# CM06-230309Thu

2023년 3월 9일 목요일 12:50

in class quiz
from 11am till 12:15pm
open everything except being helped by any human being
in class quiz

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open everything except being helped by any human being

covers upto CM#6

CM06-230309Thu

Announcements

1. Quiz #1 will be taken on Tue. Mar. 14th. from 11am till 12:15pm.
2. HW#4 is assigned and HW#5 will be posted soon. HWs #4 and 5 are due at 10:59am, Tue. Mar. 14th.
3. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm . Passwords are ...
4. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

1.

$f_{X(t_1)}(x_1) = \int_{-\infty}^{\infty} f_{X(t_1),X(t_2)}(x_1,x_2)\,dx_2 = \int_{-\infty}^{\infty} f_{X(t_1+\Delta),X(t_2+\Delta)}(x_1,x_2)\,dx_2 = f_{X(t_1+\Delta)}(x_1)$

Overview (Chapter numbers from the main textbook by Papoulis)

9. General Concepts
7. Sequence of Random Variables
10. Random Walks and Other Applications
11. Spectral Representation
12. Spectrum Estimation
13. Mean-Square Estimation

Review

1. How to characterize a RP?
2. How to compute an ensemble average?
3. a single RP and its stationarity: strict-sense, to order n, wide-sense
4. multiple RPs
   a. joint stationarity:
   b. independent RPs
   c. uncorrelated RPs
   d. orthogonal RPs
5. Gaussian RVector

$R_{XX}(t_1,t_2) \triangleq E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1,x_2)\,dx_1\,dx_2$

2 RPs → multiple RPs

$R_{X,X}(t_1,t_2) = R_{X(t_1),X(t_2)}$

$E[X(t_1)Y(t_2)] = E[X(t_1)]E[Y(t_2)] \quad \forall (t_1,t_2)$

$\rho_{XX}(t_1,t_2) \triangleq \frac{C_{XX}(t_1,t_2)}{\sqrt{C_{XX}(t_1,t_2)}\sqrt{C_{XX}(t_2,t_2)}} = E\left\{\left(\frac{X(t_1)-\mu_X(t_1)}{\sigma_X(t_1)}\right)\left(\frac{X(t_2)-\mu_X(t_2)}{\sigma_X(t_2)}\right)\right\}, \quad |\rho_{XX}(t_1,t_2)| \le 1$

$E[X(t_1)Y(t_2)] = 0, \quad \forall (t_1,t_2)$

$\sigma_{XX}(t_1) \triangleq \sqrt{Var(X(t_1))} = \sqrt{C_{XX}(t_1,t_1)}$

indep $\Rightarrow E[XY] = E[X]E[Y]$ (converse is not true.)

$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy$

$f_{X,Y}(x,y) = f_X(x)f_Y(y)$

$= \int_{-\infty}^{\infty} x f_X(x)\,dx \times \int_{-\infty}^{\infty} y f_Y(y)\,dy = E[X]E[Y]$

Preview

1. Gaussian RPs
   a. Property of Gaussian RVector
   b. a Gaussian RP
      i. WSS Gaussian RP <=> SSS Gaussian RP
   c. multiple jointly Gaussian RPs
2. Poisson RPs
   a. a counting process
      i. a DT counting process
      ii. a CT couting process
   b. an arrival process associated with a counting process satisfying ...

Two jointly distributed RPs are uncorrelated if...
Two jointly distributed RPs are orthogonal if...

$R_{XY}(t_1,t_2) = \mu_X(t_1)\mu_Y(t_2)$
$= E[X(t_1)Y(t_2)] = E[X(t_1)]E[Y(t_2)] \quad \forall (t_1,t_2)$
$\Leftrightarrow C_{XY}(t_1,t_2) = 0, \quad \forall (t_1,t_2)$

$R_{XY}(t_1,t_2) = 0, \quad \forall (t_1,t_2)$

cf) uncorrelated at specific $(t_1,t_2)$
orthogonal " " $(t_1,t_2)$

## 6.5 GAUSSIAN RANDOM PROCESSES

- Def. A continuous random process $X(t)$ is Gaussian, if the RVs $X_1 = X(t_1), ..., X_N = X(t_N)$ for any $N$ and for any $\underline{t} \triangleq t_1, ..., t_N]^T$, are jointly Gaussian,
that is, the joint CF $X(t) \triangleq [X(t_1), ..., X(t_N)]'$ is given by

$$\Phi_{\underline{X}(\underline{t})}(\underline{\omega}) = \exp\left(j\underline{\omega}^T \underline{\mu}_{\underline{X}(\underline{t})} - \frac{\underline{\omega}^T C_{\underline{X}(\underline{t})\underline{X}(\underline{t})} \underline{\omega}}{2}\right)$$

where

$$\underline{\mu}_{\underline{X}(\underline{t})} \triangleq \mathbf{E}[X(t)]$$

and

$$[C_{\underline{X}(\underline{t})\underline{X}(\underline{t})}]_{i,j} \triangleq C_{XX}(t_i, t_j) = Cov\{X(t_i), X(t_j)\}$$

