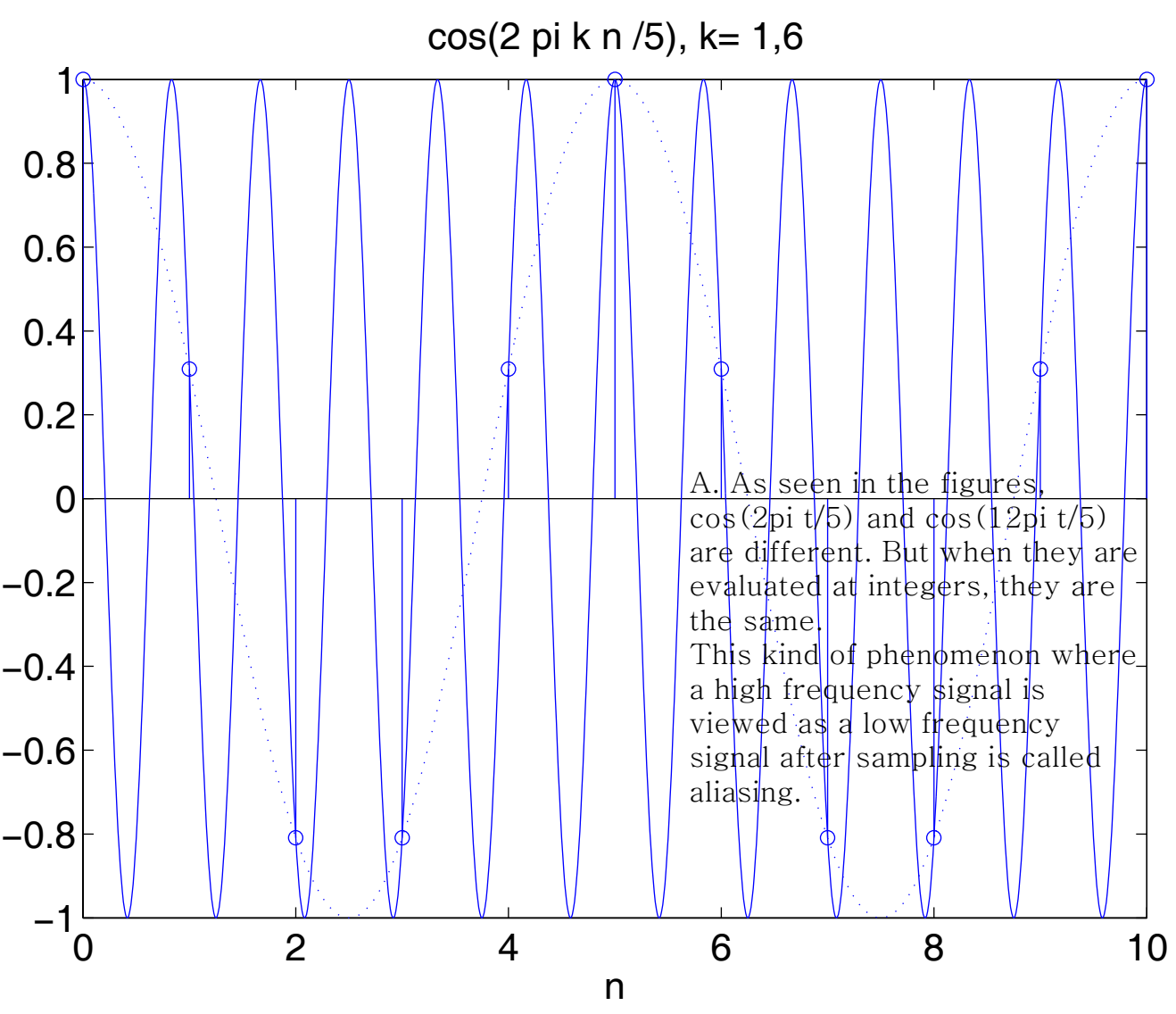

cos(2 pi n /5)
Q. There are only N distinct complex exponential functions with period N. However, there are countably infinite periodic complex exponential functions with period T. Why?
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
2
4
6
8
10
n


cos(2 pi k n /5), k= 1,6
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
2
4
6
8
10
n
A. As seen in the figures, cos(2pi t/5) and cos(12pi t/5) are different. But when they are evaluated at integers, they are the same.
This kind of phenomenon where a high frequency signal is viewed as a low frequency signal after sampling is called aliasing.

