CM02 measure-theoretic def. of a random XX

2023년 2월 23일 목요일 12:54

CM02-230223Thu.

Announcements

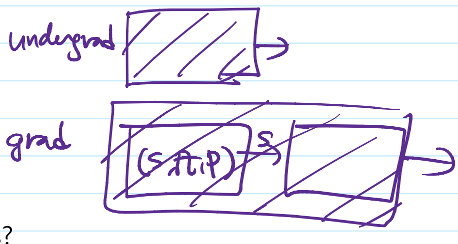
1. PDF for CM01 and handout for CM02 are uploaded at http://cisl.postech.ac.kr/class/eece574/schedule.htm . Passwords are …
2. However, no recorded Video for CM01 available.
3. HW#1 will be posted on the PLMS and will be due at 10:59am on Tuesday Feb. 28.

Corrections, Clarifications, Common mistakes, Q&A

1.

Overview (Chapter numbers from the main textbook by Papulis)

9. General Concepts
7. Sequence of Random Variables
10. Random Walks and Other Applications
11. Spectral Representation
12. Spectrum Estimation
13. Mean-Square Estimation



Review

1. unofficial Pre-requisites:
   a. undergraduate-level Prob, RV, RVec
   b. undergraduate-level Digital Comm.
2. Extracredit Project
3. What is a non-measure-theoretic definition of a random process?
4. What is a full probabilitistic description of a random process?

Preview

1. measure-theoretic definition of a random XX
2. Kolmogorov's existence/extension theorem
3. a sample path/function and an underlying probability space
4.
5. …
6. …

Chapter 6
RANDOM PROCESSES - TEMPORAL CHARACTERISTICS

Handout by Joon Ho Cho & Kyeongcheol Yang
EECE 302, Spring 2022 based on P. Z. Peebles, *Probability, Random Variables and Random Signal Principles, International 4th ed.*

6.0 INTRODUCTION

- Scalar, vector, matrix, sequence, discrete-time signal, continuous-time signal,...
- Random variable, random vector, random matrix, a sequence of random variables
- Example of random time waveforms in a binary communication system
  - the bit stream: a desired random signal
  - noise: an undesired random waveform
- random process = stochastic process

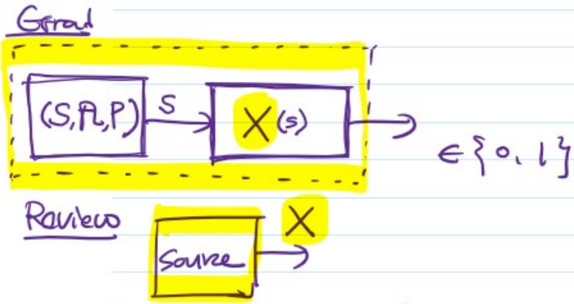
  the description of random waveforms in a probabilistic sense

6.1 THE RANDOM PROCESS CONCEPT

- the concept of random variable
  - Given a probability space $(S, \mathcal{A}, P)$, a random variable $X(s)$ is a function from the sample space S to the real line.
  - Instead of $X(s)$, we use the notation X for a random variable.
- the concept of random process (undergraduate)
  - Given a probability space $(S, \mathcal{A}, P)$, a random process $X(t, s)$ is a mapping from the sample space S to a set of time functions.
  - a sample function

$$x(t, s)$$

  - Roughly speaking, the family of all such functions, denoted $X(t, s)$, is called a random process.
  - Instead of $X(t, s)$, we use the notation $X(t)$ for a random process.
  - Example. $P(\{H\}) = 1/3, P(\{T\}) = 2/3.$
    $X(t, H) = \cos(2\pi t), \forall t$
    $X(t, T) = \sin(2\pi t), \forall t$



$s \in S$

Prob of your ~~noticing~~ observing 0 or 1?

$P(s \in A)$

$P: \mathcal{A} \to [0,1]$

$f(x) = x^2 + 2x$

$\boxed{f: \mathbb{R} \to \mathbb{R} \quad f(x) = x^2 + 2x.}$

- A random process $X(t, s)$ represents a family or ensemble of time functions when t and s are variables.
- Each member time function is called a sample function, ensemble member, or sometimes a realization of the process.

