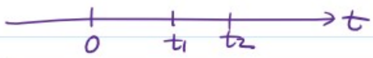
2023년 3월 7일 화요일 12:27

CM05-230223Tue

Announcements

1. Handouts and annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm . Passwords are ...
2. Lecture videos are available at the cisl.postech YouTube channel. Please add clickable time stamps.
3. HW#02 and #03 were due at 10:59am Today. HW#04 will be posted soon.

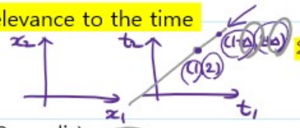
$X(t)$



Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

1. stationarity = invariance under time translation = irrelevance to the time origin/reference
2. 2-D CDF vs. 2-D time plane

$f_{X(t_1),X(t_2)}(x_1,x_2)$



$t_2 - t_1 = 1$

stationarity $\Rightarrow$ translation-invariant probabilistic properties

Overview (Chapter numbers from the main textbook by Papoulis)

9. General Concepts
7. Sequence of Random Variables
10. Random Walks and Other Applications
11. Spectral Representation
12. Spectrum Estimation
13. Mean-Square Estimation

$R_{XX}(t_1,t_2)$ $\quad R_{XX}: \mathbb{R}^2 \to \mathbb{R}$

$= R_{XX}(t_1+\Delta, t_2+\Delta)$ $\quad \forall\Delta, \forall(t_1,t_2)$

$= R_{XX}(t_2-t_1)$ $\quad R_{XX}: \mathbb{R} \to \mathbb{R}$

If at least stationary to order 1, then $\mu_{X(t)}$, $E[X^2(t)]$ are all const. fts.

S2 $\Rightarrow$ S1

Review
$$f_{X(t_2)}(x_2) = \int_{-\infty}^{\infty} f_{X(t_1),X(t_2)}(x_1,x_2)\,dx_2 = \int_{-\infty}^{\infty} f_{X(t_1+\Delta),X(t_2+\Delta)}(x_1,x_2)\,dx_2 = f_{X(t_2+\Delta)}(x_2)$$

1. stationarity
   a. the stationarity to order 1, 2, 3, etc.
   b. the strict-sense stationarity
   c. the invariance of the n-th order CDFs/PDFs/PMFs/CFs under time translation: time difference when n=2
   d. 1st includes 2nd includes ... includes SSS
2. the mean function/path, the (auto)correlation function, the (auto)covariance function
3. a wide-sense stationarity
   a. The mean function and the correlation function, consequently, the covariance function are translation invariant.
   b. The WS stationarity includes the stationarity to order 2.
   c. cf) a wide-sense covariance stationary RP $= C_{XX}(\tau)$
4.

cf) $\mu_X(t) \triangleq E[X_t]$

$R_X(s,t) \triangleq E[X(s)X(t)]$

$C_X(s,t) \triangleq Cov(X_s, X_t)$

$f_{X,n}(x_1,t_1;\ \dots\ ; x_n,t_n)$
$\triangleq P(X_{t_1}\le x_1, X_{t_2}\le x_2, \dots, X_{t_n}\le x_n)$

Def. A RP X_t is called second-order if $E\{X_t^2\}<\infty, \forall t$

real-valued RP

Def $\boxed{C_{XX}(t,t+\tau)} \triangleq E[(X(t)-\mu_X(t))(X(t+\tau)-\mu_X(t+\tau))]$

centered

Lemma $C_{XX}(t,t+\tau) = R_{XX}(t,t+\tau) - \mu_{X(t)}\mu_{X(t+\tau)}$

centered.

Def

Counter example Consider WSS RP $X(t)$ $\mu_X(t)=0, \forall t$

$Y(t) = X(t) + f(t)$ ft of t

$Y(t) - \mu_Y(t) = Y(t) - f(t) = X(t)$

Preview

- One, two, many, ...
- Multiple random processes are independent.
- Multiple random processes are jointly stationary.
- Multiple random processes are WSS.
- vector-valued RP

6.5 Gaussian RPs

6.6 Poisson (counting) RPs

- As a CT discrete RP: a counting process; a Levy process,
- As a DT RP: an arrival process, an interarrival process, a renewal process
- nonstationary but ergodic process

time average of a random process

stationarity and ergodicity of Gaussian and Poisson RPs

ergodicity in XX sense

a. the time average of a random process
   i. the time average of a determinstic power signal
   ii. vs. an ensemble average of a RP
b. an ergodic vs. a non-ergodic RP