EECE233 Signals and Systems

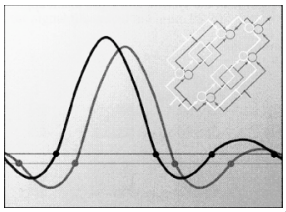
Lecture Note #8

Prof. Joon Ho Cho

(Slides thanks to Qing Hu*, D. Boning, D. Freeman, T. Weiss, J. White, and A. Willsky)

- Covers O & W pp. 284-309
- Derivation of the CT Fourier Transform pair
- Examples of Fourier Transforms
- Fourier Transforms of Periodic Signals
- Properties of the CT Fourier Transform

* These lecture notes are based on the lecture notes used in 6.003 taught Prof. Qing Hu at M.I.T. These notes are adopted for EECE233 under the permission of Prof. Qing Hu.



앞서 Lecture Note #7에서 LPF, BPF, HPF등의 개념을 배웠는데, 문제는 이들은 입력이 (다른 주기도 허용하여) complex sinusoids의 linear combination일때만 의미가 있었다. 그렇다면 이제는
Q. 주기함수 가 아닌 입력 신호가 들어가면 어떻게 될까?
라는 의문이 들 것이다.
이 Lecture note에서는 이 문제에 대해 배운다.

Fourier Transform

- We have shown that Fourier series are useful in analyzing periodic signals, but many (most) signals are aperiodic. Need a more general tool — *Fourier transform*.

Fourier's own derivation of the CT Fourier transform

- $x(t)$ – an aperiodic signal
  – view it as the limit of a periodic signal as $T \to \infty$
- The harmonic components are spaced $\omega_0 = 2\pi/T$ apart, as $T \to \infty$, and $\omega_0 \to 0$, then $\omega = k\omega_0$ becomes continuous

$$\Downarrow$$

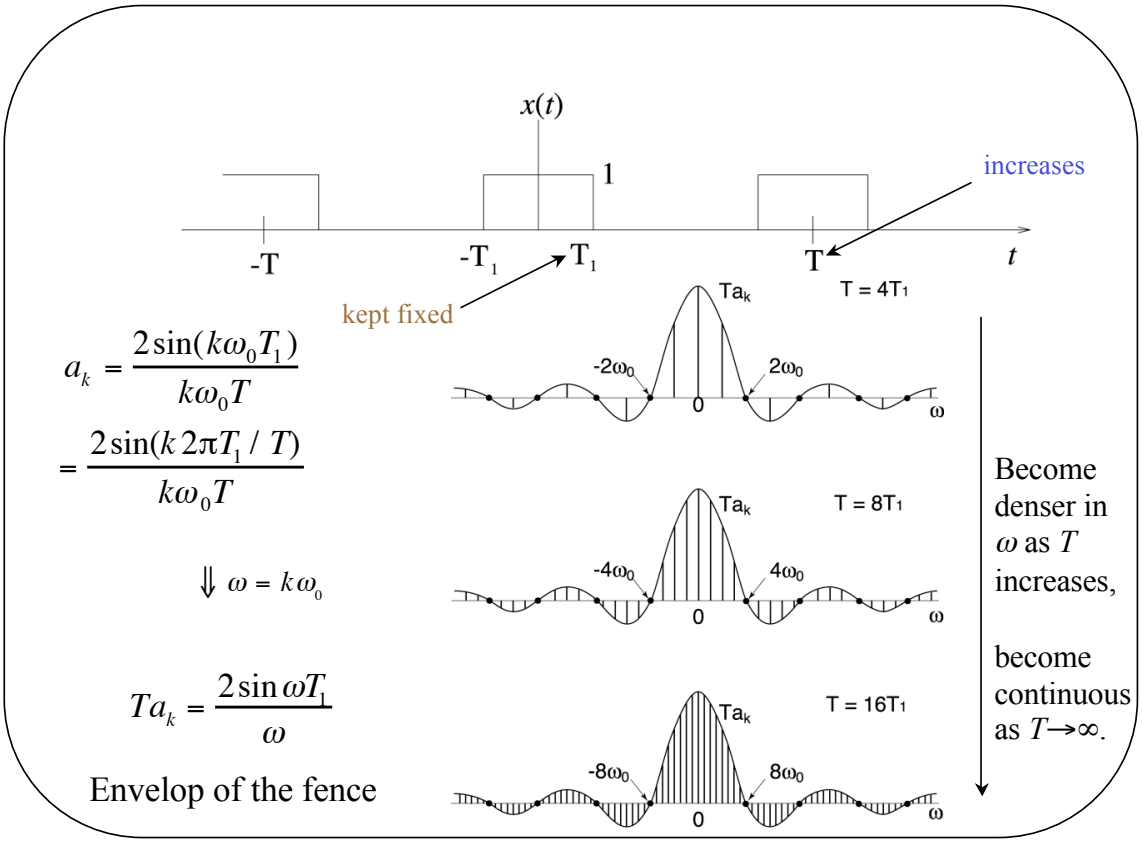
Fourier series $\longrightarrow$ Fourier integral

