Quiz#1

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23-1quiz1

EECE 574: Probability and Random Processes

Quiz #1 (Mar. 14th, 2023)

Time allowed: 75 minutes

Student #: ______________________________

Name: ______________________________

Problem 1: _____ / 10

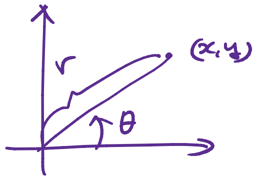
Problem 2: _____ / 10

Problem 3: _____ / 15

Total Score: _____ / 35

1. (10 points) Suppose that X and Y are i.i.d. Gaussian random variables following $\mathcal{N}(0, P)$ for some $P > 0$. Let $R \triangleq \sqrt{X^2 + Y^2}$ and $\Theta \triangleq \text{atan2}(Y, X)$, where $\text{atan2}(y, x)$ is defined as the four-quadrant inverse tangent. Answer the following questions.

(a) (5 points) Derive the PDFs of R and Θ, and show that the random variables are independent.



$$f_{R,\Theta}(r,\theta) = f_{X,Y}(r\cos\theta, r\sin\theta)\, r$$

$$= \frac{1}{2\pi P} \exp\left(-\frac{(r\cos\theta)^2}{2P}\right) \exp\left(-\frac{(r\sin\theta)^2}{2P}\right) r$$

$$= \underbrace{\frac{1}{2\pi}}_{} \underbrace{\frac{r}{P} \exp\left(-\frac{r^2}{2P}\right)}_{}$$

$$= f_\Theta(\theta) \quad f_R(r)$$

(b) (5 points) Find $\mathbf{E}[R]$, $\mathbf{E}[R^2]$, $\underbrace{\mathbf{E}[\cos(\Theta+\phi)]}_{=0}$, and $\underbrace{\mathbf{E}[\cos(2\Theta+\phi)]}_{=0}$ for some real constant ϕ.

2. (10 points) Recall the Euler's identity given by

$$e^{\mathrm{j}\phi} = \cos\phi + \mathrm{j}\sin\phi,$$

where $\mathrm{j} \triangleq \sqrt{-1}$ and ϕ is a real constant. In other words,

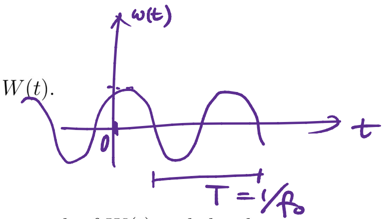
$$\mathrm{Re}\{e^{\mathrm{j}\phi}\} = \cos\phi \text{ and } \mathrm{Im}\{e^{\mathrm{j}\phi}\} = \sin\phi,$$

where $\mathrm{Re}\{\cdot\}$ is the real part operator and $\mathrm{Im}\{\cdot\}$ is the imaginary part operator. Using X, Y, R and Θ in Problem 1, we define a random process $W(t)$ as

$$W(t) \triangleq R\cos(2\pi f_0 t + \Theta),$$

where $f_0 > 0$ is a constant. Answer the following questions.

(a) (1 point) Sketch a sample path of the random process $W(t)$.



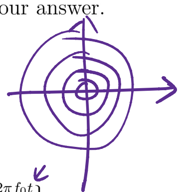
(b) (1 point) Using the results in Problem 1, derive the mean path of $W(t)$ and sketch its graph.

$E[W(t)] = E[R]\,E[\cos(2\pi f_0 t + \Theta)] = 0, \ \forall t$

(c) (2 points) Using the results in Problem 1, derive the auto-correlation function of $W(t)$ and sketch its graph.

$R_{WW}(t+\tau, t) = P\cos(2\pi f_0 \tau)$

(d) (1 point) Is $W(t)$ wide-sense stationary? Justify your answer.



(e) (5 points) Show that $W(t)$ can be rewritten as

$$W(t) = \mathrm{Re}\{(X + \mathrm{j}Y)e^{\mathrm{j}2\pi f_0 t}\}.$$

$= \mathrm{Re}\{(X + jY)(\cos(2\pi f_0 t) + j\sin(2\pi f_0 t))\}$

$= X\cos(2\pi f_0 t) - Y\sin(2\pi f_0 t) = \sqrt{X^2+Y^2}\cos(2\pi f_0 t + \mathrm{atan2}(Y,X))$

$= R\cos(2\pi f_0 t + \Theta)$

3. (15 points) Answer the following questions by using X, Y, R and Θ in Problem 1 and $W(t)$ in Problem 2. You may use the results in Problems 1 and 2.

$\sigma^2 = P$

(a) (5 points) Derive the 1st-order PDF of $W(t)$.

$$f_{W(t)}(w) = \frac{1}{\sqrt{2\pi P}} \exp\left(-\frac{w^2}{2P}\right)$$

(b) (5 points) Derive the 2nd-order PDF of $W(t)$.

$$f_{W(t_1),W(t_2)}(w_1, w_2) = \underline{\qquad\qquad}$$

$$W(t_1) = \mathrm{Re}\{\boxed{(X+jY)\, e^{j2\pi f_0 t_1}}\} \quad Z = Z_r + jZ_i$$

$$W(t_2) = \mathrm{Re}\{(X+jY)\, e^{j2\pi f_0 (t_1 - t_1 + t_2)}\}$$

$$= \mathrm{Re}\{\boxed{(X+jY)\, e^{j2\pi f_0 t_1}}\, e^{j2\pi f_0 (t_2 - t_1)}\}$$

$$Z e^{j\cdots}$$

$$= Z_r \cos(2\pi f_0 (t_2 - t_1)) - Z_i \sin(2\pi f_0 (t_2 - t_1))$$

$$\mathrm{Re}\{z\} = \frac{z + z^*}{2}$$

(c) (4 points) Is $W(t)$ a Gaussian random process? Justify your answer.

$$Y = C\, X$$

Q. Is Gaussian R vector? Const vector, Gaussian RV

$$W(t_1), W(t_2), \cdots W(t_N)$$

$$W(t_i) = \mathrm{Re}\{(X+jY)\, e^{j2\pi f_0 t_i}\}$$

$$=$$

(d) (1 point) Is $W(t)$ strict-sense stationary? Justify your answer.