- the concept of random process (graduate)
  - Given a probability space, we have a collection of measurable functions, i.e., random variables $(X_t(s))_t$ from the sample space into the real line.
- Two points of view:
  - When t is a variable and s is fixed, a random process represents a single time function.
  - When s is a variable and t is fixed, a random process represents a random variable.
    - $\mathbf{E}[X(t_1)]$ called the ensemble average as well as the expected or mean value of the random process (at time t_1)
    - The mean value of a process may not be constant, but a function of time in general.
  - When s is a variable and $t_1, t_2, ..., t_N$ is fixed, a random process represents a random vector.

$$X_i = X(t_i, s) = X(t_i)$$

Classification of Random Processes

- continuous-parameter vs. discrete-parameter RP
  - a continuous-parameter RP: $X(t, s)$ has an uncountable set for t.
    - There are uncountably infinite measurable functions or random variables $X_t(s)$ parameterized by t.
    - When t means time, a continuous-parameter RP is often denoted by $X(t)$ and is called a continuous-time (CT) RP.
    - $X_s(t)$ for a specific s as a CT signal called a sample path of the RP and is often denoted by $x(t)$.
  - a discrete-parameter RP: $X(t, s)$ has a countable set for t.
    - There are countably infinite measurable functions or random variables $X_t(s)$ parameterized by t.
    - When t means time, a discrete-parameter RP is often denoted by $X[n]$ and is called a discrete-time (DT) RP.
    - $X_s(t)$ for a specific s as a DT signal called a sample path of the RP and is often denoted by $x[n]$.
- continuous vs. discrete RP
  - a continuous RP: $X(t)$ at each t or $X[n]$ at each n is a continuous RV.
  - a discrete RP: $X(t)$ at each t or $X[n]$ at each n is a discrete RV.

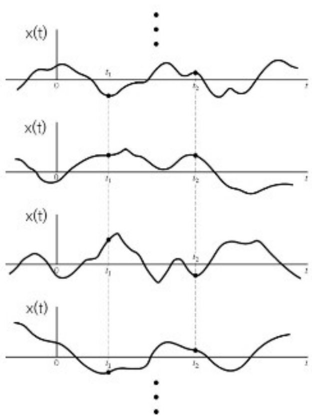


Figure 6.1-1 Sample paths of a CT continuous RP $X(t)$

$P(s \in A)$ $\quad$ $f(x) = x^2 + 2x$

$P: \mathcal{A} \to [0,1]$

ex/ $S = \{1, 2, \cdots, 6\}$

$\mathcal{A} = \{\phi, \{1\}, \cdots \{6\}, \{1,2\}, \{\cdots\}, \{5,6\}, \cdots \{1,2,3,\cdots 6\}\}$

$P(\phi) = 0$

$P(\{1\}) = 1/6$

$P(\{1,2\}) = 1/3 \quad \cdots \quad P(\{1,2,\cdots 6\}) = 1.$

$A_i \in \mathcal{A}.$

$A_1 = \{1\}$

$A_2 = \{1, 3\}$

$A_3 = \{6\}$

$A_4 = \phi. \quad \cdots$

(iii) Countable Additivity. If $A_i \cap A_j = \phi \quad \forall i \neq j$, then $P(\bigcup_{i \in I} A_i) = \sum_{i \in I} P(A_i)$. where I is a discrete set.

Review. Def. A RV $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$ ← finite

$\boxed{} \to \underline{X}$ $\quad F_{\underline{X}}(x)$, $f_{\underline{X}}(x)$, $P_{\underline{X}}(x)$

Grad $\boxed{(S, \mathcal{A}, P)} \xrightarrow{s} \boxed{\underline{X}(s)} \to$ $\quad$ $s \to \boxed{X_1(s)}, \boxed{X_2(s)}, \vdots, \boxed{X_n(s)}$

$\underline{X}(s) = \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix}$

Review a DT RP.