heuristic (발견적인, 보통 not rigorous but correct in some limited sense의 의미로 쓰임)

이것은 countably infinite한 것들의 극한(?)이 어떤 조건하에서는 uncountably infinite한 것이라는 주장으로 엄밀한 수학적 증명이 필요하다.
그러나....공학적으로는 큰 문제없다.

Motivating Examples: Rectangular pulse & Square wave

Def. The periodic extension with period T of a signal x(t) is given by
y(t) = \sum_{k=-\infty}^{\infty}x(t-kT).



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T=4T_1에서의 k=4에서의 a_k에 associated되어 있는 complex sinusoidal function의 주파수와

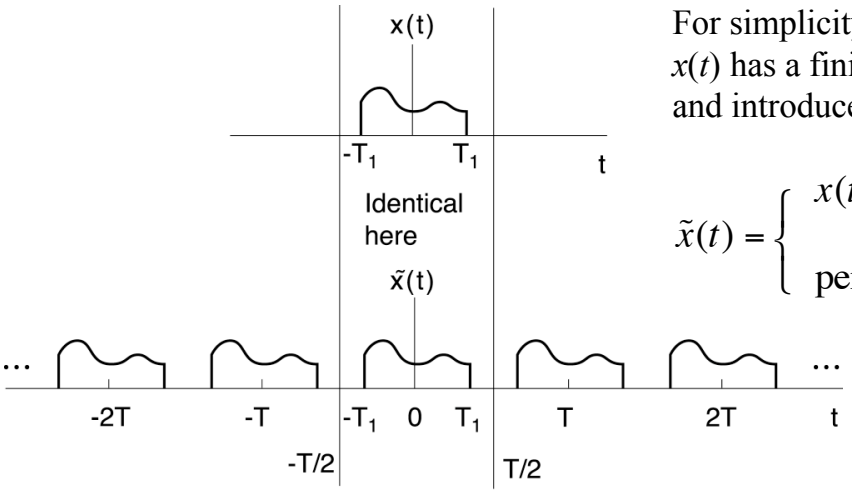
T=8T_1에서의 k=8에서의 a_k에 associated되어 있는 complex sinusoidal function의 주파수가 서로 같다.

따라서 그림들과 같이 같은 주파수에 해당하는 a_k들이 세로로 align이 되도록 그리면

T가 무한대로 감에 따라 envelope는 일정하나 단위 주파수 내에 점점 더 많은 a_k들이 나타나게 된다.

그리고 T를 곱하지 않았지만 성분의 크기들은 점점더 작아진다. 왜냐하면 한 주기 동안의 에너지는 일정하고 Parseval's relation에 의해서 a_k들의 제곱합이 일정해야 하기 때문이다.

So, on with the derivation of FT ...