6.7 Complex RPs

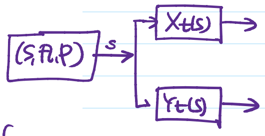
- complex envelope process
- proper-complex vs. improper-complex RPs

∘ complementary correlation function

Statistical Independence and Joint Stationarity

- Two processes $X(t)$ and $Y(t)$ are statistically independent if the random variable group $X(t_1), X(t_2), ..., X(t_N)$ is independent of the group $Y(t'_1), Y(t'_2), ..., Y(t'_M)$ for any M, N and any choice of times $t_1, ..., t_N, t'_1, ..., t'_M$.
- More formally, two processes $X(t)$ and $Y(t)$ are statistically independent if for any M, N and any $t_1, ..., t_N, t'_1, ..., t'_M$,

$$\begin{aligned} f_{X,Y}(x_1, ..., x_N, y_1, ..., y_M; t_1, ..., t_N, t'_1, ..., t'_M) \\ = f_X(x_1, ..., x_N; t_1, ..., t_N) f_Y(y_1, ..., y_M; t'_1, ..., t'_M) \end{aligned} \quad (6.2\text{-}6)$$



- Def. Two random processes $X(t)$ and $Y(t)$ are called jointly stationary to order .... if ...
- Def. Two random processes $X(t)$ and $Y(t)$ are called jointly wide-sense stationary if
  - each process is WSS; and
  - their cross-correlation function, defined by

$$R_{XY}(t_1, t_2) = \mathbf{E}[X(t_1)Y(t_2)]$$

is a function only of time differences and not absolute time; that is, if

$$R_{XY}(t, t+\tau) = \mathbf{E}[X(t)Y(t+\tau)] = R_{XY}(\tau)$$

def. The cross-correlation ft of $X(t)$ & $Y(t)$

e.g. $R_{XY}(t_1, t_2) \triangleq E[X(t_1)Y(t_2)]$

$R_{XY}(t_1,t_2) = R_{XY}(t_1+t_0, t_2+t_0)$

$= R_{XY}(t_2 - t_1)$

□ A random vector vs A vector-valued RP

$$\underline{X} \triangleq \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

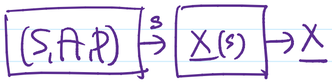
$f_{\underline{X}}(\underline{x})$

$F_{\underline{X}}(\underline{x})$

$P_{\underline{X}}(\underline{x})$

$\Phi_{\underline{X}}(\underline{\omega})$

$\vdots$



real RVs

$$\underline{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_{200}(t) \end{bmatrix}$$

$E[\underline{X}(t)] = \mu_{\underline{X}}(t)$ ← vector-valued mean path

$E[\underline{X}(t_1)\underline{X}(t_2)^T] = R_{\underline{X}(t_1)\underline{X}(t_2)}$ ← auto-correlation ft

$= R_{\underline{X}\underline{X}}(t_1, t_2)$

cf) $E[\underline{X}(t_1)^T \underline{X}(t_2)] = tr\{R_{\underline{X}\underline{X}}(t_1,t_2)\}$

$tr(AB) = tr(BA)$

Def $\underline{X}(t)$ is WSS

if (i) $\mu_{\underline{X}}(t) = \mu_{\underline{X}}, \forall t$

(ii) $\exists\ R_{\underline{X}\underline{X}}(\tau)$ such that

$R_{\underline{X}\underline{X}}(t_1,t_2) = R_{\underline{X}\underline{X}}(\tau)$ ← $\tau = t_2 - t_1$

/ex/ $\underline{Z}(t) = \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}$

$\underline{Z}(t)$ is WSS

$\Rightarrow$ (i) $\mu_{\underline{Z}}(t) = \begin{bmatrix} \mu_X(t) \\ \mu_Y(t) \end{bmatrix}$ → const

$$\Rightarrow \text{(i) } \mu_{\underline{Z}}(t) = \begin{bmatrix} \mu_X(t) \\ \mu_Y(t) \end{bmatrix} \rightarrow \text{const}$$

$$\text{(ii) } R_{\underline{Z}\underline{Z}}(t_1, t_2) = \begin{bmatrix} R_{XX}(t_1,t_2) & R_{XY}(t_1,t_2) \\ R_{YX}(t_1,t_2) & R_{YY}(t_1,t_2) \end{bmatrix}$$

