CM03-230228Tue

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Announcements

1. Handouts, annotated handouts, and lecture videos are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm .
   Passwords are ...
2. HW#1 was due at 10:59am Today. HW#2 will be posted soon.

Corrections; Clarifications; Common mistakes, misunderstandings, misconceptions, myth; Frequently Asked Questions

1. a CT random process having an uncountably infinite number of jointly distributed random variables can be defined
   a. on a finite interval only -> a time-limited CT RP
   b. on the entire real line. -> a time-unlimited CT RP

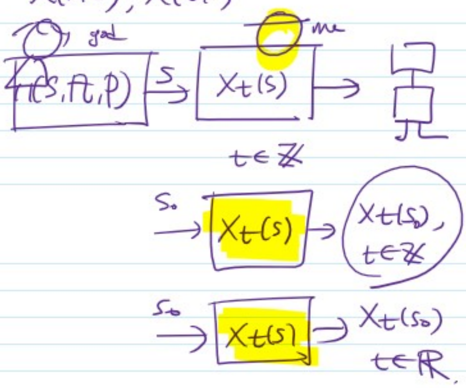
$X_t(s)$ $\quad t \in \mathbb{Z}.$ $\quad t \in \mathbb{N}$

2. a DT RP, a sequence of RVs vs. a random vector
   a. DT signal vs. a sequence of RVs
   b. a random vector that has a definite length vs. a variable length

$X(s,t)$, $X(t,s)$

3. Practical/Engineering meaning of Infinity
   a. for a sufficiently large N: "as N tends to infinity"
   b. for sufficiently far past and future: "t₩in {-₩Infinity, ₩infty}"

latex $\quad t \in \{-\infty, \infty\}$ $\quad t \in \overline{\mathbb{R}}$



4. Implication of Kolmogorov's existence theorem
   a. It means that a random process can also be well defined in the measure-theoretic sense by using the terms of a probability space and a selection/mapping rule.
   b. It introduces the notion of a SAMPLE PATH/FUNCTION, which is a very important and convenient way of understanding what is going on.

5. For a DT RP or a CT time-unlimited RP, the Kolmogorov existence theorem means that we can observe only a single sample path because X_t(s) has -₩Infinity<t<₩infty.
   a. On one hand, it appears to mean that the world is pre-deterimed, which is a way to interpreting the theorem.
      - Big bang, big crush, and CT signal
      - It is the case with a deterministic random process.
   b. On the other hand, it appears to imply the existence of parallel universes, which is a way to interpreting the theorem.
      - But in my universe ...
   c. However, in many engineering prblems, we do not need to perplex ourselves because an infinity does not mean the infinity.

Overview (Chapter numbers from the main textbook by Papulis)

1. General Concepts
2. Sequence of Random Variables
3. Random Walks and Other Applications
4. Spectral Representation
5. Spectrum Estimation
6. Mean-Square Estimation

Review

1. Non-measure-theoreric definition of a random process
2. Measure-theoretic definition of a random XX
   a. underlying probability space and a mapping rule
   b. Kolmogorov's extension theorem
3. How can we characterize a random process?

Preview

1. more classification of RPs
2. n-th order CDF/PDF, n-th order conditional CDF/PDF
   a. a deterministic RP ....

RANDOM PROCESSES - TEMPORAL CHARACTERISTICS 6 - 11

6.2 STATIONARITY AND INDEPENDENCE

- a full characterization of a RP
  - Q. Recall how to fully characterize a random vector $\underline{X}$, which is a collection of finite number of jointly distributed random variables?

    A.
    - Method 1. Provide a probability space $(S, \mathcal{A}, P)$ and a mapping rule $\underline{X}(s)$ or, equivalently, the induced probability space $(\mathbb{R}^n, \mathcal{B}^n, P_{\underline{X}})$.
    - Method 2. Provide a joint CDF/PDF/CF of the finite number of random variables.
      - $X \to$ Provide the CDF $F_X(x)$.
      - $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \to$ Provide the joint CDF $F_{X_1,X_2}(x_1, x_2)$.
      - $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \to$ Provide the joint CDF $F_{X_1,X_2,X_3}(x_1, x_2, x_3)$.
      - $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{10^6} \end{bmatrix} \to$ Provide the joint CDF $F_{X_1,X_2,\cdots,X_{10^6}}(x_1, x_2, \cdots, x_{10^6})$.

– Q. Now, how can we fully characterize a CT random process $X(t)$, a collection of *infinite* number of jointly distributed random variables?

– A.