For simplicity, assume $x(t)$ has a finite duration, and introduce a periodic $\tilde{x}(t)$

$$\tilde{x}(t) = \begin{cases} x(t) & \frac{-T}{2} < t < \frac{T}{2} \\ \in \text{periodic} & |t| > \frac{T}{2} \end{cases}$$

As $T \to \infty$, $\quad x(t) = \tilde{x}(t)$ for all t

다음 세 페이지의 슬라이드에서와 같은 유도가 있고
straightforward한 유도가 있다.

Q. x(t) = \int a_\omega e^(j\omega t)d\omega가 된다고 가정하면

x(t)와
a_\omega의 관계는?

A. a_omega = \frac{1}{2\pi}X(j=omega)

where
X(s)는 x(t)의 Laplace transform.

We call X(j=omega) the Fourier transform of x(t).

이 결과를 얻기 위한 Fundamental assumption in FT 은

\int_{-\infty}^{\infty}e^{j(\omega-\omega_0)t}dt
=
2\pi \delta(\omega-\omega_0)

2\pi \delta(\omega) = \delta(f)
where f=2\pi \omega

Derivation (continued)

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \qquad \omega_o = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jk\omega_0 t} dt$$

$$\uparrow$$

$$\tilde{x}(t) = x(t) \text{ in this interval}$$

$$= \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt \qquad (1)$$

Since $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

then Eq. (1) $\Rightarrow$ $a_k = X(jk\omega_0)/T$

which is the sampled value of X(j omega) at 2\pi k/T times 1/T.

Derivation (continued)

Thus, for $-\frac{T}{2} < t < \frac{T}{2}$

$$x(t) = \tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \underbrace{\frac{1}{T} X(jk\omega_0)}_{a_k} e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} \omega_0 X(jk\omega_0) e^{jk\omega_0 t}$$

는다는 것!

$$\Downarrow$$

As $T \to \infty$, $\omega_o \to 0$, $\sum\omega_o \to \int d\omega$, and $k\omega_o = \omega$, we get the CT FT pair

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \qquad \text{Synthesis equation}$$

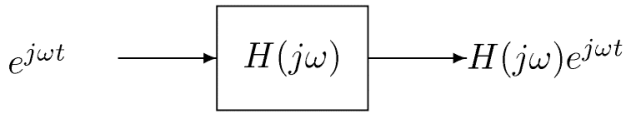
— "sum" of $e^{j\omega t}$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \qquad \text{Analysis equation}$$

radian frequency외에 Herz단위를 쓴 synthesis 및 analysis equation이 있으며 통신에서는 주로 이를 사용한다특징은 synthesis equation에서 1/(2\pi)가 나타나지 않는다.

여기서 \omega_0 = 2\pi /T로 놓는다. 그러면 구분구적법에 의해

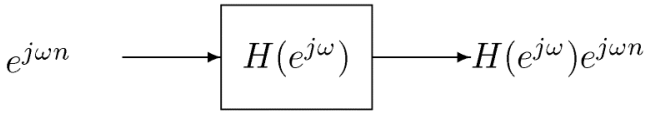
With CTFT, now the Frequency Response of an LTI System makes complete sense



$(s = j\omega)$

CT Frequency response: $H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$

FT of $h(t)$



$(z = e^{j\omega})$

DT Frequency response: $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$

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So far, the arguments have been heuristic. From now, more rigorous.

Q. Existence of FT, 즉 The set of all CT signals의 부분집합으로 aperiodic이면서 complex sinusoidal functions 의 linear combination으로 나타낼 수 있는 것들의 집합에 우리는 관심이 있다. 이 부분 집합은???

For what kinds of signals can we do *FT*?

(1) It works also even if $x(t)$ is infinite duration, but satisfies:

a) Finite energy $\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$

In this case, there is *zero* energy in the error

$e(t) = x(t) - \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega$ Then $\int_{-\infty}^{\infty} |\, e(t)\,|^2\, dt = 0$

b) Dirichlet conditions (including $\int_{-\infty}^{+\infty} |x(t)|^2 dt < \infty$)

(i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega = \frac{1}{2}[x(t-0) + x(t+0)$

$= \begin{cases} x(t) & \text{at points of continuity} \\ \text{midpoint at discontinuity} \end{cases}$

(ii) Gibb' s phenomenon

c) By allowing impulses in $x(t)$ or in $X(j\omega)$, we can represent even *more* signals

E.g. It allows us to consider *FT* for *periodic* signals

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Q. Analysis equation이 모든 omega 에서 정의된다고 해서 자동으로 이들을 이용한 synthesis equation이 (almost) everywhere x(t)와 같은가?

