Chapter 9
GENERAL CONCEPTS

Handout by Joon Ho Cho
based on A. Papoulis & S. U. Pillai, *Probability, Random Variables and Stochastic Processes, 4th ed.*

EECE 574, Spring 2021

9.1 Definitions

9.2 Systems with Stochastic Inputs

9.3 The Power Spectrum

9.4 Discrete-Time Processes

9.5 APPENDIX Continuity, Differentiation, Integration

9.6 APPENDIX Shift Operators and Stationary Processes

9.0 Introduction

- Scalar, vector, matrix, sequence, discrete-time signal, continuous-time signal,...
- Random variable, random vector, random matrix, a sequence of random variables
- Example of random time waveforms in a binary communication system
  - the bit stream: a desired random signal
  - noise: an undesired random waveform
- *random process = stochastic process*

  the description of *random waveforms* in a probabilistic sense
- Characterization of RPs in the time domain

  autocorrelation, cross-correlation, and covariance functions
- Description of RPs in the frequency domain
  - Fourier transform (See Appendix D)
  - The transform may not exist.
  - The spectral analysis of random processes requires a bit more subtlety than do deterministic signals.

Outline of 9.1

- Definition
  - two points of view
- Kolmogorov's extension theorem and a full characterization of a CT random process
  - first-order pdf/cdf
  - second-order pdf/cdf
  - nth-order pdf/cdf
  - theorem
- A simple classification of random processes
  - Example. Poisson random process
- Stationarity
  - stationary vs. non-stationary processes
    - stationarity to order one, two, and n
    - strict-sense stationary and non-stationary processes
  - expectation of second-order processes
    - mean process
    - auto-correlation and auto-covariance functions

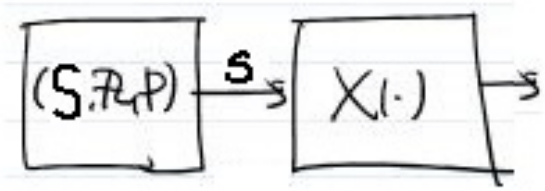
    * wide-sense stationary (WSS) and covariance stationary processes

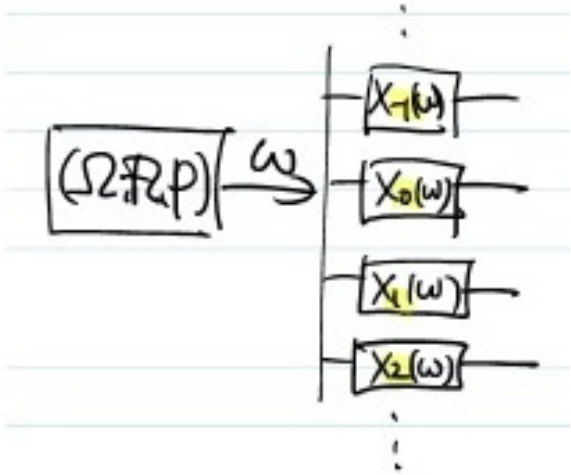
- Two joint distributed random processes
  - statistically independent processes
  - expectation of second-order processes
    * cross-correlation and cross-covariance functions
    * uncorrelated and orthogonal processes
    * jointly WSS processes
- Gaussian random processes
  - WSS Gaussian random processes

9.1 Definitions

Definition of a continuous-time random process

- Recall the definition of a random variable:
  - Given a probability space $(S, \mathcal{A}, P)$, a random variable $X(s)$ is a function from the sample space S into the (extended) real line.
  - Instead of $X(s)$, we use the notation X for a random variable.

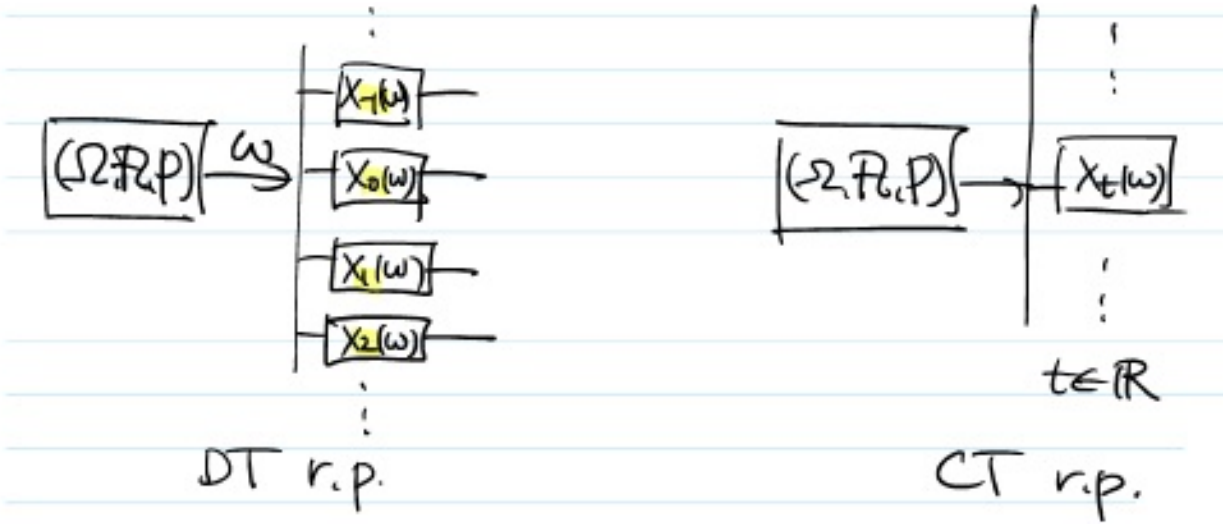




- Recall the definition of a random sequence:
  - Given a probability space $(S, \mathcal{A}, P)$, a random sequence $X_n(s)$ is a collection of integer-indexed countable number of random variables.

– Instead of $X_n(s)$, we use the notation X_n for a random sequence.

- Def.
  - Given a probability space $(S, \mathcal{A}, P)$, a continuous-time random process $X_t(s)$ is a collection of real-indexed uncountable number of random variables.
  - Instead of $X_t(s)$, we often use the notation X_t for a CT random process.
  - Sometimes, called a continuous-parameter random process.

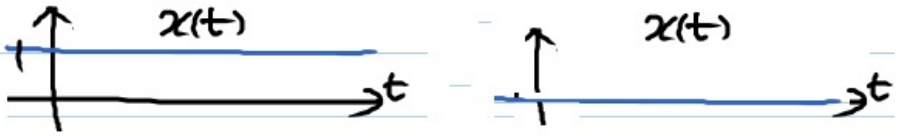


- Def.
  - Given s_0, the sample function/path of a CT random process $X_t(s)$ is given by a deterministic function $X_t(s_0)$ of time t.
  - Instead of $X_t(s_0)$, we often use the notation $x(t, s_0)$ for a CT random process.
  - Each member time function is called a *sample function*, an ensemble member, or sometimes a realization of the process.

ex/

$X(t) = A$, $t \in (-\infty, \infty)$ where A is a discrete random variable w/ $p_X(0) = 1/2$, $p_X(1) = 1/2$

realization of a random process ← sample path.



ex) $\{X = x\}$

$\{X(t) = x(t)\}$

$Pr(\{X(t) = 0, \forall t\}) = 1/2$

$Pr(\{X(t) = 1, \forall t\}) = 1/2$.

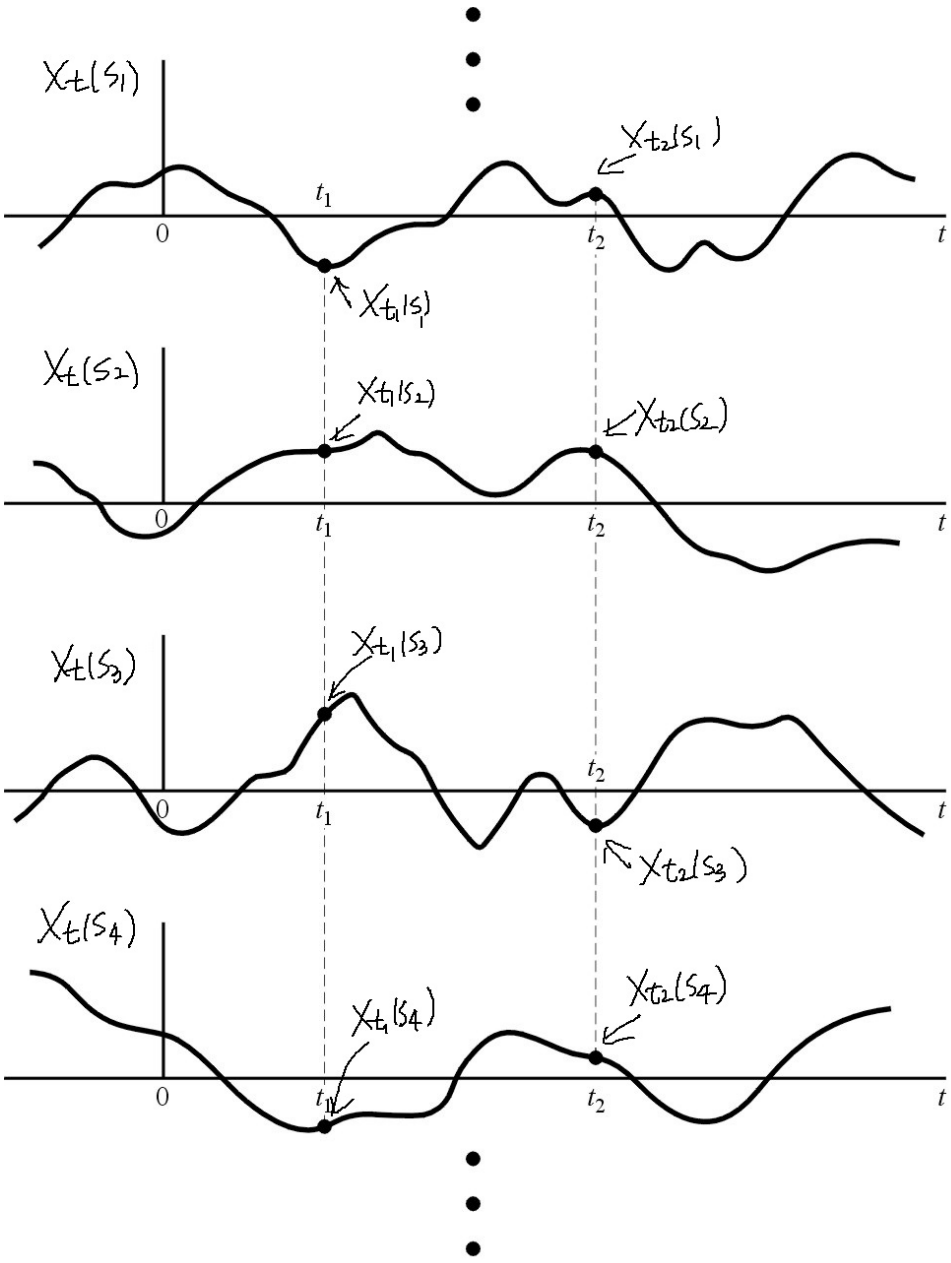
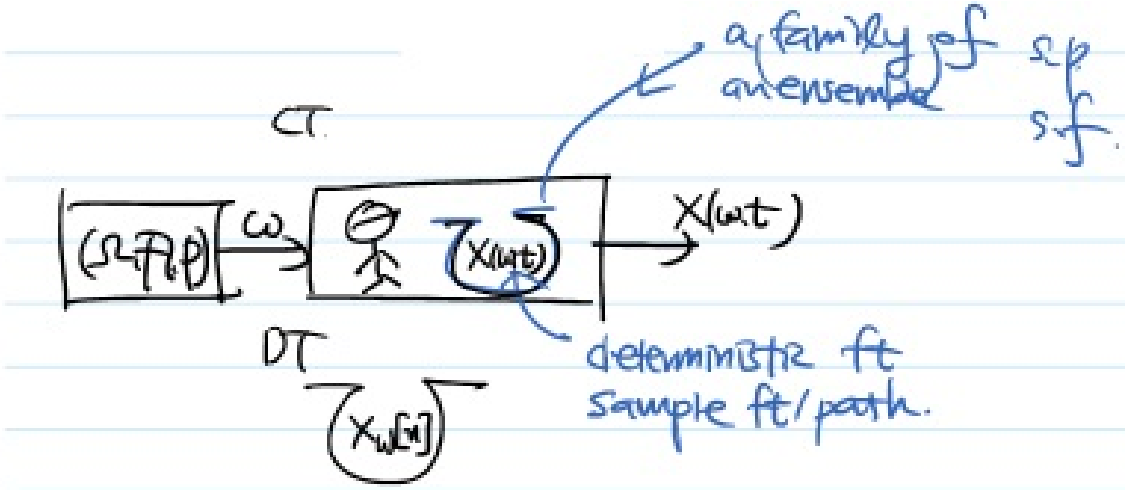


Figure 9.1-1 Sample paths of a continuous-time random process.

- Alternative Def.
  - Given a probability space $(S, \mathcal{A}, P)$, a continuous-time random process $X(t, s)$ is a mapping from the sample space S into a set of continuous-time (CT) sample functions.