Grad $\boxed{} \to (X[n])_{n \in \mathbb{Z}} \to X_n$

$\boxed{(S, \mathcal{A}, P)} \xrightarrow{s} \boxed{X(s, n)} \to$ $\quad X_n(s)$

$\boxed{(S, \mathcal{A}, P)} \to$ $\vdots$ $\boxed{X_{-1}(s)} \to$ $\boxed{X_0(s)} \to$ $\boxed{X_1(s)} \to$ $\vdots$

Fix $s \in S$, $X_n(s_0)$, $n \in \mathbb{Z}$

$X_{s_0}[n] \in$ a deter- DT signal

Review $\boxed{} \to X(t)$

Grad $\boxed{(S, \mathcal{A}, P)} \xrightarrow{s} \boxed{X(t, s)} \to$

$\boxed{(S, \mathcal{A}, P)} \xrightarrow{s} \boxed{X(\cdots)} \to$

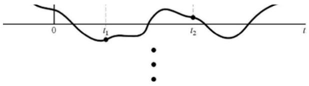


Figure 6.1-1 Sample paths of a CT continuous RP $X(t)$

Errab $\boxed{(S, \mathcal{A}, P)} \xrightarrow{s} \boxed{X(t,s)} \to$

$\boxed{(S, \mathcal{A}, P)} \xrightarrow{s} \boxed{X(s,t)} \to$

Fix $s_0 \in S$, then

$X(t, s_0)$ is a deterministic

$X(s_0, t)$ CT $\underbrace{\text{function of}}_{\text{signal}}$ t.

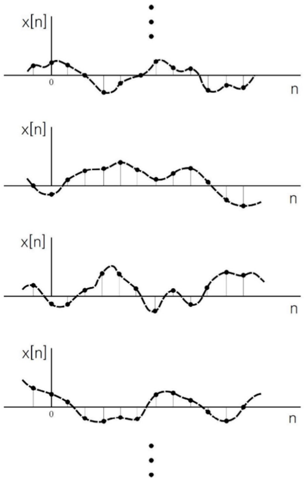


Figure 6.1-2 Sample paths of a DT continuous RP $X[n]$ obtained by uniformly sampling a CT continuous RP

Def. Sample path

$X(s_0, t)$

$X(s_0, n)$

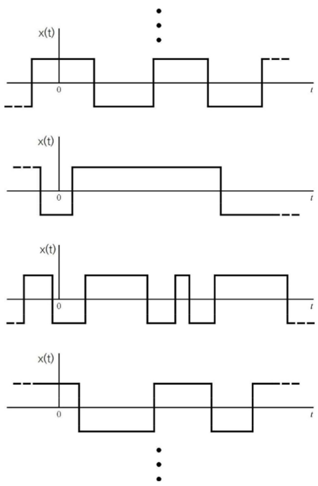


Figure 6.1-3 Sample paths of a CT discrete RP $X(t)$

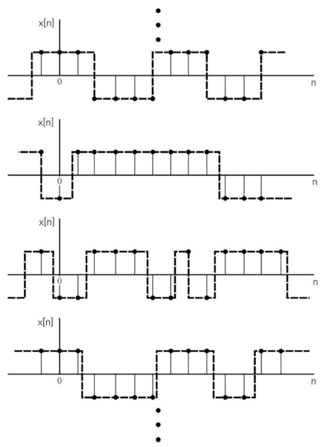


Figure 6.1-4 Sample paths of a DT discrete RP $X[n]$ obtained by uniformly sampling a CT discrete RP

- a digital process
  - It is for a discrete RP. However,
    - In circuits, a digital signal/RP often means a *CT discrete* signal/RP.
    - In communications, a digital signal/RP often means a *DT discrete* signal/RP w/ or w/o additional quantization.

Deterministic and Nondeterministic Processes

- A random process can be described by the form of its sample functions.
- If future values of any sample function cannot be predicted exactly from observed past values, the process is called nondeterministic.
- A process is called deterministic if future values of any sample function can be predicted exactly from observed past values.

$$X(t) = A\cos(\omega_0 t + \Theta)$$

where A, ω_0, or Θ (or all) may be random variables.