$R_{XY}(\tau)$

$R_{YX}(\tau)$

$$R_{XY}(\tau) = R_{YX}(-\tau).$$

6.5 GAUSSIAN RANDOM PROCESSES

- Def. A continuous random process $X(t)$ is Gaussian, if the RVs $X_1 = X(t_1), ..., X_N = X(t_N)$ for any N and for any $\underline{t} \triangleq [t_1, ..., t_N]^T$, are jointly Gaussian,
  that is, the joint CF $\underline{X}(\underline{t}) \triangleq [X(t_1), ..., X(t_N)]^T$ is given by
  $$\Phi_{\underline{X}(\underline{t})}(\underline{\omega}) = \exp\left(j\underline{\omega}^T \mu_{\underline{X}(\underline{t})} - \frac{\underline{\omega}^T C_{\underline{X}(\underline{t})\underline{X}(\underline{t})} \underline{\omega}}{2} \right)$$
  where
  $$\mu_{\underline{X}(\underline{t})} \triangleq \mathbf{E}[\underline{X}(\underline{t})]$$
  and
  $$[C_{\underline{X}(\underline{t})\underline{X}(\underline{t})}]_{i,j} \triangleq C_{XX}(t_i, t_j) = Cov\{X(t_i), X(t_j)\}.$$
- Lemma. If $C_{\underline{X}(\underline{t})\underline{X}(\underline{t})}$ is invertible, then $X(t)$ is a Gaussian RP iff the joint PDF is given by
  $$f_{\underline{X}(\underline{t})}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^N \det(C_{\underline{X}(\underline{t})\underline{X}(\underline{t})})}} \exp\left(-\frac{1}{2}(\underline{x} - \mu_{\underline{X}(\underline{t})})^T C^{-1}_{\underline{X}(\underline{t})\underline{X}(\underline{t})} (\underline{x} - \mu_{\underline{X}(\underline{t})}) \right)$$
  for any N and for any $\underline{t} \triangleq [t_1, ..., t_N]^T$.

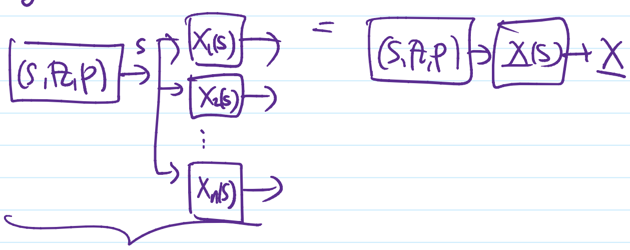
independent and identically distributed

/ex/ $X_t = A + Bt + t^2$ where A & B are i.i.d. $N(0,1)$ RVs.

(i) $\mu_X(t)$
(ii) $R_X(s,t)$
(iii) $C_X(s,t)$
(iv) $f_{X,1}(x,t) =$
⋮

Def. A RVector $\underline{X}$ of length n is a collection of jointly distributed RVs indexed by $1, 2, \cdots, n$.

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix},$$



Def. A RVector $\underline{X}$ consisting of n real RVs is called Gaussian if

$$\Phi_{\underline{X}}(\underline{\omega}) \triangleq E\left[e^{j\underline{\omega}^T \underline{X}} \right] =$$
$$\exp\left(+j \underline{\mu}_{\underline{X}}^T \underline{\omega} - \frac{1}{2} \underline{\omega}^T C_{\underline{X}\underline{X}} \underline{\omega} \right)$$

for some $\underline{\mu}_{\underline{X}} \in \mathbb{R}^n$ &

for some $\mu_{\underline{X}} \in \mathbb{R}^n$ &

$C_{\underline{X}\underline{X}} \in \mathbb{R}^{n \times n}$ &

$\underbrace{C_{\underline{X}\underline{X}} = C_{\underline{X}\underline{X}}^T}\ \underbrace{\succeq 0}_{\text{positive semi-definite}}$.

If $C_{\underline{X}\underline{X}} \succ 0$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^n \det(C_{\underline{X}\underline{X}})}} \cdot \exp\left(-\frac{1}{2}(\underline{x}-\mu_{\underline{X}})^T C_{\underline{X}\underline{X}}^{-1}(\underline{x}-\mu_{\underline{X}})\right).$$