  - Instead of $X(t, s)$, we often use the notation $X(t)$ for a CT random process.
  - Instead of $X(t, s_0)$, we often use the notation $x(t)$ for a sample path.
  - A random process $X(t, s)$ represents a family or ensemble of time functions when t and s are variables.
  - Roughly speaking, the family of all such functions, denoted $X(t, s)$, is called a *random process*.

Q. $X_{t_1}(s)$?

A. a random variable.

$X_0(s)$ a r.v., $X_1(s)$ another r.v., $X_{2.5}(s).s)$ another r.v.

a CT r.p. consists of uncountably infinite # of jointly distributed random variables.

- Example.

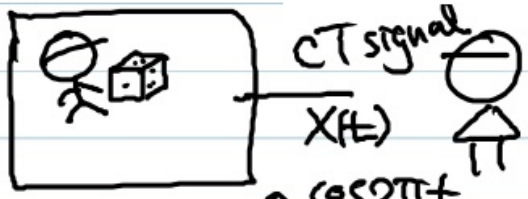
  a die-tossing experiment

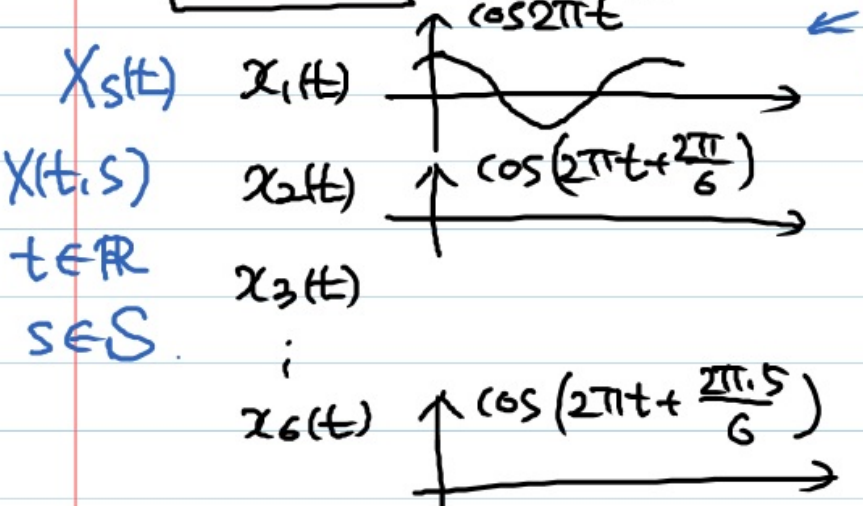
$(S, \mathcal{A}, P)$ $\quad S = \{1, 2, \cdots, 6\}$

$\mathcal{A} = 2^S$

$P(\{1\}) = P(\{2\}) = \cdots = P(\{6\}) = 1/6.$

$P(\{6\}) = 1/6.$





- Two points of view:
  - When t is a variable and s is fixed, a random process represents a single time function.
  - When s is a variable and t is fixed, a random process represents a random variable.
  - When s is a variable and $t_1, t_2, ..., t_N$ is fixed, a random process represents a random vector.

$$X_i = X(t_i, s) = X(t_i) \tag{9.1-1}$$

  - When t and s are both fixed, a random process represents a mere number.

Kolmogorov's Extension Theorem and Full Characterization of a Random Process

- Given a particular time t_1, the distribution function associated with the random variable $X_1 = X(t_1)$ is given by

$$F_X(x_1; t_1) \triangleq F_{X(t_1)}(x_1) = P\{X(t_1) \leq x_1\}.$$

  This cdf and its derivative

$$f_X(x_1; t_1) = \frac{d}{dx_1} F_X(x_1; t_1)$$

  are called the first-order cdf and pdf of $X(t)$, respectively.

- For two RVs $X_1 = X(t_1)$ and $X_2 = X(t_2)$, the joint cdf is given by

$$F_X(x_1, x_2; t_1, t_2) \triangleq F_{X(t_1),X(t_2)}(x_1, x_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}.$$

  This cdf and its partial derivative

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2; t_1, t_2)$$

  are called the second-order cdf and pdf of $X(t)$, respectively.

- For N RVs $X_i = X(t_i), i = 1, 2, \ldots, N$, the joint cdf is given by

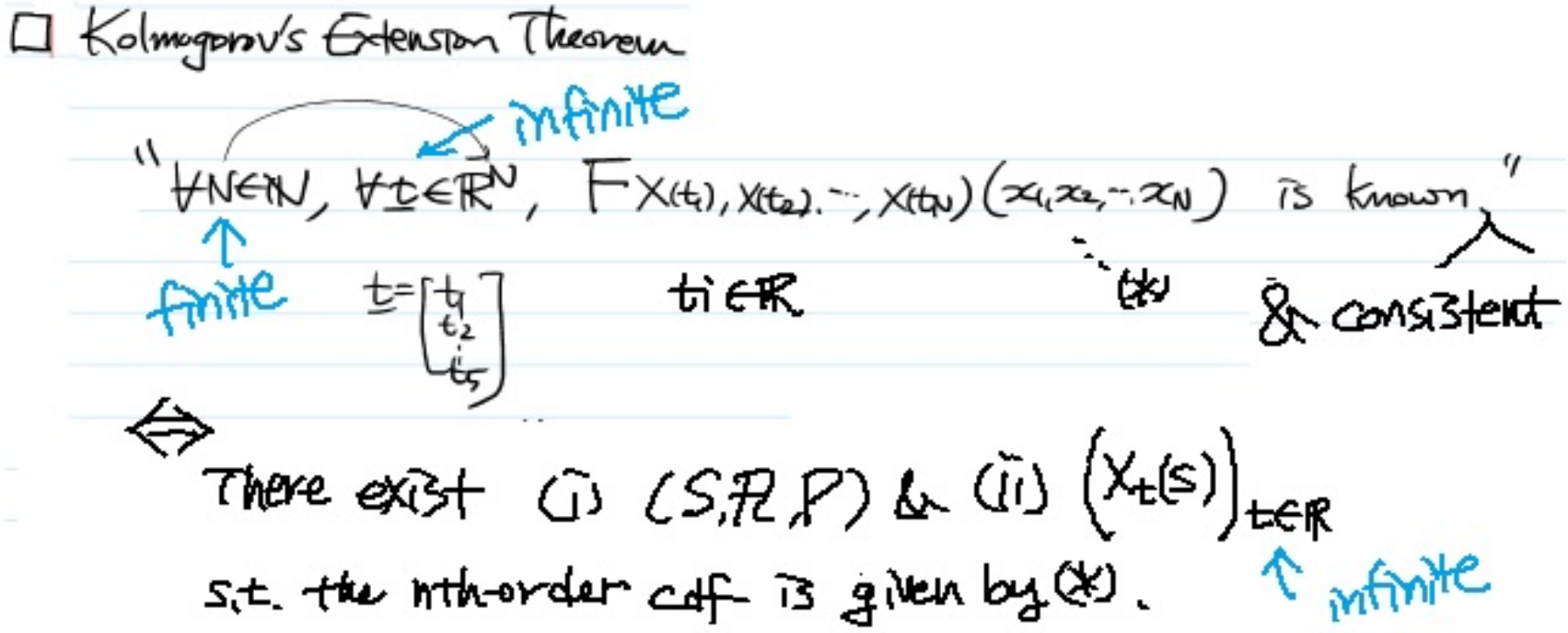
$$\begin{aligned} F_X(x_1, ..., x_N; t_1, ..., t_N) &\triangleq F_{X(t_1),X(t_2),\cdots,X(t_N)}(x_1, x_2, \cdots, x_N) \\ &= P\{X(t_1) \leq x_1, ..., X(t_N) \leq x_N\}. \end{aligned}$$

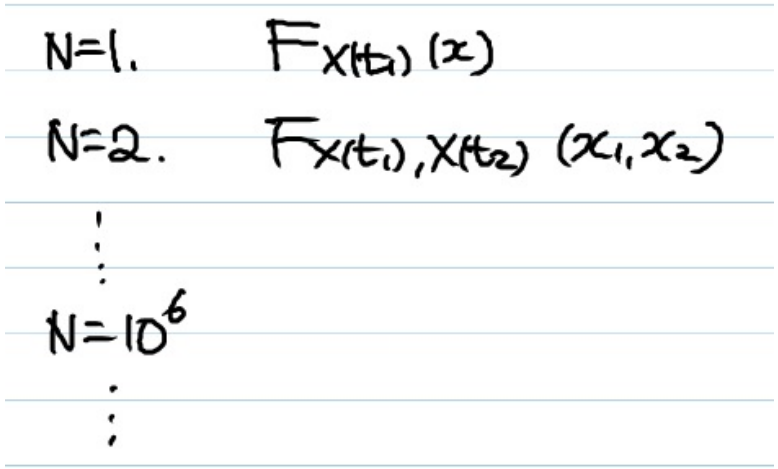
This cdf and its partial derivative

$$f_X(x_1, ..., x_N; t_1, ..., t_N) = \frac{\partial^N}{\partial x_1 \cdots \partial x_N} F_X(x_1, ..., x_N; t_1, ..., t_N)$$

are called the Nth-order cdf and pdf of $X(t)$, respectively.

- Theorem





- Q. Find the Nth-order pmf of $X(t) = A, \forall t$, where A is a discrete random variable.

A simple classification of random processes

- A random process can be described by whether involved random variables are continuous or discrete.
  - a continuous-time/parameter (CT) random process $X_t(s)$
    - a continuous random process: $X_{t_1}(s)$ is a continuous RV for all t_1
    - a discrete random process: $X_{t_1}(s)$ is a discrete RV for all t_1
    - else
  - a discrete-time/parameter (DT) random process $X_n(s)$
    - a continuous random process: $X_{n_1}(s)$ is a continuous RV for all n_1
    - a discrete random process: $X_{n_1}(s)$ is a discrete RV for all n_1
    - else
  - a digital signal $\approx$ a CT discrete signal
- A random process can be described by the form of its sample functions.
  - If future values of any sample function cannot be predicted exactly from observed past values, the process is called nondeterministic.
  - A process is called deterministic if future values of any sample function can be predicted exactly from observed past values.
  - If a random process is a deterministic signal, then it is a deterministic random process. However, the converse is not always true.

– Example

$$X(t) = A\cos(\omega_0 t + \Theta)$$

where A, ω_0, or Θ (or all) may be random variables, is a deterministic random process but not a deterministic signal.

ex/ $X(t) = A\cos(2\pi f_c t + \Theta)$

where A & Θ are jointly distributed r.v.'s w/ $f_{A\Theta}(a,\theta)\cdots$

$f_A(a) \leftarrow$ Rayleigh. $\rightsquigarrow$ $f_\Theta(\theta)$. ind.

$\Pr(\{X(t) = a_0\cos(2\pi f_c t + \theta_0), \forall t\}) = 0.$

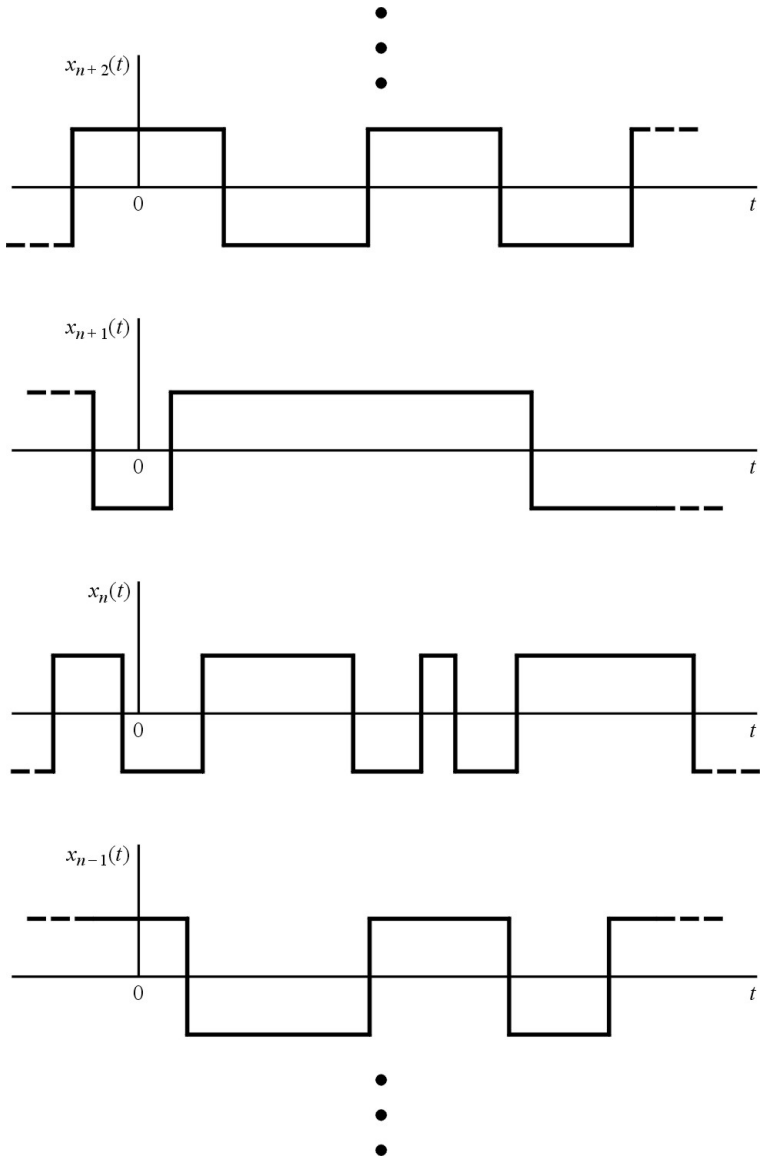


Figure 9.1-2 A CT discrete random process.