- Lemma. If $C_{\underline{X}(\underline{t})\underline{X}(\underline{t})}$ is invertible, then $X(t)$ is a Gaussian RP iff the joint PDF is given by

$$f_{\underline{X}(\underline{t})}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^N \det(C_{\underline{X}(\underline{t})\underline{X}(\underline{t})})}} \exp\left(-\frac{1}{2}(\underline{x} - \underline{\mu}_{\underline{X}(\underline{t})})^T C^{-1}_{\underline{X}(\underline{t})\underline{X}(\underline{t})} (\underline{x} - \underline{\mu}_{\underline{X}(\underline{t})})\right)$$

for any $N$ and for any $\underline{t} \triangleq [t_1, ..., t_N]^T$.

SS Stationary Gaussian

$\underline{X}(\underline{t}) = \begin{bmatrix} X(t_1) \\ \vdots \\ X(t_N) \end{bmatrix}$

$[C_{\underline{X}(\underline{t})\underline{X}(\underline{t})}]_{ij}$

ex/ $X_t = A + Bt + t^2$ where A & B are i.i.d. $N(0,1)$ RVs.

Q1. Is this RP stationary?

(i) $\mu_{X(t)}$
(ii) $R_X(s,t)$
(iii) $C_X(s,t)$
(iv) $f_{X,t}(x,t) =$

(i) $E[X(t)] = \mu_X, \forall t$
(ii) $Cov[X(t_i), X(t_j)]$
$= C_{XX}(t_i, t_j)$
$= C_{XX}(t_i + \Delta, t_j + \Delta)$ $\forall \Delta$
(i) & (ii) $\Leftrightarrow$ (i) & $R_{XX}(t_i, t_j) = R_X(t_j - t_i), \forall i, j$



- For a WSS RP

$$\mu_{X(t)} = \mu_X, \forall t$$
$$C_{XX}(t_i, t_j) = C_{XX}(t_1 - t_j),$$
$$R_{XX}(t_i, t_k) = R_{XX}(t_i - t_k).$$

- A WSS Gaussian process is also strictly stationary.

**Example**: $X(t)$ is a WSS Gaussian RP with

$$R_{XX}(\tau) = 25e^{-3|\tau|} - 16.$$

Specify the joint PDF for three RVs $X(t_i)$, $i = 1, 2, 3$, defined at times $t_i = t_0 - (i-1)/2$.

(solution)

(1) Mean: $\bar{X} = \pm 4$, since

$$\lim_{\tau \to \infty} R_{XX}(\tau) = \bar{X}^2$$

(2) Covariance:

$$\begin{aligned} C_{XX}(t_k - t_j) &= R_{XX}(t_k - t_j) - \bar{X}^2 \\ &= 25e^{-3|k-j|/2} + 16 - 16 \end{aligned}$$



(3) Covariance matrix:

$$C_{\underline{X}(\underline{t})\underline{X}(\underline{t})} = 25 \begin{bmatrix} 1 & e^{-3/2} & e^{-6/2} \\ e^{-3/2} & 1 & e^{-3/2} \\ e^{-6/2} & e^{-3/2} & 1 \end{bmatrix}.$$

These values completely determine the joint PDF.

- Examples
  - CT continuous: a Gaussian random process with mean path $\mu_X(t)$ and auto-covariance function $C_{XX}(t_1, t_2)$
  - CT continuous: a Wiener or a Brownian motion process $W(t)$ with parameter $\alpha$
    - independent increment
    - $t_1 < t_2 \Longrightarrow N(t_2) - N(t_1) \sim N(0, \alpha(t_2 - t_1))$, i.e., a Gaussian RV with mean 0 and variance $\alpha(t_2 - t_1)$
    - $W(0) = 0$
    - It has a continuous sample path with probability 1.

example of non-stationary Gaussian RP.

- Def. Two random processes $X(t)$ and $Y(t)$ are jointly Gaussian, if the RVs $X_1 = X(t_1), ..., X_N = X(t_N), Y_1 = Y(t'_1), ..., Y_M = Y(t'_M)$ defined at times $t_1, ..., t_N$ for $X(t)$ and times $t'_1, ..., t'_M$ for $Y(t)$, are jointly Gaussian for any $N$, $t_1, ..., t_N$, $M$, $t'_1, ..., t'_M$.