A. No. CTFT가 존재한다는 것과 synthesis equation이 잘 정의된다는 것과는 별개이다.

우리는 CTFS에서과 마찬가지로

Analysis equation에 의해 구해진 CTFT로 다시 x(t)는 almost everywhere equal로 synthesize할 수 있는 집합의 두 부분집합을 배운다.

1. the set of all finite-energy CT aperiodic signals.

2. the set of CT aperiodic signals that satisfies 3 Dirichlet conditions

CTFT가 존재한다는 것은
analysis equation이 모든 값에서 잘 정의된다는 것, 즉 적분이 수렴한다는 의미이다. 이럴 sufficient conditions를 알아보면...

1. the set of absolutely integrable CT aperiodic signals

2. the set of square integrable CT aperiodic signals: 하지만 여기서는 Fourier Plancherel transform으로 Fourier transform의 정의를 확장해야 성립한다.

3. ...

CTFT와 inverse CTFT를 구별하여 생각해야 함.

Fact 1. x(t)가 absolutely integrable하면 X(j\omega)는 언제나 존재한다. 그러나 inverse CTFT인 \hat x(t)가 존재하는 것은 아니다.
absolutely integrable하다는 것을 x(t) \in L^1(R)이라 표현한다.
Fact 2. x(t)와 X(j\omega)가 모두 absolutely integrable하면 inverse CTFT \hat x(t)가 존재하고 연속이며,x(t)와 \hat x(t)가 almost every where equal이다.

Fact3. 문제는 x(t)는 L^1 function인데 X(j\omega)가 L^1 function이 아닌 중요한 x(t)가 있다는 것이다. 예를 들어 square pulse. 다행히 이들 function중 상당수가 L^2 function인데, L^2 function에 대해서는
Fourier-Plancherel transform이라는 것이 정의되어 (간단히 이것도 구별없이 Fourier transform 이라 부른다.) L^2 function인 Fourier-Plancherel transform (FPT)이 존재하고
inverser Fourie-Plancherel transform (iFPT)이 또한 존재하여

FPT X(j\omega)와 FT에서 -T부터 T까지 적분한것의 오차의 에너지가 0으로 수렴하며

FT에서 -T부터 T까지 적분한것의 iFT와 x(t)와의 오차의 에너지도 0으로 수렴한다.

Example #1

(a) $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$\Downarrow$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega t} d\omega \quad \text{— Synthesis equation for } \delta(t)$$

(b) $x(t) = \delta(t - t_0)$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t - t_0) e^{-j\omega t} dt$$

$$= e^{-j\omega t_o} \quad \text{— Linear phase shift in } \omega$$

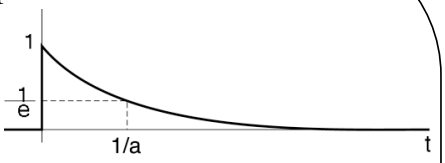
X(f) = X(j2\pi f)로 정의하면, \delta(t) <-> 1
e^{j2\pi f_0t} <-> \delta(f-f_0)이다.

이것이 Fundamental assumption in Fourier transform to handle a delta ft이다.

주의. e^{j=omega}의 어떤 harmonic 들을 모두 countable sum하면 impulse train이 나오고 여기서는 e^{j=omega}를 모든 주파수에서 uncountable sum (=integrate)하면 impulse가 나온다는 말씀.