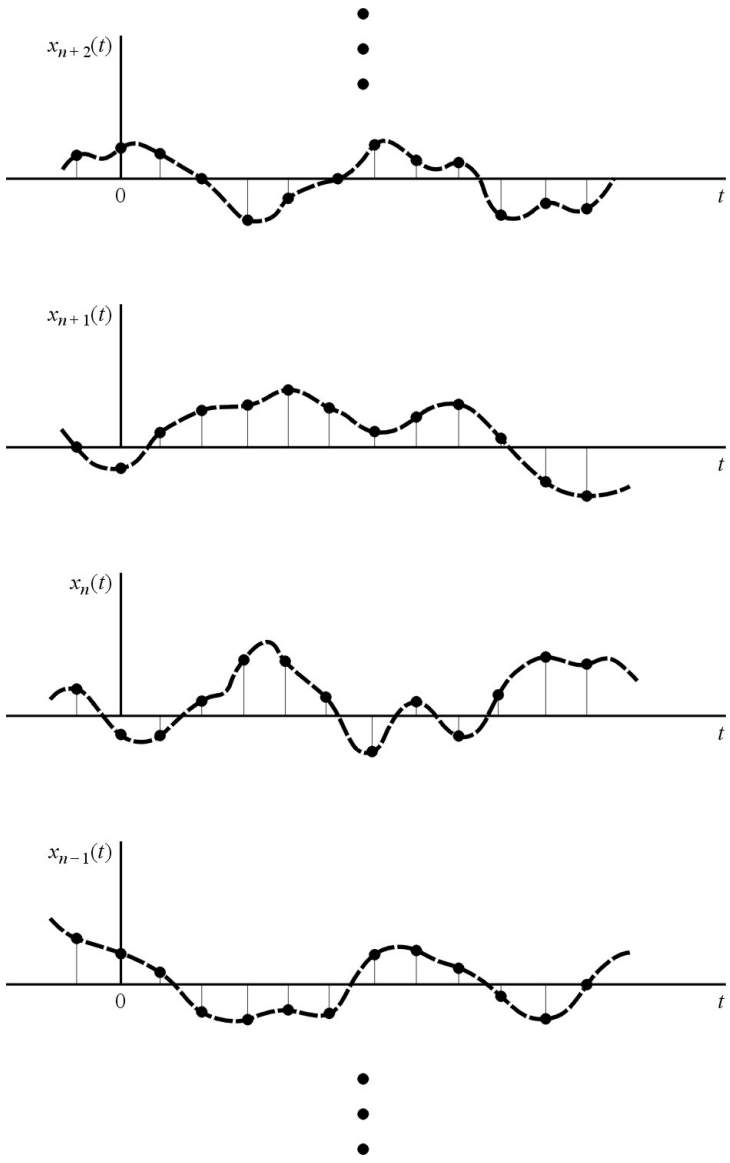


Figure 9.1-3 A DT continuous random process.

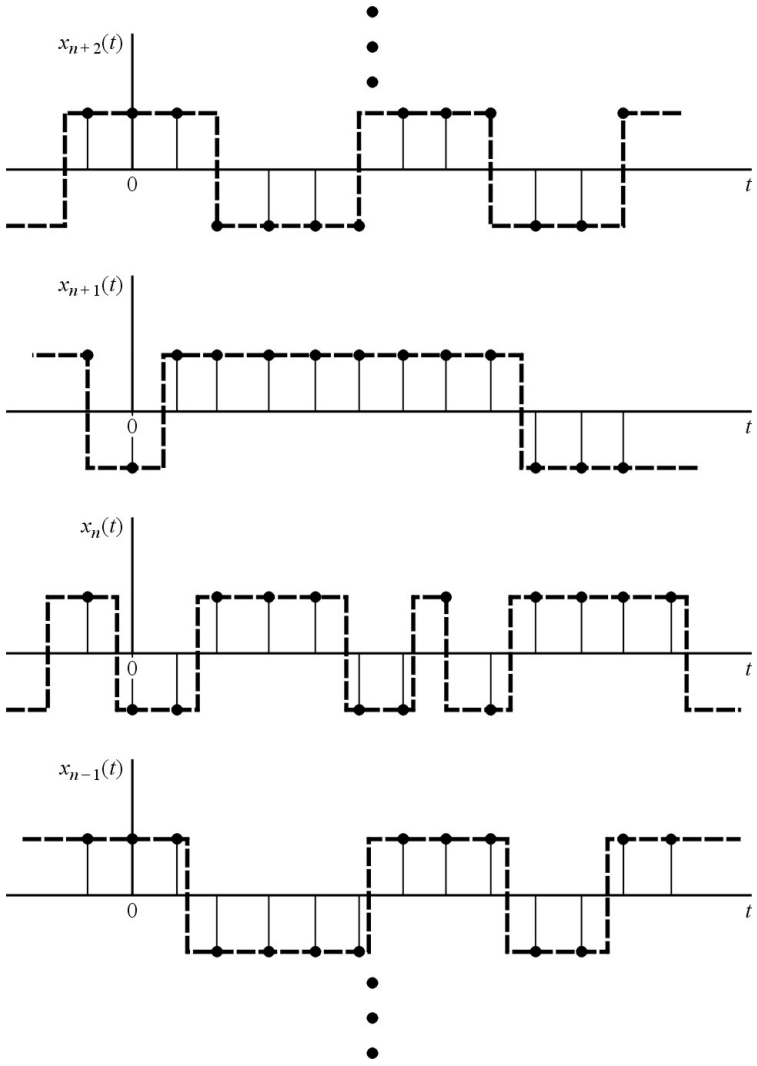


Figure 9.1-4 A DT discrete random process.

- Example
  - The Poisson process also known as the Poisson counting process is considered as an important example of a continuous-time discrete RP.
  - It describes the number of times that some event has occurred as a function of time, where the events occur at random times.
    * the arrival of a customer at a bank or supermarket check-out register
    * the occurrence of a lightening strike within some prescribed area
    * the failure of some component in a system
    * the emission of an electron from the surface of a light-sensitive material (photodetector)
  - Let $X(t)$ represent the number of event occurrences in the time interval $(0, t)$ with
$$P[X(t) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \qquad k = 0, 1, 2...,$$
which is a Poisson random variable. Then $X(t)$ has integer-valued, nondecreasing sample functions
  - To define the Poisson process we shall require the value of λ and two conditions:
    * The event occur only once in any vanishingly small interval of time. In essence, only one event can occur at a time.
    * Occurrence times are statistically independent so that the number that occurs

in any given time interval is independent of the number in any other nonoverlapping time interval.

– Q. Is a Poisson process fully characterized by the value of λ and these two conditions?

– A. Yes. We can derive any nth-order pmf consistently.

– The probability of exactly k_1 occurrences over a time interval $(0, t_1)$ is

$$P[X(t_1) = k_1] = \frac{(\lambda t_1)^{k_1} e^{-\lambda t_1}}{k_1!} \qquad k_1 = 0, 1, 2, ... \tag{9.1-2}$$

– The conditional probability of k_2 occurrences over $(0, t_2)$ given that k_1 events occurred over $(0, t_1)$ is just the probability that $k_2 - k_1$ events occurred over (t_1, t_2), which is

$$P[X(t_2) = k_2 | X(t_1) = k_1] = \frac{[\lambda(t_2 - t_1)]^{k_2 - k_1} e^{-\lambda(t_2 - t_1)}}{(k_2 - k_1)!} \tag{9.1-3}$$

– The joint probability of k_2 occurrences at time t_2 and k_1 occurrences at time t_1 is

$$\begin{aligned} P(k_1, k_2) &= P[X(t_2) = k_2 | X(t_1) = k_1] \cdot P[X(t_1) = k_1] \\ &= \frac{(\lambda t_1^{k_1})[\lambda(t_2 - t_1)]^{k_2 - k_1} e^{-\lambda t_2}}{(k_2 - k_1)!} \qquad k_2 \geq k_1 \end{aligned} \tag{9.1-4}$$

– Similarly, any nth-order pmf can be obtained.

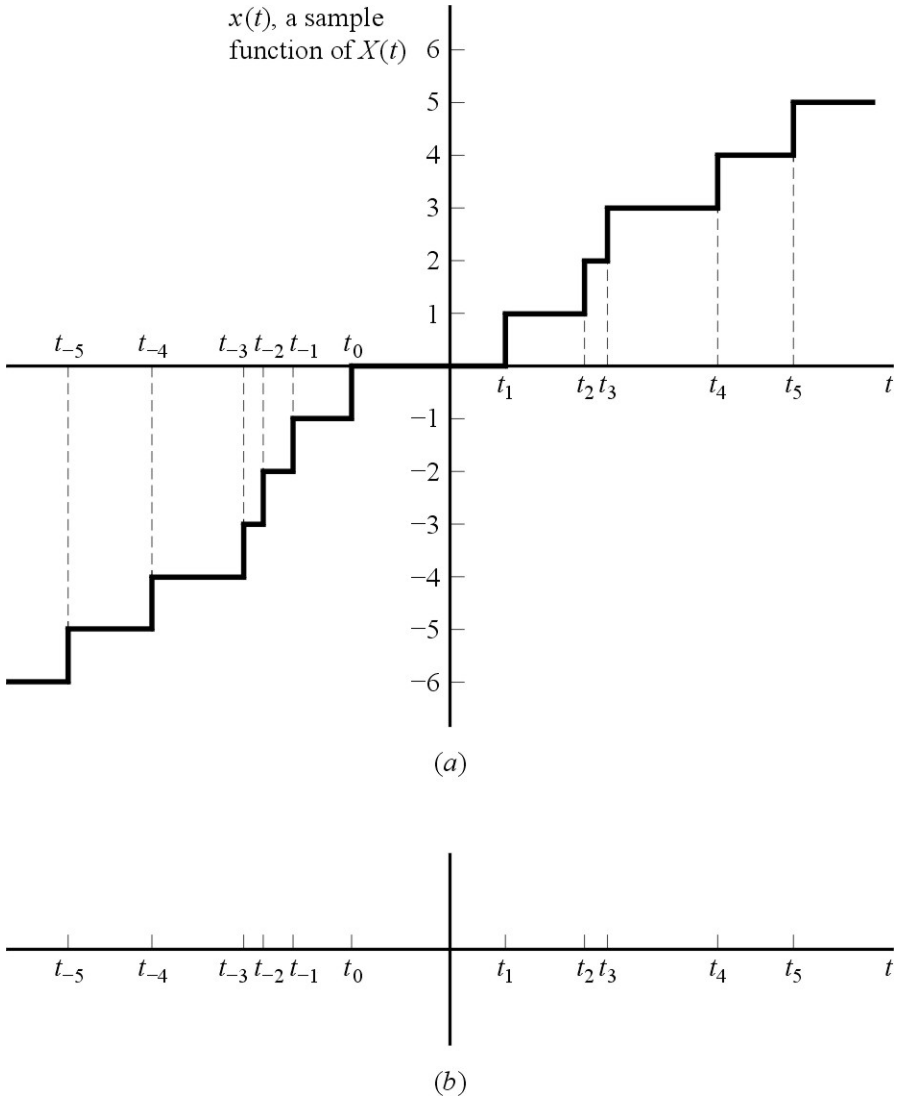
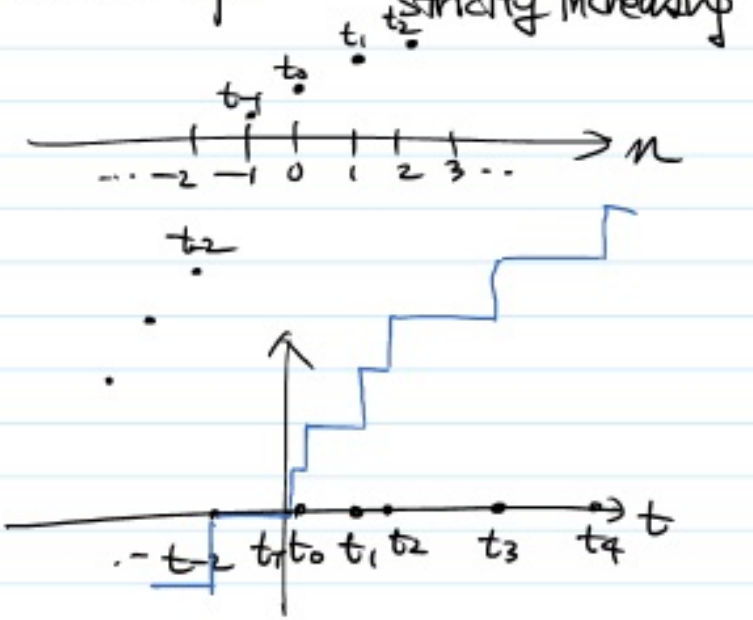

x(t), a sample function of X(t)
6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
t−5
t−4
t−3
t−2
t−1
t0
t1
t2
t3
t4
t5
t
(a)
(b)

Figure 9.1-5

$X(0)=0$ $\quad X(t_1)\le X(t_2),\ \forall t_1\le t_2$ $\quad \in \mathbb{Z}^+$

□ Poisson Random Process $X(t)$ CT discrete r.p.