**Joint Characteristic Function of $N$ Jointly Gaussian RVs**

- The joint characteristic function of $N$ Gaussian RVs

$$\Phi_{\underline{X}}(\underline{\omega}) = \exp\left(+\mathrm{j}\underline{\mu}_{\underline{X}}^T\underline{\omega} - \frac{\underline{\omega}^T\mathbf{C}_{\underline{X}\,\underline{X}}\underline{\omega}}{2}\right).$$

- The first and second order moments determine the characteristic function.
- No matrix inverse is involved.
  cf) pdf → a degenerate Gaussian random vector
- Uncorrelatedness implies $\mathbf{C}_{\underline{X}\,\underline{X}} = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_N^2)$, so

$$\Phi_{\underline{X}}(\underline{\omega}) = \Pi_{i=1}^N \Phi_{X_i}(\omega_i),$$

  which means independence.
- $\underline{Y} = A\underline{X}$ implies that $\underline{Y}$ is Gaussian because

$$\Phi_{\underline{Y}}(\underline{\omega}) = \mathbf{E}[e^{\mathrm{j}\underline{\omega}^T(A\underline{X})}] = \mathbf{E}[e^{\mathrm{j}(A^T\underline{\omega})^T\underline{X}}] = \Phi_{\underline{X}}(A^T\underline{\omega}),$$

  that is,

$$\Phi_{\underline{Y}}(\underline{\omega}) = \exp\left(+\mathrm{j}(A\underline{\mu}_{\underline{X}})^T\underline{\omega} - \frac{\underline{\omega}^T(A\mathbf{C}_{\underline{X}\,\underline{X}}A^T)\underline{\omega}}{2}\right)$$

  which is the characteristic function of a Gaussian random vector with mean vector $A\underline{\mu}_{\underline{X}}$ and covariance matrix $A\mathbf{C}_{\underline{X}\,\underline{X}}A^T$.

$\Phi_{\underline{X}}(A^T\underline{\omega}) = \exp\left(\mathrm{j}(A^T\underline{\omega})^T\underline{\mu}_{\underline{X}} - \frac{1}{2}(A^T\underline{\omega})^T C_{XX}(A^T\underline{\omega})\right)$

$= \exp\left(\mathrm{j}(A\underline{\mu}_{\underline{X}})^T\underline{\omega} - \frac{1}{2}\underline{\omega}^T(AC_{XX}A^T)\underline{\omega}\right)$

$\underline{\omega}^T A\underline{X} = (\underline{\omega}^T A)\underline{X} = (A^T\underline{\omega})^T\underline{X}$

# Poisson RPs

(i) a counting process $N_t$ — DT: $N[t]$, CT: $N(t)$, $N[m]$

Def. A counting process

(i) a DT counting process $t_0 \in \mathbb{Z}_+$ (cf) $\mathbb{Z}_{++} = \mathbb{N}$

$N[t_0] = \begin{cases} 0 & \text{if } t_0 = 0 \\ k & \text{if there are } k \text{ occurrences (arrivals) in } 0 < t \le t_0 \end{cases}$

$S_1$
$S_2$
$S_3$
$S_m \le S_{m+1}$
$S_m < S_{m+1}$

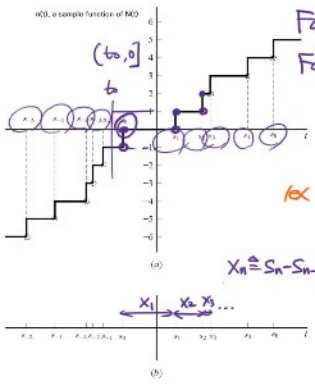
(ii) a CT counting process $t \in \mathbb{R}_+$ (cf) $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$

$N(t_0) = \begin{cases} 0 & \text{if } t_0 = 0 \\ k & \text{if there are } k \text{ arrivals (occurrences) in } 0 < t \le t_0 \end{cases}$

right-continuous

(iii) extensions to $t \in \mathbb{R}, \mathbb{Z}$






For $t_0 > 0$, $N(t_0) = m \Leftrightarrow$ $m$ arrivals for $0 < t \le t_0$

For $t_0 < 0$, $N(t_0) = m \Leftrightarrow$ $-m$ arrivals for $t_0 < t \le 0$

$t \in (\ ,\ ]$

Properties of a counting process

(i) $N_t$ is integer-valued for all $t$.