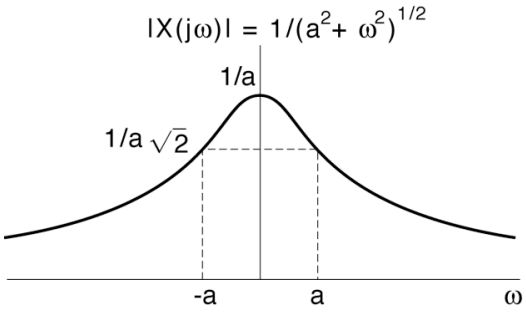
Example #2: Exponential function

$$x(t) = e^{-at} u(t), \quad a > 0$$

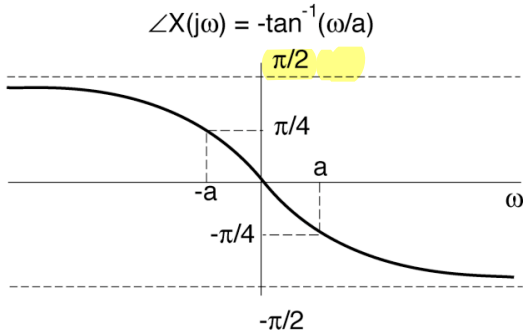


$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt = \int_0^{+\infty} \underbrace{e^{-at} e^{-j\omega t}}_{e^{-(a+j\omega)t}} dt$$

$$= -\left(\frac{1}{a + j\omega}\right) e^{-(a+j\omega)t} \Big|_0^{\infty} = \frac{1}{a + j\omega}$$



Even

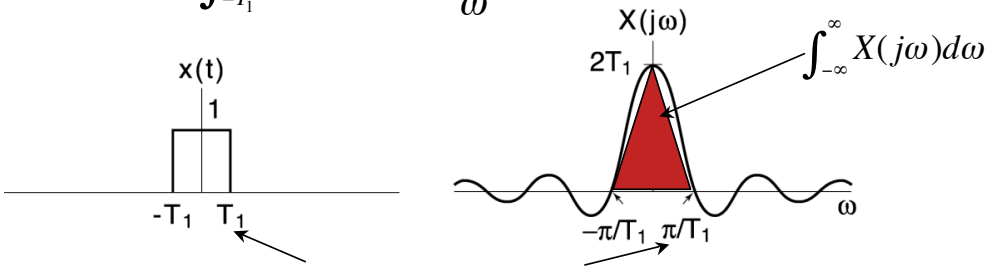


Odd

right-sided exponential function은 circuit theory, control에서 아주 중요한 신호이다.
이의 CTFT를 반드시 기억하자.

Example #3: A square pulse in the time-domain

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt = \frac{2\sin\omega T_1}{\omega}$$



Note the inverse relation between the two widths ⇒ Uncertainty principle

Useful facts about CTFT's

$$X(0) = \int_{-\infty}^{+\infty} x(t)\,dt$$

Example above: $\int_{-\infty}^{\infty} x(t)\,dt = 2T_1 = X(0)$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\,d\omega$$

Ex. above: $x(0) = 1 = \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty} X(j\omega)\,d\omega}_{2\pi} = \frac{1}{2\pi}\cdot\frac{1}{2}\left(2T_1 \times \frac{2\pi}{T_1}\right)$

$= \frac{1}{2\pi} \times$ (Area of the triangle)

We have a duality. To obtain the value of x(t) at t=0, integrate the CTFT and divide it by 2\pi .

X(0) is not the DC component.
2X(0)Delta_f is the DC component around |f| \leq Delta_f.

strictly time-limited signal은 무한대까지 주파수 성분을 갖는다.

strictly band-limited signal은 무한대까지 time 성분을 갖는다.

Q. True or False: A CT signal is either time-limited or band-limited.
A. No.
There are CT signals neither time-limited nor band-limited. Ex. Gaussian pulse in the next slide.

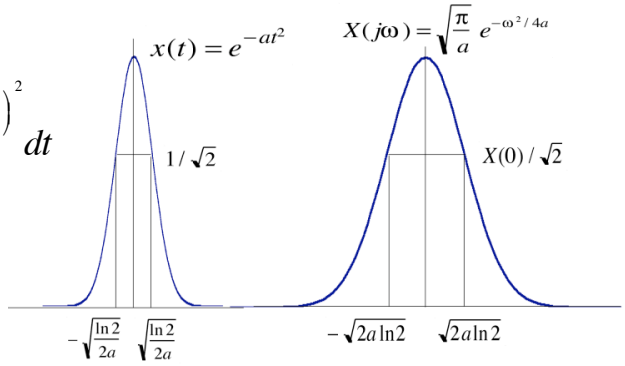
Example #4: $x(t) = e^{-at^2}$ — A Gaussian, important in probability, optics, etc.