- (point process) sample path
  - DT r.p
  - CT r.p (arrival process)
    - discrete

ex/ Poisson arrival process.

(i) $n(t_N, t_{N+1}) = k \in \mathbb{Z}^+$ Poisson r.v. w/ rate λt, $t = t_{N+1} - t_N$

(ii) $n(t_a, t_b) \perp\!\!\!\perp n(t_c, t_d)$ are indep.

$<$ $<$

nonoverlapping

Q. Is this a full characterization of a Poisson np?
A. Yes.

(i) $N=1$, $t_1>0$. $p_{X(t_1)}(k_1) = \Pr(X(t_1)=k_1) = \Pr(X(t_1)-X(0)=k_1)$

$$= \frac{e^{-\lambda t_1}}{k_1!}(\lambda t_1)^{k_1}, \qquad k_1 \in \mathbb{Z}_+.$$

(ii) $N=2$, $0<t_1<t_2$, $p_{X(t_1),X(t_2)}(k_1,k_2) = p_{X(t_1)}(k_1)\, p_{X(t_2)|X(t_1)}(k_2|k_1)$

$$= \cdot \Pr(X(t_2)-X(t_1)=k_2-k_1 \mid X(t_1)=k_1)$$

$$= \cdot \Pr(X(t_2)-X(t_1)=k_2-k_1 \mid X(t_1)-X(0)=k_1)$$

$$= \frac{e^{-\lambda t_1}}{k_1!}(\lambda t_1)^{k_1} \cdot \frac{e^{-\lambda(t_2-t_1)}\{\lambda(t_2-t_1)\}^{k_2-k_1}}{(k_2-k_1)!}$$

(iii) N th is given.

$0<t_1<t_2<\cdots<t_N<t_{N+1}$

$$p_{X(t_1),X(t_2),\cdots,X(t_{N+1})}(k_1,k_2,\cdots,k_{N+1}) = p_{X(t_1),\cdots,X(t_N)}(k_1,\cdots,k_N)$$

$$\times p_{X(t_{N+1})|X(t_1),\cdots,X(t_N)}(k_{N+1}|k_1,\cdots,k_N)$$

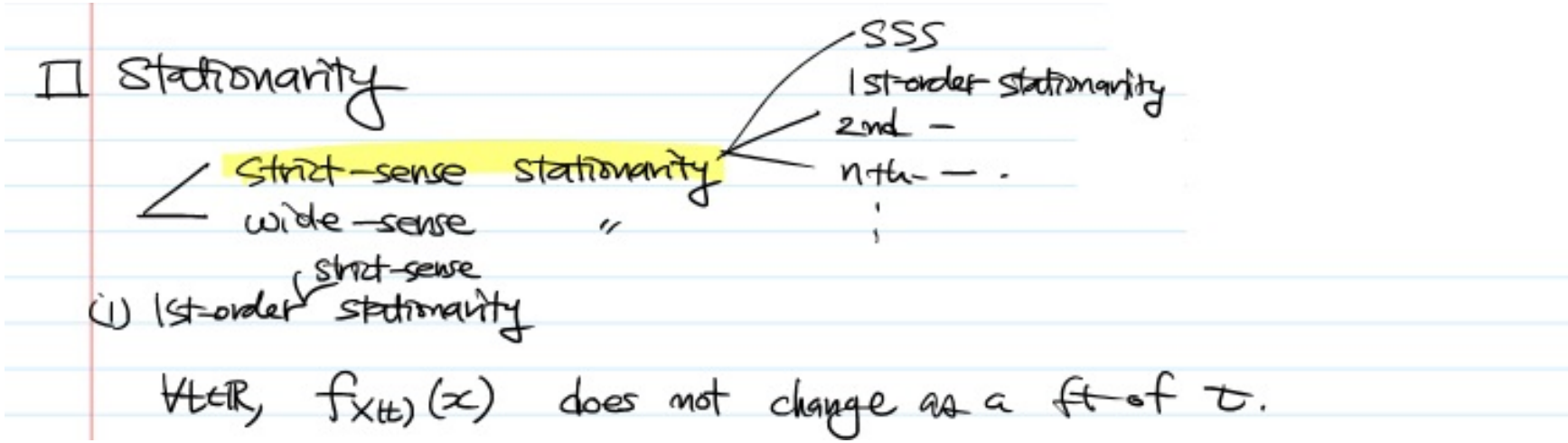
$$= \cdot \frac{e^{-\lambda(t_{N+1}-t_N)}\{\lambda(t_{N+1}-t_N)\}^{k_{N+1}-k_N}}{(k_{N+1}-k_N)!}$$

STATIONARITY

- Def. A random process is said to be stationary if its statistical properties do not change *with time*.
  - There are several "levels" of stationarity, all of which depend on the density functions of the random variables of the process.
- Def. Otherwise, it is said to be non-stationary.
- Def. A random process is called stationary to order one (or *first-order stationary*) if its first-order density function does not change with a shift in time origin, that is,

$$f_X(x_1; t_1) = f_X(x_1; t_1 + \Delta) \tag{9.1-5}$$

for any t_1 and any real number Δ, i.e., $f_X(x_1; t_1)$ is not a function of t_1.

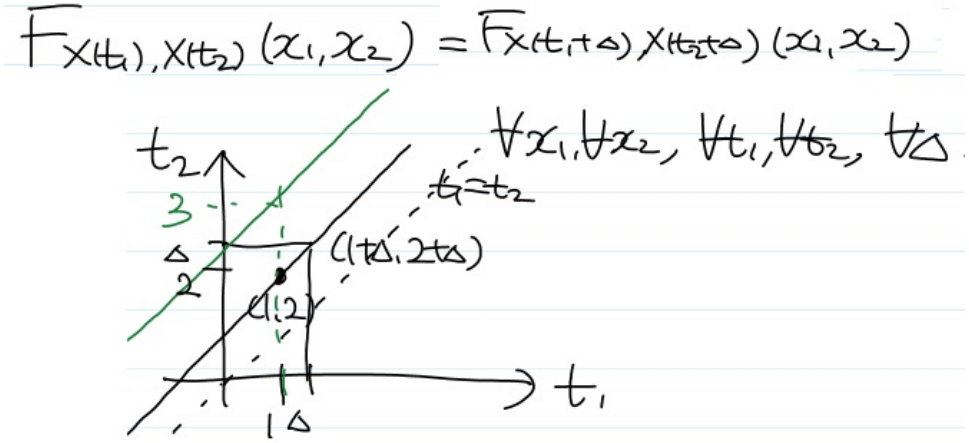


- Def. A process is called stationary to order two (or *second-order stationary*) if its second-order density function satisfies

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta) \tag{9.1-6}$$

for all t_1, t_2, and Δ, i.e., $f_X(x_1, x_2; t_1, t_2)$ is a function of $t_1 - t_2$.

  - The joint density is a function of time differences but not absolute time (by letting $\Delta = -t_1$).
  - A second-order stationary RP is a first-order stationary because the second-order density function determines the lower, first-order, density.



- Def. A random process is stationary to order N if its Nth-order density function is invariant to a time origin shift; that is, if

$$f_X(x_1, ..., x_N; t_1, ..., t_N) = f_X(x_1, ..., x_N; t_1 + \Delta, ..., t_N + \Delta) \tag{9.1-7}$$

for all $t_1, ..., t_N$ and Δ.

- Stationarity of order N implies stationarity to all orders $k \leq N$.
- Def. A process stationary to *all* orders $N = 1, 2, ...,$ is called *strict-sense stationary* (SSS).

(ii) 2nd-order SS stationarity

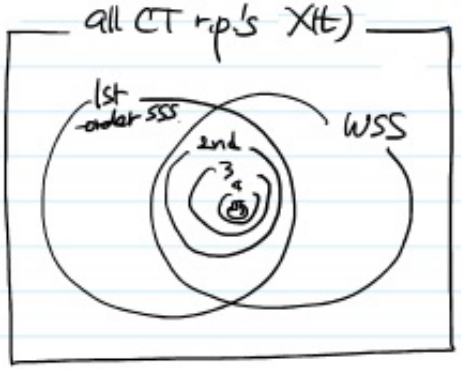
$\forall (t_1, t_2)$, $\forall \tau \in \mathbb{R}$, $f_{X(t_1),X(t_2)}(x_1, x_2) = f_{X(t_1+\tau),X(t_2+\tau)}(x_1, x_2) \quad \forall (x_1, x_2) \in \mathbb{R}^2$

parameterized only by $t_1 - t_2$.

"translation invariance"

(iii) Nth order SSS.

$\forall \underline{t} \in \mathbb{R}^N$, $\wedge$ $\forall \tau \in \mathbb{R}$. $f_{X(t_1),\cdots,X(t_N)}(x_1, \cdots, x_N) = f_{X(t_1+\tau),\cdots,X(t_N+\tau)}(x_1, \cdots, x_N)$, $\forall \underline{x}$



WSS

(i) $E\{X(t)\} = \mu_X, \forall t$

(ii) $E\{X(t_1)X(t_2)\} = R_{XX}(t_1 - t_2), \forall t_1, t_2$

(iv) $X(t)$ is Nth order SSS $\forall N$ $\Rightarrow$ $X(t)$ is SSS.

- Expectation of a random process?
  - Q. What is $E[X(t)]$?
  - Q. How can we compute $E[X(t)]$?
- Def. The expected/mean value of a process $X(t)$ is defined as

$$E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x) dx.$$

  - It is a function of time t and not a constant in general. So, it is called the mean path/process of $X(t)$.
  - It is the first moment and often denoted by $\mu_X(t)$.
  - $E[X(t)]$ is called the ensemble average to be contrasted to the time average of the random process.
  - If the process is stationary to order one,
    - then $E[X(t)]$ is a constant function, i.e., $\mu_X(t) = \mu_X, \forall t$.
      - If $X(t)$ is stationary to order 1, then $f_X(x_1; t_1)$ is not a function of t_1.
      - Hence, the process mean value $E[X(t)]$ is a constant:

$$E[X(t)] = \bar{X} = \text{constant}. \tag{9.1-8}$$

Proof. For $X_1 = X(t_1)$,

$$E[X_1] = E[X(t_1)] = \int_{-\infty}^{\infty} x_1 f_X(x_1; t_1) dx_1. \tag{9.1-9}$$

For $X_2 = X(t_2)$,

$$E[X_2] = E[X(t_2)] = \int_{-\infty}^{\infty} x_1 f_X(x_1; t_2) dx_1 \tag{9.1-10}$$

By letting $t_2 = t_1 + \Delta$, we get

$$E[X(t_1 + \Delta)] = E[X(t_1)] \tag{9.1-11}$$

* However, the converse is not true in general.

- Def. The expected/mean power of a process $X(t)$ is defined as

$$E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_{X(t)}(x) dx.$$

  – It is a second moment and often denoted by $P_{XX}(t)$.
  – It is a function of time t and not a constant in general.
  – If the process is stationary to order one,
    * then $E[X^2(t)]$ is a constant function, i.e., $P_{XX}(t) = P_{XX}, \forall t$.
    * However, the converse is not true in general.

- Def. The auto-correlation function of a random process $X(t)$ is defined as

$$E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2) dx_1 dx_2. \tag{9.1-12}$$

  – It is a second moment and often denoted by $R_{XX}(t_1, t_2)$.

- It is a function of t_1 and t_2, and not a function of $t_1 - t_2$ in general.
- $E[X^2(t)] = R_{XX}(t,t)$
- If the process is stationary to order two,
  - then $E[X(t)] = \mu_X(t)$ and $E[X^2(t)] = R_{XX}(t,t)$ are constants for all t.
  - then $E[X(t_1)X(t_2)]$ is a function of time difference $t_1 - t_2$ not absolute times, i.e., there exists a single variable function $R_{XX}(\tau)$ s.t.

$$E[X(t_1)X(t_2)] = R_{XX}(t_1,t_2) = R_{XX}(t_1 - t_2).$$

  - However, the converse is not true in general.

• Def. The auto-covariance function of a random process $X(t)$ is defined as

$$E[\{X(t_1) - \mu_X(t_1)\}\{X(t_2) - \mu_X(t_2)\}] = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2). \quad (9.1\text{-}13)$$

- It is a second moment and often denoted by $C_{XX}(t_1,t_2)$.
- It is a function of t_1 and t_2, and not a function of $t_1 - t_2$ in general.
- $E[\{X(t) - \mu_X(t)\}^2] = C_{XX}(t,t) = R_{XX}(t,t) - \mu_X^2(t)$
- If the process is stationary to order two,
  - then $C_{XX}(t_1,t_2)$ is a function of time difference $t_1 - t_2$ not absolute times, i.e., there exists a single variable function $C_{XX}(\tau)$ s.t.

$$C_{XX}(t_1,t_2) = C_{XX}(t_1 - t_2) = R_{XX}(t_1 - t_2) - \mu_X^2.$$

  - However, the converse is not true in general.