* Method 1. Provide a probability space $(S, \mathcal{A}, P)$ and a mapping rule $X(t, s)$.
* Method 2. Provide all possible joint CDF/PDF/CF of any finite number of random variables.
  · Theorem. (Kolmogorov's Extension/Consistency Theorem) A CT random process is fully characterized by providing a joint CDF $F_{X(t_1),X(t_2),\cdots,X(t_N)}(x_1, x_2, \cdots, x_N)$ for every N-tuple $(t_1, t_2, \cdots, t_N) \in \mathbb{R}^N$ for every $N \in \mathbb{N}$.
  (1) $F_{X(t)}(x)$ for every $t \in \mathbb{R}$,
  (2) $F_{X(t_1),X(t_2)}(x_1, x_2)$ for every pair $(t_1, t_2) \in \mathbb{R}^2$,
  (3) $F_{X(t_1),X(t_2),X(t_3)}(x_1, x_2, x_3)$ for every triplet $(t_1, t_2, t_3) \in \mathbb{R}^3$,
  $(\vdots)$
  and they are all consistent after any marginalization.
  · Proof. Omitted.
  · Note that this method works even though there are *uncountably infinite* number of jointly distributed random variables but only the joint CDFs of *any finite* number of random variables are provided.
  · Similarly, a DT RP can be fully characterized.

Distribution and Density Functions

- CT case

1. Given a particular time t_1, the distribution function associated with the random variable $X_1 = X(t_1)$ is defined as

$$F_X(x_1; t_1) \triangleq P(X(t_1) \le x_1) = F_{X(t_1)}(x_1)$$

for any real number x_1.

(i) $X(s) = 1, \forall s$
$X = 1$
(ii) $\{X = 1\}$
$\{s \in S : X(s) = 1\}$

2. For two RVs $X_1 = X(t_1)$ and $X_2 = X(t_2)$, the second-order joint distribution function is

$$F_X(x_1, x_2; t_1, t_2) \triangleq P(X(t_1) \le x_1, X(t_2) \le x_2) = F_{X(t_1),X(t_2)}(x_1, x_2)$$

3. For N RVs $X_i = X(t_i), i = 1, 2, \ldots, N$, the Nth-order joint distribution function is

$$F_X(x_1, ..., x_N; t_1, ..., t_N) \triangleq P(X(t_1) \le x_1, ..., X(t_N) \le x_N) = F_{X(t_1),...,X(t_N)}(x_1, ..., x_N) \quad (6.2\text{-}1)$$

4. joint density functions

$$f_X(x_1; t_1) = dF_X(x_1; t_1)/dx_1 \quad (6.2\text{-}2)$$
$$f_X(x_1, x_2; t_1, t_2) = \partial^2 F_X(x_1, x_2; t_1, t_2)/(\partial x_1 \partial x_2) \quad (6.2\text{-}3)$$
$$f_X(x_1, ..., x_N; t_1, ..., t_N) = \partial^N F_X(x_1, ..., x_N; t_1, ..., t_N)/(\partial x_1 \cdots \partial x_N) \quad (6.2\text{-}4)$$

mean of the RP X(t) = mean function/path of the RP X(t)

$$\mu_{X(t)} = \boxed{\int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx}$$

cf) $x_1 \quad x_2 \; x_3 \quad x_M$
$m_1 \quad m_2 \; \cdots \; m_M$

$$\bar{x} = \sum_{i=1}^{M} x_i \frac{m_i}{\sum_{j=1}^{M} m_j}$$

Recall. Given a RV X,
$E\{X\} =$
$E(X)$
$E[X] = \sum_{i=1}^{M} p_i x_i$
$= \sum_{i=1}^{M} x_i p_i$

- Similar for DT cases. Repeat above 1-4 yourself.
- Def. Given a RP $X(t)$, the deterministic signal

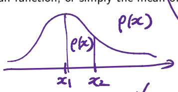
$t \in \mathbb{R}$

$$E[X(t)] \qquad \mathbf{E}[X(t)] =$$

as a function of t is called the mean path, the mean function, or simply the mean of $X(t)$.

– can also be defined for DT RPs.

cf. a sample path, a sample function



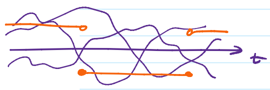
$$\bar{x} \simeq \sum_{i=1}^{M'} x_i' \frac{m_i'}{\int_{-\infty}^{\infty} \rho(x)\,dx} = \frac{\int_{-\infty}^{\infty} x \rho(x)\, dx}{\int_{-\infty}^{\infty} \rho(x)\, dx}$$

p(x)

Examples

- Fully characterize the following RPs.

1. DT discrete: a Bernoulli random process with p
   – $X[n]$ is a Bernoulli RV with p.
   – $X[n]$'s are i.i.d.
   – Q. Let $N = X[1] + X[2] + \cdots X[n]$ and $p = \lambda/n$. Find $P(N = 0)$, $P(N = 1)$, and $P(N \ge 2)$. Also find the limits of $P(N = 0)$, $P(N = 1)$, and $P(N \ge 2)$ as n tends to ∞.



2. DT continuous: a DT white Gaussian RP with power σ^2
   – $X[n] \sim \mathcal{N}(0, \sigma^2)$.