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-a\left[t^2 + j\frac{\omega}{a}t + \left(\frac{j\omega}{2a}\right)^2\right] + a\left(\frac{j\omega}{2a}\right)^2} dt$$

$$= \underbrace{\left[\int_{-\infty}^{\infty} e^{-a\left(t + \frac{j\omega}{2a}\right)^2} dt\right]}_{\sqrt{\pi}/a} \cdot e^{-\frac{\omega^2}{4a}}$$

$$= \sqrt{\frac{\pi}{a}}\, e^{-\frac{\omega^2}{4a}}$$



Also a Gaussian!

(Pulse width in t)•(Pulse width in ω)
$\Rightarrow \Delta t \cdot \Delta\omega \sim (1/a^{1/2}) \cdot (a^{1/2}) = 1$

Uncertainty Principle! Cannot make *both* Δt and $\Delta\omega$ arbitrarily small.

반드시 기억할 것:

standard Gaussian random variable N(0,1)의 characteristica function은 여전히 bell curve이다.
In other words,
the CTFT of a bell curve is a bell curve.

If you multiply the standard deviations of the two bell curves of a CTFT pair, then the product is a constant.

As a rectangular pulse, a Gaussian pulse can be used to approximate the Dirac delta function when the width decreases and height increase with the product fixed.

이제 CTFS와 CTFT를 연결해 보자.

CT Fourier Transforms of Periodic Signals

Suppose $X(j\omega) = \delta(\omega - \omega_0)$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\delta(\omega-\omega_0)e^{j\omega t}d\omega = \frac{1}{2\pi}e^{j\omega_0 t}$$

— periodic in t with frequency ω_o

That is $\boxed{e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0)}$

— All the energy is concentrated in one frequency — ω_o

e^{j2\pi f_0 t} <-> \delta(f-f_0) 이다.

More generally, if $x(t) = x(t+T)$, then

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \longleftrightarrow X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$ Discrete spectra

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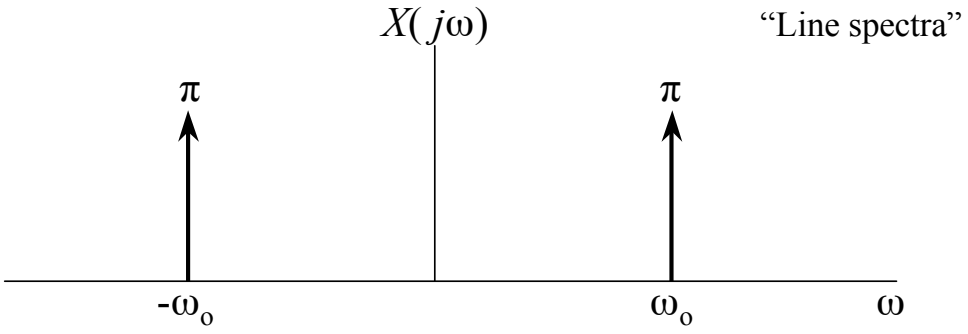
Example #4:

$$x(t) = \cos\omega_0 t = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

$$\updownarrow$$

$$X(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$



"Line spectra"

f를 쓸 경우도 생각해 보자.

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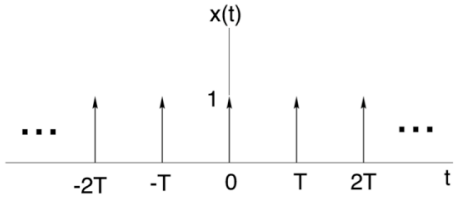
Example #5:

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$ — Old friend, sampling function

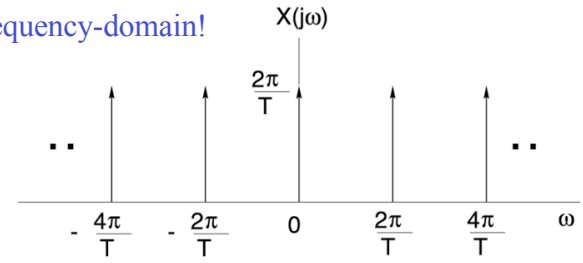
$$x(t) \longleftrightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_o t}dt = \frac{1}{T}$$



$$\Downarrow x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T} e^{jk\omega_o t}$$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T}\delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_o})$$

Same function in the frequency-domain!