Second-order Moments

$$\mu_X(t) \triangleq E\{X(t)\} \triangleq \int_{-\infty}^{\infty} x f_{X(t)}(x)\,dx$$ mean function. (path)

$$R_{XX}(t_1,t_2) \triangleq E\{X(t_1)X(t_2)\} \triangleq \int\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1,x_2)\,dx_1\,dx_2$$

$R_X(t,t) = E\{X^2(t)\}$ average power

auto-correlation function

$$C_{XX}(t_1,t_2) \triangleq E\left[\{X(t_1)-\mu_X(t_1)\}\{X(t_2)-\mu_X(t_2)\}\right]$$

$$\triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_1-\mu_X(t_1))(x_2-\mu_X(t_2)) f_{X(t_1),X(t_2)}(x_1,x_2)\,dx_1\,dx_2$$

auto-covariance function

cf) $R_{XY} = C_{XY} + \mu_X\mu_Y$

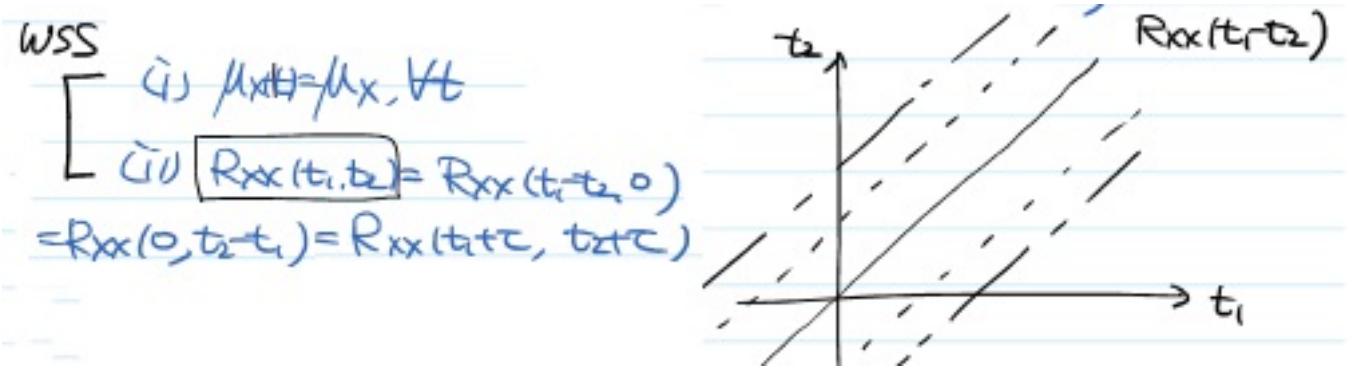
$C_{XY} = R_{XY} - \mu_X\mu_Y$ $\quad C_X(t_1,t_2) = R_X(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$

- Def. A process is called wide-sense stationary (WSS) if

$$E[X(t)] = \bar{X} = \text{constant}, \forall t, \tag{9.1-14a}$$

$$E[X(t)X(t+\tau)] = R_{XX}(\tau) \tag{9.1-14b}$$

  - The stationarity to order two is often more restrictive than necessary, and a more relaxed form of stationarity is desirable.
  - A process stationary to order two satisfies above two conditions and is WSS. However, the converse is not necessarily true.



- Def. A process is called covariance stationary if

$$C_{XX}(t+\tau, t) = C_{XX}(\tau) \tag{9.1-14}$$

  - If WSS, then covariance stationary.
  - However, the converse is not necessarily true.

– Properties

(i) 1st-order SSS $\Rightarrow \mu_{X(t)} = \mu_{X(0)}, \forall t$. $\mu_{X(t)} \triangleq E\{X(t)\} = \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx$

(ii) 2nd-order SSS $\Rightarrow$ (i) $E\{X(t)\} = \text{const}, \forall t$

(ii) $E\{X(t_1)X(t_2)\}$ $R_{XX}(t_1,t_2) = \iint x_1 x_2 f_{X(t_1),X(t_2)}(x_1,x_2)\, dx_1\, dx_2$

$= a$ ft of τ.

$= R_{XX}(\tau)$, where $t_1 - t_2 = \tau$ $\tau = t_1 - t_2$

$\exists\ R_{XX}(\tau)$

(ii)' $C_{XX}(t_1,t_2)$ is a ft of τ.

$\because C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2)$

$\mu_X \quad \mu_X$

- Example 1. Find the mean and auto-correlation of a Poisson process.
  - The mean and second moment of $X(t)$ are formally computed as follows:

$$E[X(t)] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k) dx$$
$$= \sum_{k=0}^{\infty} \frac{k(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t$$

$$E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t(1+\lambda t)$$

  - The autocorrelation function of the process can be similarly found as ....

- Example 2. Show that the random process

$$X(t) = A\cos(\omega_0 t + \Theta)$$

is WSS, if it is assumed that A and ω_0 are constants and Θ is a uniformly distributed RV on the interval $(0, 2\pi)$.

Proof. The mean value is

$$E[X(t)] = \int_0^{2\pi} A\cos(\omega_0 t + \theta) \frac{1}{2\pi} d\theta = 0.$$

The autocorrelation function becomes

$$\begin{aligned} R_{XX}(t, t+\tau) &= E[A\cos(\omega_0 t + \Theta)\ A\cos(\omega_0 t + \omega_0 \tau + \Theta)] \\ &= \frac{A^2}{2}\cos(\omega_0 \tau) \end{aligned}$$

Therefore, $X(t)$ is WSS.

- Example 3.

ex/ $X(t) = A\cos(\omega t + \Theta)$ where A & Θ are independent, Θ is uniform on $[0, 2\pi)$

Q. $X(t)$ is WSS?

(i) $E\{X(t)\} = E\{A\cos(\omega t+\Theta)\} = E\{A\} \cdot E\{\cos(\omega t+\Theta)\} = E\{A\}\cdot 0 = 0, \forall t$

(ii) $E\{X(t_1)X(t_2)\} = E\{A\cos(\omega t_1+\Theta)\cdot A\cos(\omega t_2+\Theta)\}$

$= E\{A^2\} E\{\cos(\omega t_1+\Theta)\cos(\omega t_2+\Theta)\}$

$= E\{A^2\} E\left[\frac{1}{2}\{\cos(\omega(t_1+t_2)+2\Theta) + \cos(\omega(t_1-t_2))\}\right]$

$= E\{A^2\}\frac{1}{2}\cos(\omega(t_1-t_2))$

A. Yes.

Q. $X(t)$ is 1st-order stationary? A. Yes.

Q. $X(t)$ is 2nd- " " ? A.

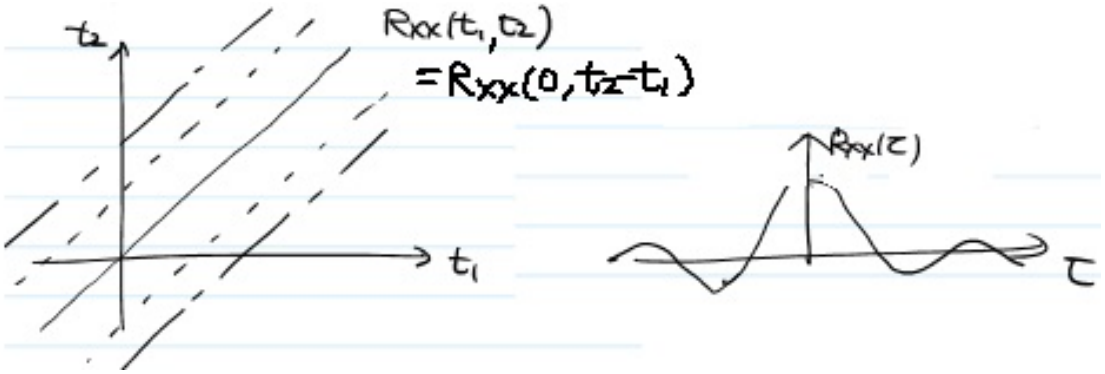
Properties of Autocorrelation Function

- The autocorrelation function of a real-valued random process $X(t)$ is defined by

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] \tag{9.1-15}$$

By letting $t_2 = t$ and $t_1 = t_2 + \tau$ with τ a real number,

$$R_{XX}(t + \tau, t) = E[X(t + \tau)X(t)] \tag{9.1-16}$$

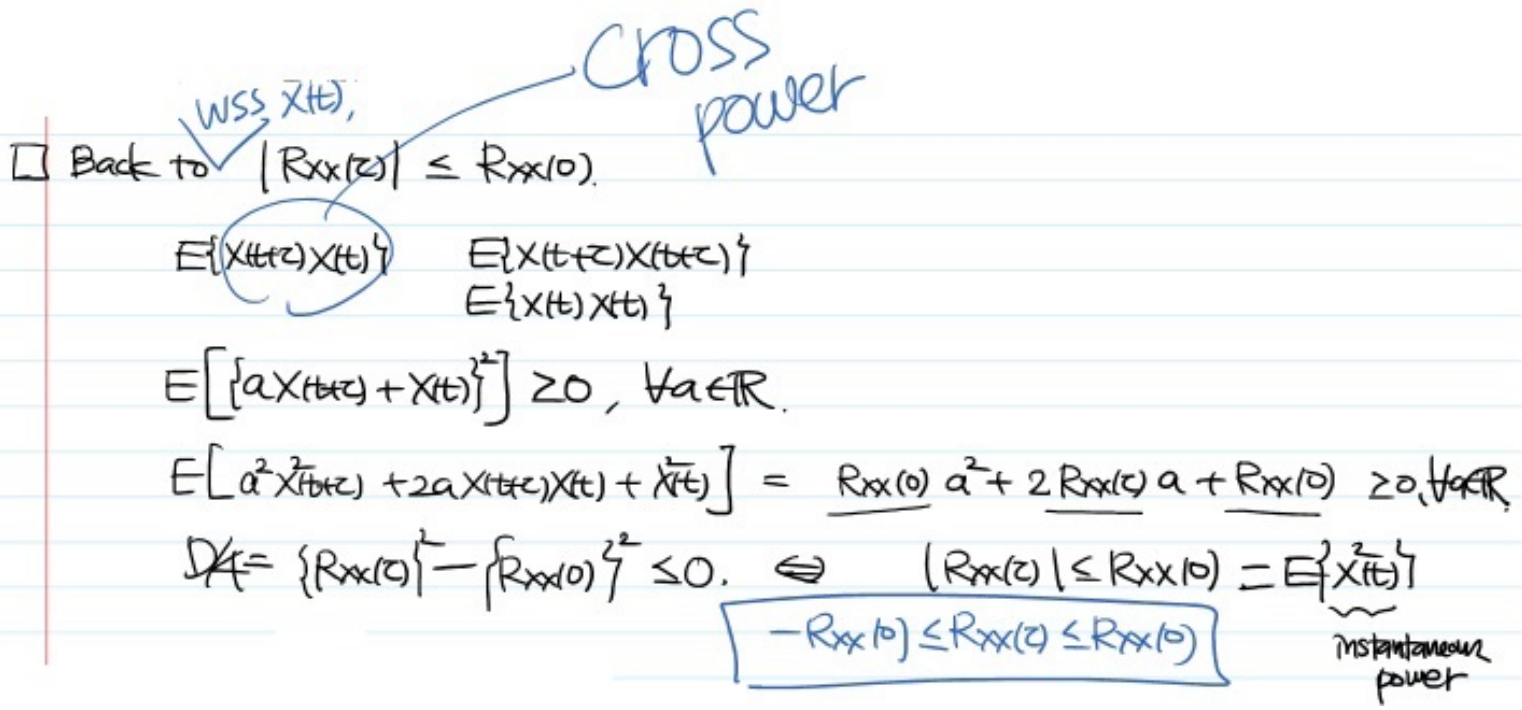


- Assume that $X(t)$ is at least WSS. Then

$$R_{XX}(\tau) = E[X(t + \tau)X(t)] \tag{9.1-17}$$

(1) $R_{XX}(\tau)$ is bounded by its value of the origin:

$$|R_{XX}(\tau)| \leq R_{XX}(0)$$


WSS X(t),
cross power
Back to |Rxx(τ)| ≤ Rxx(0).
E{X(t+τ)X(t)}
E{X(t+τ)X(t+τ)}
E{X(t)X(t)}
E[{aX(t+τ)+X(t)}²] ≥0, ∀a∈R.
E[a²X²(t+τ)+2aX(t+τ)X(t)+X²(t)] = Rxx(0)a²+2Rxx(τ)a+Rxx(0) ≥0, ∀a∈R
Δ= {Rxx(τ)}²−{Rxx(0)}² ≤0. ⇔ |Rxx(τ)| ≤ Rxx(0) = E{X²(t)}
−Rxx(0) ≤ Rxx(τ) ≤ Rxx(0)
instantaneous power

(2) An autocorrelation function has even symmetry.

$$R_{XX}(-\tau) = R_{XX}(\tau)$$

(3) The bound in (1) is equal to the mean-squared value called the power in the process.

$$R_{XX}(0) = E[X^2(t)]$$

(4) If $E[X(t)] = \bar{X} \neq 0$ and $X(t) = \bar{X} + N(t)$ is at least WSS with no periodic components, and $N(t)$ is zero-mean with $R_{NN}(\tau) \to 0$ as $|\tau| \to \infty$, then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2$$

(5) If $X(t)$ has a periodic component, then $R_{XX}(\tau)$ will have a periodic component with the same period.

(6) If $X(t)$ is ergodic, zero-mean, and has no periodic component, then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = 0$$

(7) $R_{XX}(\tau)$ *cannot have an arbitrary shape.*
This fact will be more apparent when the *power density spectrum* is introduced in the next chapter.

Example: Given that the autocorrelation function is

$$R_{XX}(\tau) = 25 + \frac{4}{1 + 6\tau^2}.$$

for a stationary ergodic process $X(t)$ with no periodic components, find its mean and variance.

(solution)

From Property (4), the mean value is $E[X(t)] = \bar{X} = \sqrt{25} = \pm 5$.

The variance is

$$\begin{aligned} \sigma_X^2 &= E[X^2(t)] - (E[X(t)])^2 \\ &= R_{XX}(0) - \bar{X}^2 \\ &= (25 + 4) - 25 \\ &= 4. \end{aligned}$$

Example: Let $X(t)$ be a WSS random process with autocorrelation function

$$R_{XX}(\tau) = e^{-a|\tau|}, \quad a > 0$$

Determine the autocorrelation function of $Y(t) = X(t)\cos(\omega_0 t + \Theta)$.

(solution)

$$\begin{aligned} R_{YY}(t+\tau, t) &= E[Y(t+\tau)Y(t)] \\ &= \frac{1}{2} R_{XX}(\tau)\cos(\omega_0 \tau) \\ &= R_{YY}(\tau) \end{aligned}$$

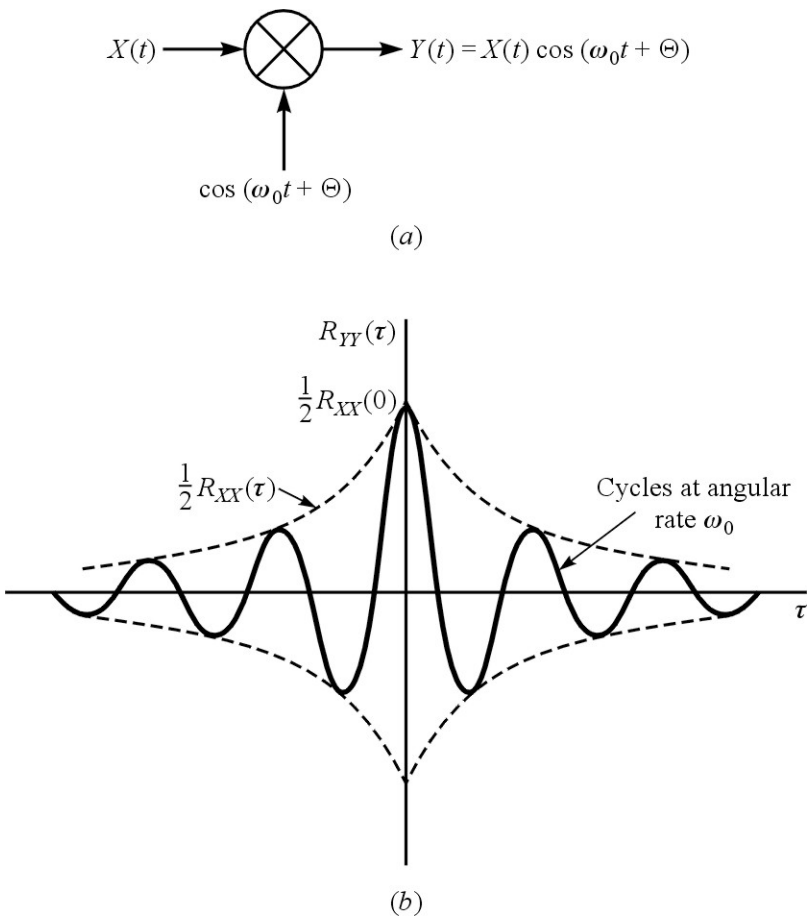


Figure 9.1-6 (a) Amplitude modulation of a carrier with random phase by a random process $X(t)$. (b) A plot of the autocorrelation function of $Y(t)$.

Covariance Functions

- The *autocovariance function* of two real-valued RPs $X(t)$ and $Y(t)$ is defined by

$$\begin{aligned} C_{XX}(t+\tau,t) &\triangleq E[\{X(t+\tau)-E[X(t+\tau)]\}\{X(t)-E[X(t)]\}] \\ &= R_{XX}(t+\tau,t)-E[X(t+\tau)]E[X(t)] \end{aligned}$$

- The *variance* of a real-valued RP $X(t)$ is defined by $C_{XX}(t,t)$.

  For a WSS RP,

$$\sigma_X^2 = E[\{X(t)-E[X(t)]\}^2] = R_{XX}(0)-\bar{X}^2 \qquad (9.1\text{-}18)$$

- positive semi-defininteness of a correlation/covariance function

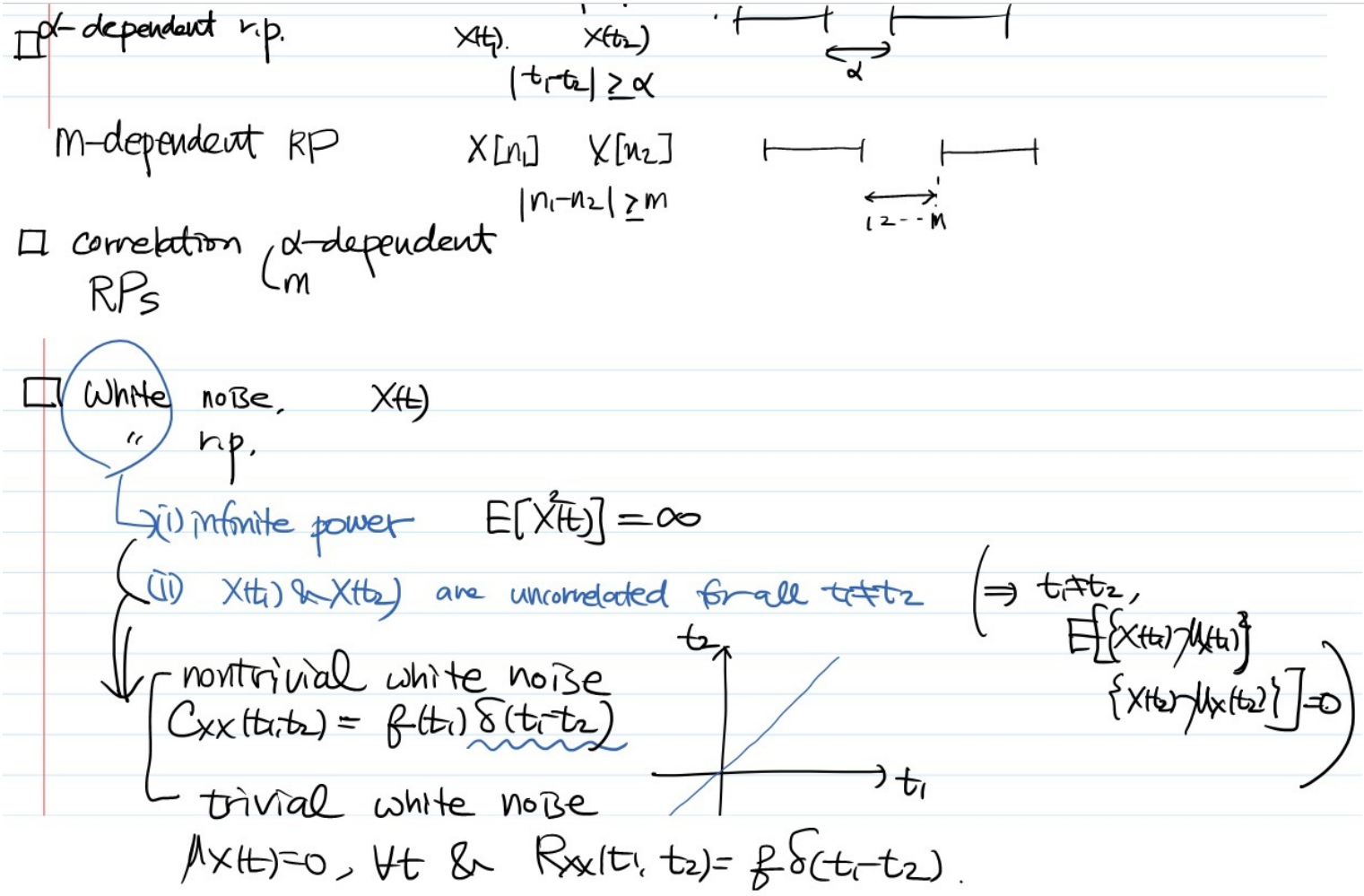
□ $R_{xx}(t_1,t_2)$ & $C_{xx}(t_1,t_2)$ are positive semi-definite functions of t_1 & t_2.

$\forall\ a_i, a_j \quad t_i\ t_j \qquad \sum_{ij} a_i a_j^* R_{xx}(t_i,t_j) \geq 0.$

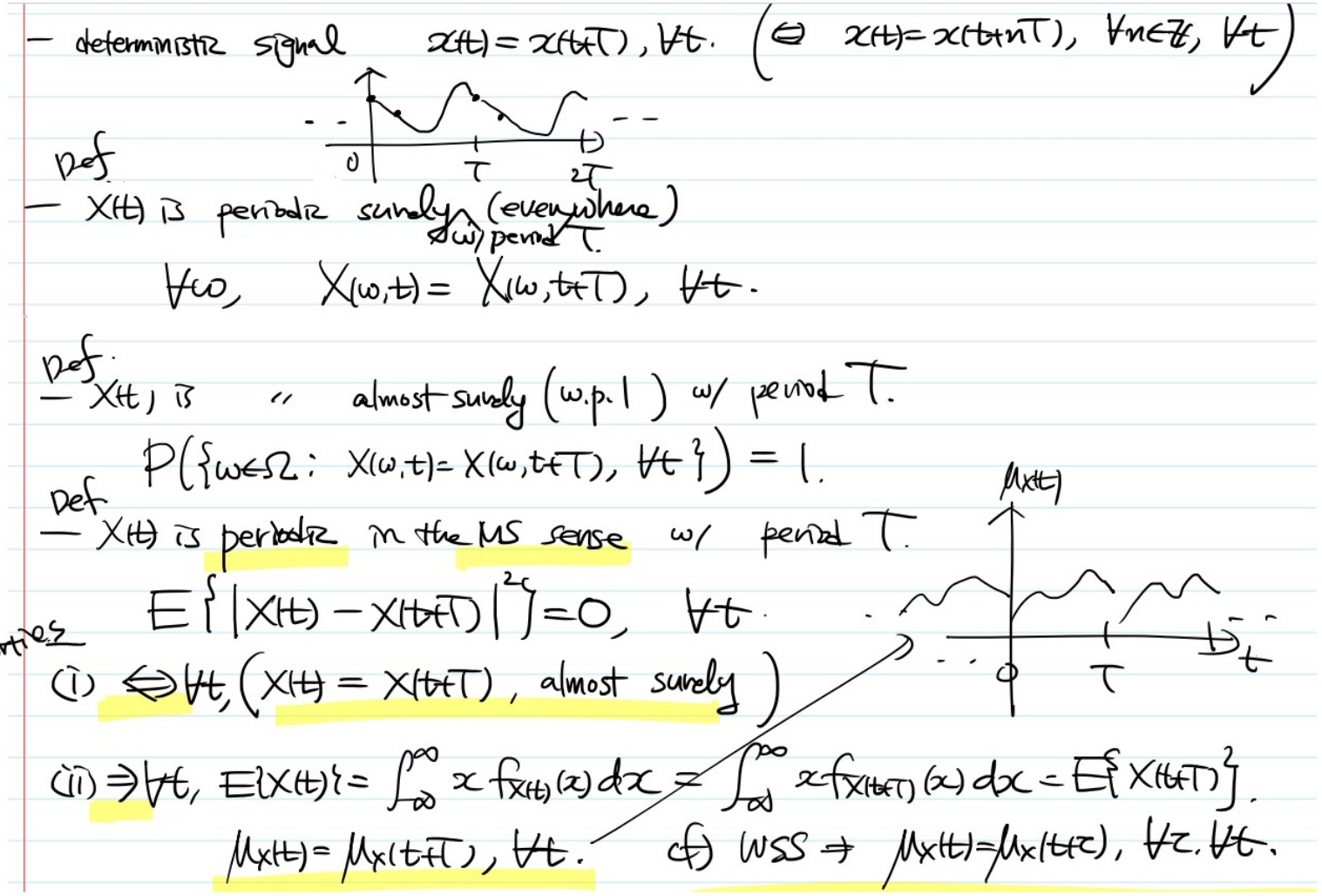
$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_i R_{xx}(t_i,t_j) a_j^* = \underline{a}^H R_{\underline{x}(t)\,\underline{x}(t)}\,\underline{a}$$

$$\begin{bmatrix} R_{xx}(t_1,t_1) & & \\ & \ddots & \\ & & R_{xx}(t_N,t_N) \end{bmatrix}$$

- α-dependent random process



- mean-square period random process



(iii) $R_{XX}(t_1,t_2) = E\{X(t_1)X(t_2)\} = E\{X(t_1+mT)\,X(t_2+nT)\} = R_{XX}(t_1+mT, t_2+nT)$ $\forall m,n$.

Recall $E\{(aW+Z)^2\} \geq 0$, $\forall a \in \mathbb{R}$.