Note in this case, periodic in both time domain (with a period *T*) and frequency domain (with a period $2\pi/T$)

\omega대신에 f를 쓰면 X(f)에서는 2\pi가 없다.

Properties of the CT Fourier Transform

1) Linearity $ax(t) + by(t) \longleftrightarrow aX(j\omega) + bY(j\omega)$

2) Time Shifting $x(t - t_0) \longleftrightarrow e^{-j\omega t_o} X(j\omega)$

Proof: $$\int_{-\infty}^{\infty} x(\underbrace{t - t_o}_{t'})e^{-j\omega t}dt = e^{-j\omega t_o}\underbrace{\int_{-\infty}^{\infty} x(t')e^{-j\omega t'}dt'}_{X(j\omega)}$$

FT magnitude unchanged

$$\left|e^{-j\omega t_0}X(j\omega)\right| = \left|X(j\omega)\right|$$

Linear change in *FT* phase

$$\angle(e^{-j\omega_0 t}X(j\omega)) = \angle X(j\omega) - \omega t_0$$

x(t)를 time shifting을 하며 X(j\omega)에 linear phase shift가 추가된다.

CTFT Properties (continued)

3) Conjugate Symmetry

$$x(t) \text{ real} \longleftrightarrow X(-j\omega) = X^*(j\omega)$$

$$\Downarrow$$

$$|X(-j\omega)| = |X(j\omega)| \qquad \textit{Even}$$

$$\angle X(-j\omega) = -\angle X(j\omega) \qquad \textit{Odd}$$

Or $\quad \mathrm{Re}\{X(-j\omega)\} = \mathrm{Re}\{X(j\omega)\} \qquad \textit{Even}$

$$\mathrm{Im}\{X(-j\omega)\} = -\mathrm{Im}\{X(j\omega)\} \qquad \textit{Odd}$$

When $x(t)$ is real (all the physically measurable signals are *real*), the negative frequency components do *not* carry any additional information from the positive frequency components. $\omega \geq 0$ will be sufficient.

Lemma. x(t)가 real이면
X(j\omega)는 conjugate symmetrical 하고
its magnitude는 even, its phase는 odd, its real part는 even. its imaginary part는 odd이다. 그림을 그려 보자.

The Properties Keep on Coming ...

4) Time-Scaling $\quad x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$

E.g. $a > 1 \rightarrow at > t$
compressed in time ↔ stretched in frequency

5) Time-Reverse

$$\Downarrow a = -1$$

$$x(-t) \longleftrightarrow X(-j\omega)$$

$$\Downarrow$$

a) $x(t)$ real and even $\quad x(t) = x(-t) = x^*(t)$

$$\Rightarrow X(j\omega) = X(-j\omega) = X^*(j\omega) \text{ — Real \& even}$$

b) $x(t)$ real and odd $\quad x(t) = -x(-t) = x^*(t)$

$$\Rightarrow X(j\omega) = -X(-j\omega) = -X^*(j\omega) \text{ — Purely imaginary \& odd}$$

c) $\quad X(j\omega) = \mathrm{Re}\{X(j\omega)\} + j\,\mathrm{Im}\{X(j\omega)\}$

For real $\quad x(t) = Ev\{x(t)\} + Od\{x(t)\}$

Q. audio signal을 빠르게 play하면 고음화 되는데 어떻게 하면 높이는 일정하게 하면서 빠르게 play할 수 있을까?

-> MATLAB project에서 구현?!

일반적으로 X(j\omega)는 complex valued function of \omega이다.

이 관계 c)는 communication system에서 real bandpass signal을 이해하는데 있어 매우 중요하다.

반드시 기억할 것

x(t) <-> X(j\omaga)
x(-t) <-> X(-j\omega)
x(t)^* <-> X(-j\omega)^*
x(-t)^* <-> X(j\omega)^*