$\Leftrightarrow E\{W^2\}a^2 + 2E\{WZ\}a + E\{Z^2\} \geq 0$, $\forall a \in \mathbb{R}$

$\Rightarrow D/4 \leq 0 \Leftrightarrow (E\{WZ\})^2 \leq E\{W^2\}E\{Z^2\}$.

Let

$W = X(t_1) - X(t_1+T)$ $\quad$ $W = X(t_1) - X(t_1+mT)$

$Z = X(t_2)$ $\quad$ $Z = X(t_2)$

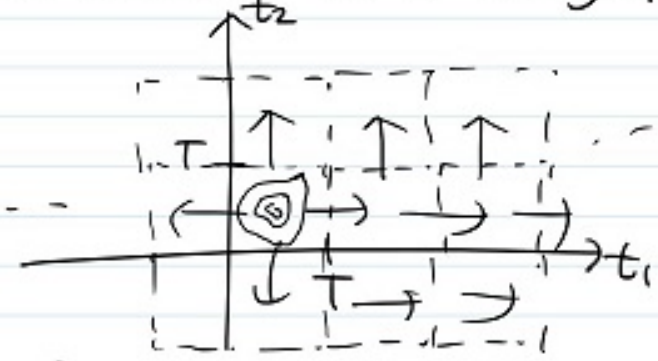
Then, $\left(E\{WZ\}\right)^2 \leq E\{W^2\}E\{Z^2\}$

$0 \leq \{R_{XX}(t_1,t_2) - R_{XX}(t_1+T,t_2)\}^2 \leq 0$

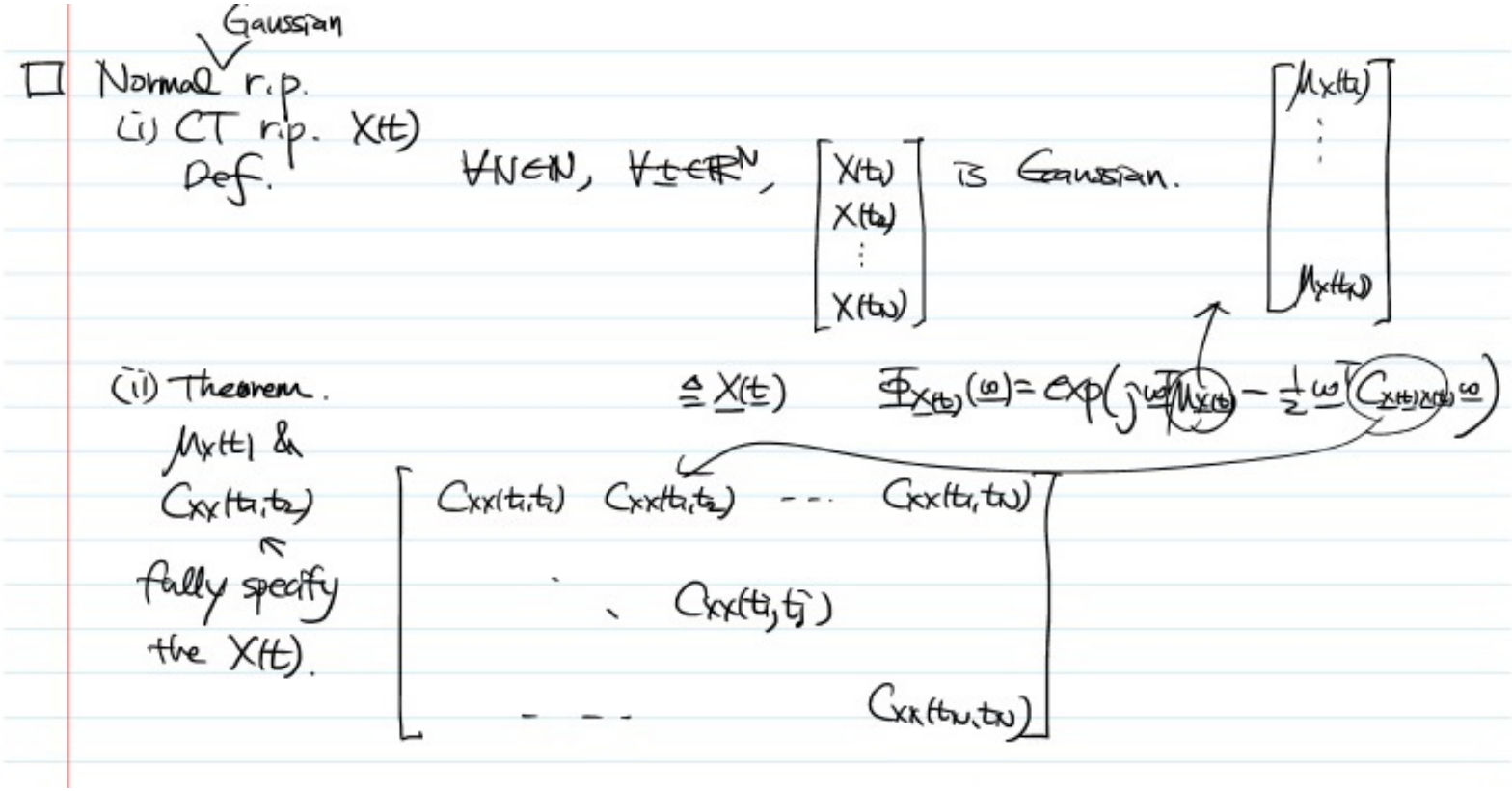
Hence, $\therefore$ $R_{XX}(t_1,t_2) = R_{XX}(t_1+T,t_2)$

Similarly, $R_{XX}(t_1,t_2) = R_{XX}(t_1+mT,t_2) = R_{XX}(t_1+mT, t_2+nT)$

autocorrelation fct is doubly periodic



GAUSSIAN RANDOM PROCESSES



- Def. A continuous random process $X(t)$ is gaussian, if N RVs $X_1 = X(t_1), ..., X_N = X(t_N)$ are jointly gaussian for any N and any $t_1, ..., t_N$,

that is, the joint pdf of $X_1 = X(t_1), ..., X_N = X(t_N)$ is given by

$$f_X(x_1, ..., x_N; t_1, ..., t_N) = \frac{\exp\{-(1/2)[x - \bar{X}]^t [C_X]^{-1} [x - \bar{X}]\}}{\sqrt{(2\pi)^N |[C_X]|}} \qquad (9.1\text{-}19)$$

where

$$\bar{X}_i = E[X_1] = E[X(t_i)] \qquad (9.1\text{-}20)$$

and $[C_X]$ is the covariance matrix whose (i,k)-entry is

$$\begin{aligned} C_{ik} = C_{X_i X_k} &= E[(X_i - \bar{X}_i)(X_k - \bar{X}_k)] \\ &= E[\{X(t_i) - E[X(t_i)]\}\{X(t_k) - E[X(t_k)]\}] \\ &= C_{XX}(t_i, t_k) \qquad (9.1\text{-}21) \\ &= R_{XX}(t_i, t_k) - E[X(t_i)]E[X(t_k)]. \qquad (9.1\text{-}22) \end{aligned}$$

- For a WSS RP

$$\begin{aligned} \bar{X}_i \;=\; E[X(t_i)] \;&=\; \bar{X} \qquad \text{(constant)} \\ C_{XX}(t_i, t_k) \;&=\; C_{XX}(t_i - t_k) \\ R_{XX}(t_i, t_k) \;&=\; R_{XX}(t_i - t_k) \end{aligned}$$

- A WSS gaussian process is also strictly stationary.

— WSS Gaussian rp $X(t)$?

(i) Gaussianity $\Rightarrow$ $\mu_X(t)$, $C_{XX}(t_1,t_2)$.

(ii) WSS $\Rightarrow$ μ_X, $C_{XX}(t_1-t_2)$

1st-order $X(t_1) \sim N(\mu_X, \underbrace{C_{XX}(t_1,t_1)}_{C_{XX}(0)})$ (O) 1st-o SSS

2nd-order $\begin{bmatrix} X(t_1) \\ X(t_2) \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ \mu_X \end{bmatrix}, \begin{bmatrix} C_{XX}(t_1,t_1) & C_{XX}(t_1,t_2) \\ C_{XX}(t_2,t_1) & C_{XX}(t_2,t_2) \end{bmatrix}\right)$ (O) 2nd-o SSS

Nth-order $\begin{bmatrix} X(t_1) \\ \vdots \\ X(t_N) \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ \vdots \\ \mu_X \end{bmatrix}, \begin{bmatrix} C_{XX}(t_i,t_j)=C_{XX}(t_i-t_j) \\ =C_{XX}((t_i+\tau)-(t_j+\tau)) \end{bmatrix}\right)$ (O) Nth-order SSS

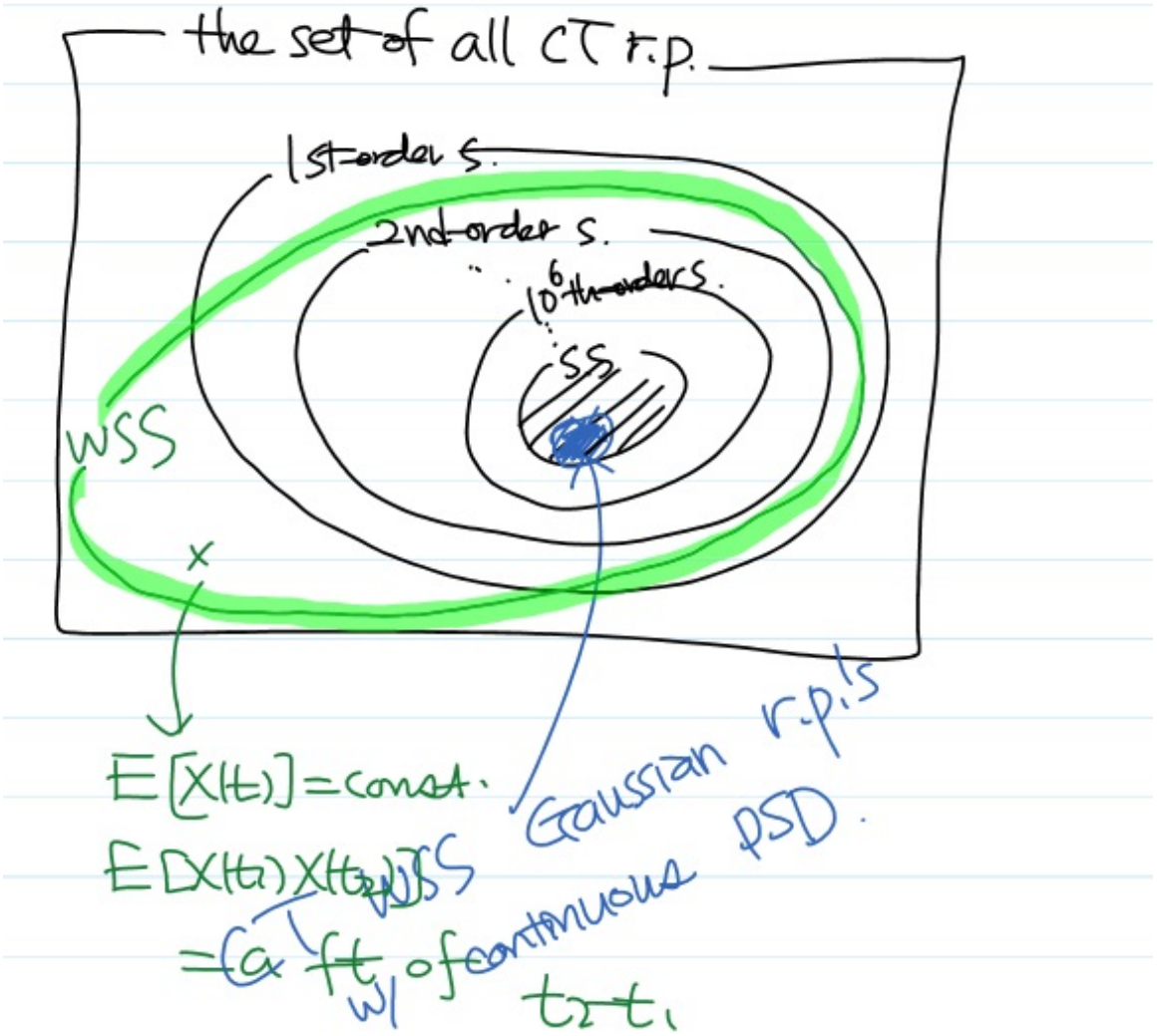
translation invariant!

$\Downarrow$

WSS Gaussian $X(t)$ $\Rightarrow$ SSS Gaussian $X(t)$.

$X(t)=X$, $\forall t$

$X \sim N(\mu_X, \sigma_X^2)$


the set of all CT r.p.
1st-order s.
2nd-order s.
6th-order s.
SS
WSS
x
E[X(t)] = const.
E[X(t1)X(t2)]
WSS
Gaussian r.p.'s
PSD.
w/ of continuous
t2-t1

Example: $X(t)$ is a WSS gaussian RP with

$$R_{XX}(\tau) = 25e^{-3|\tau|} + 16.$$

Specify the joint PDF for three RVs $X(t_i)$, $i = 1, 2, 3$, defined at times $t_i = t_0 + (i-1)/2$.

(solution)

(1) Mean: $\bar{X} = \pm 4$, since

$$\lim_{\tau \to \infty} R_{XX}(\tau) = \bar{X}^2$$

(2) Covariance:

$$\begin{aligned} C_{XX}(t_i - t_k) &= R_{XX}(t_i - t_k) - \bar{X}^2 \\ &= 25e^{-3|i-k|/2} + 16 - 16 \end{aligned}$$

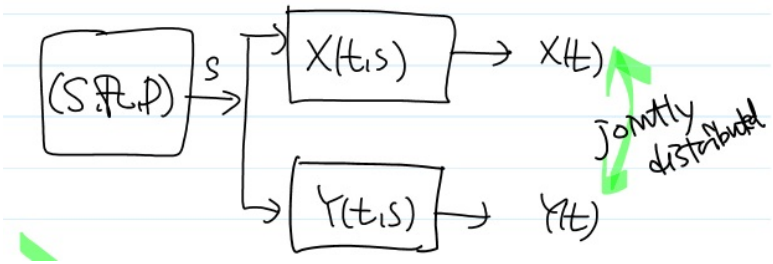
(3) Covariance matrix:

$$[C_X] = 25 \begin{bmatrix} 1 & e^{-3/2} & e^{-6/2} \\ e^{-3/2} & 1 & e^{-3/2} \\ e^{-6/2} & e^{-3/2} & 1 \end{bmatrix}.$$

These values completely determine the joint PDF.

Two Random Processes

- Def. Two random processes $X(t)$ and $Y(t)$ are jointly distributed if



- Def. Two random processes $X(t)$ and $Y(t)$ are equal if ...

Two r.p.s X(t) and Y(t) are equal.

(i) surely $X(\omega,t) = Y(\omega,t)$, $\forall \omega \in \Omega$ $\forall t \in \mathbb{R}$.

(ii) almost surely $P(\{\omega: X(\omega,t) = Y(\omega,t), \forall t\}) = 1$.

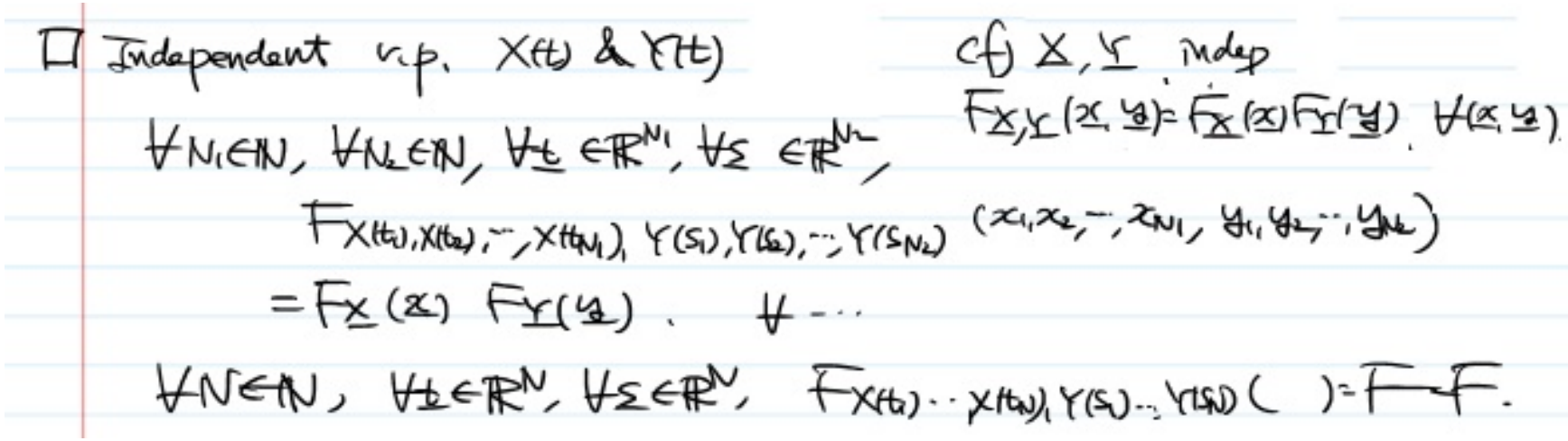
(iii) equal in the MS sense $E\{|X(t) - Y(t)|^2\} = 0$, $\forall t$

- Def. Two processes $X(t)$ and $Y(t)$ are statistically independent if

– the random variable group $X(t_1), X(t_2), ..., X(t_N)$ is independent of the group $Y(t'_1), Y(t'_2), ..., Y(t'_M)$ for any M, N and any choice of times $t_1, ..., t_N, t'_1, ..., t'_M$.

– More formally, two processes $X(t)$ and $Y(t)$ are statistically independent if for any M, N and any $t_1, ..., t_N, t'_1, ..., t'_M$,

$$f_{X,Y}(x_1, ..., x_N, y_1, ..., y_M; y_1, ..., t_N, t'_1, ..., t'_M) = f_X(x_1, ..., x_N; t_1, ..., t_N) f_Y(y_1, ..., y_M; t'_1, ..., t'_M) \quad (9.1\text{-}23)$$



• Def. Two processes $X(t)$ and $Y(t)$ are jointly strict-sense stationary if

– for any M, N and any $t_1, ..., t_M, t'_1, ..., t'_N$,

$$f_{X,Y}(x_1, ..., x_M, y_1, ..., y_N; t_1, ..., t_M, t'_1, ..., t'_N) = f_X(x_1, ..., x_M, y_1, ..., y_N; t_1 + \Delta, ..., t_M + \Delta, t'_1 + \Delta, ..., t'_N + \Delta) \quad (9.1\text{-}24)$$

- The cross-correlation function of two real-valued random processes $X(t)$ and $Y(t)$ is defined by

$$R_{XY}(t+\tau,t) = E[X(t+\tau)Y(t)]$$

- The *cross-covariance function* of two real-valued RPs $X(t)$ and $Y(t)$ is defined by

$$\begin{aligned} C_{XY}(t+\tau,t) &\triangleq E[\{X(t+\tau)-E[X(t+\tau)]\}\{Y(t)-E[Y(t)]\}] \\ &= R_{XY}(t+\tau,t) - E[X(t+\tau)]E[Y(t)] \end{aligned}$$

- Def. Two random processes $X(t)$ and $Y(t)$ are called jointly wide-sense stationary if
  - each process is WSS; and
  - their cross-correlation function is a function only of time differences and not absolute time; that is, if

$$R_{XY}(t+\tau,t) = E[X(t+\tau)Y(t)] = R_{XY}(\tau).$$

- For RPs that are at least jointly WSS

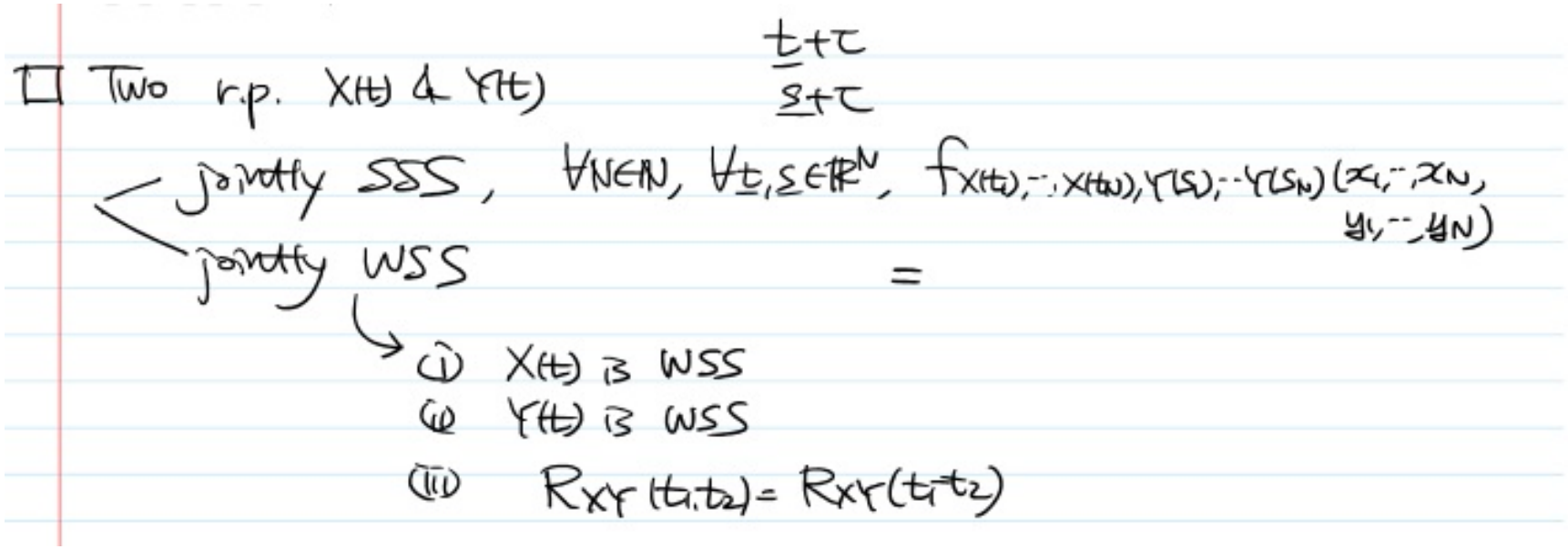
$$C_{XX}(\tau) = R_{XX}(\tau) - \bar{X}^2 \quad \text{and} \quad C_{XY}(\tau) = R_{XY}(\tau) - \bar{X}\bar{Y}$$

- Def. Two random processes $X(t)$ and $Y(t)$ are called uncorrelated if

$$C_{XY}(t_1,t_2) = 0, \forall t_1,t_2 \tag{9.1-25}$$

i.e.,

$$R_{XY}(t_1,t_2) = \mu_X(t_1)\mu_Y(t_2), \forall t_1,t_2.$$



- Def. Two random processes $X(t)$ and $Y(t)$ are called orthogonal if

$$R_{XY}(t_1, t_2) = 0, \forall t_1, t_2 \tag{9.1-26}$$

  – If $X(t)$ and $Y(t)$ are at least jointly WSS, $R_{XY}(t+\tau, t)$ is independent of absolute time, i.e.,

$$R_{XY}(\tau) = E[X(t+\tau)Y(t)] \tag{9.1-27}$$

  – If $R_{XY}(t+\tau, t) = 0$ for all τ, then $X(t)$ and $Y(t)$ are *orthogonal processes*.

  – If $X(t)$ and $Y(t)$ are statistically independent

$$R_{XY}(t+\tau, t) = E[X(t+\tau)]E[Y(t)]. \tag{9.1-28}$$

– If $X(t)$ and $Y(t)$ are statistically independent and WSS

$$R_{XY}(\tau) = \bar{X}\bar{Y} \tag{9.1-29}$$

• Properties of the cross-correlation function of at least jointly WSS processes

$$(1) \qquad R_{XY}(-\tau) = R_{YX}(\tau) \tag{9.1-30}$$

$$(2) \qquad |R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)} \tag{9.1-31}$$

$$(3) \qquad |R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)] \tag{9.1-32}$$

Remarks

(1) It describes the symmetry of $R_{XY}(\tau)$ from definition

(2) It can be proven by expanding the inequality

$$E[\{Y(t+\tau)+\alpha X(t)\}^2] \geq 0 \tag{9.1-33}$$

for any real number α

(3) The bound in (2) is tighter than the bound in (3), because the geometric mean of two positive real numbers cannot exceed their arithmetic mean, that is,

$$\sqrt{R_{XX}(0)R_{YY}(0)} \leq \frac{1}{2}[R_{XX}(0)+R_{YY}(0)] \tag{9.1-34}$$

Example: Let two RPs $X(t)$ and $Y(t)$ be defined by

$$\begin{aligned} X(t) &= A\cos(\omega_0 t)+B\sin(\omega_0 t) \\ Y(t) &= B\cos(\omega_0 t)-A\sin(\omega_0 t) \end{aligned}$$

where A and B are RVs and ω_0 is a constant. Show that $X(t)$ and $Y(t)$ are jointly WSS if A and B are uncorrelated, zero-mean RVs with the same variance σ^2.

(solution) Check that $X(t)$ and $Y(t)$ are WSS.

The cross-correlation function of $X(t)$ and $Y(t)$ can be computed as

$$R_{XY}(t+\tau,t) = -\sigma^2\sin(\omega_0\tau).$$

$X(t)$ and $Y(t)$ are jointly WSS, since $R_{XY}(t+\tau, t)$ depends only on τ.

- Two RPs are called *uncorrelated* if

$$C_{XY}(t+\tau, t) = 0 \tag{9.1-35}$$

or equivalently,

$$R_{XY}(t+\tau, t) = E[X(t+\tau)]E[Y(t)] \tag{9.1-36}$$

- Independence implies uncorrelatedness. The converse is not true in general, although it is true for *joint gaussian process*.

- Def. Two random processes $X(t)$ and $Y(t)$ are *jointly gaussian*, if the RVs $X_1 = X(t_1), ..., X_N = X(t_N)$, $Y_1 = Y(t'_1), ..., Y_M = Y(t'_M)$ defined at times $t_1, ..., t_N$ for $X(t)$ and times $t'_1, ..., t'_M$ for $Y(t)$, are jointly gaussian for any N, $t_1, ..., t_N$, M, $t'_1, ..., t'_M$.