(ii) $N_t$ is non-decreasing. $t_1 < t_2 \Rightarrow N_{t_1} \le N_{t_2}$

Remark. There are two types of counting processes.

(i) $N[m] - N[m-1]$ or $N(t_0) - N(t_0^-)$ can be $> 1$.

(ii) $N[m] - N[m-1] < 1$, $N(t) - N(t_0^-) < 1$

Remark. There are two types of counting processes.

(i) $N[m]-N[m-1]$ or $N(t_0)-N(t_0^-)$ can be $>1$.

(ii) $N[m]-N[m-1] \le 1$, $N(t_0)-N(t_0^-) \le 1$.

$\Leftrightarrow$ Each arrival has a distinct arrival instant.

In many counting processes, this is the case.

# Def. For a DT/CT counting process, we can define an associated DT RP $S_n$ as

$S_n \triangleq$ the time instant when the $n$th arrival happens.

which is called an arrival process.

$S_n < S_{n+1}$ if no simultaneous arrival

Property of an arrival process

(i) $S_n \le S_{n+1}$, $\forall n$

(ii) For a DT arrival process, $S_n \in \mathbb{N}$ & $N[S_n] = n$

(iii) " " CT " " , $S_n \in \mathbb{R}_{++}$ & $N(S_n) = n$

associated counting process

Example. (i) Let $X_n$'s be i.i.d. $B(1,p)$ i.e., a Bernoulli process.

$Y_n \triangleq \sum_{i=1}^{n} X_i$ = # of success in $1 \le i \le n$

Then $0 \le Y_{n+1} - Y_n \le 1$ $\forall n \ge 1$

(ii) Let $\tilde{X}_n$'s be i.i.d. $M(1,p)$ i.e., a multinoulli process.

$\tilde{Y}_n \triangleq \sum_{i=1}^{n} X_i$ = # of occurrences in $1 \le i \le n$.

Then $\tilde{Y}_{n+1} - \tilde{Y}_n$ can be greater than 1.

(iii)

Def. Given an arrival process (which is DT RP), we can define an associated DTRP

$X_n \triangleq S_n - S_{n-1}$

called an inter-arrival process.

Properties. $X_n \ge 0$

$X_1 = S_1 - S_0$

$X_2 = S_2 - S_1$

$\vdots$

$+) X_n = S_n - S_{n-1}$

$\therefore S_n \triangleq X_1 + X_2 + \cdots + X_n + S_0$

If $n \in \mathbb{Z}_+$, $S_0 \triangleq 0$.

If $n \in \mathbb{Z}$, $S_0 \ge 0$.

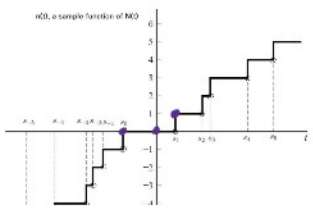
RANDOM PROCESSES - TEMPORAL CHARACTERISTICS 6 - 16

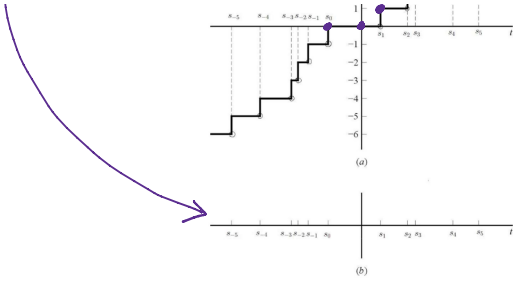
## 6.6 POISSON RANDOM PROCESS

- There are important classes of random processes:
  - Examples:
    - the arrival of a customer at a bank or supermarket check-out register
    - the occurrence of a lightening strike within some prescribed area
    - the failure of some component in a system
    - the emission of an electron from the surface of a light-sensitive material (photodetector)
  - There are 3 ways to describe such random experiments.
    1. Use a DT RP $(S_n)_{n \in \mathbb{N}}$, where $S_n$ [sec] is called the $n$th arrival/departure epoch/time.
    2. Use a DT RP $(X_n)_{n \in \mathbb{N}}$, where $X_1 = S_1$ and $X_n \triangleq S_n - S_{n-1}$ [sec] for $n > 1$ is called the $n$th inter-arrival time.
    3. Use a CT RP $(N(t))_{t \ge 0}$, where $N(t)$ means the total number of arrivals in $I < (0, t]$ and is called the counting RV.
  - related with a point process
    1. an arrival process $(S_n)_{n \in \mathbb{N}}$
       - An arrival process is a sequence of positive increasing RVs.

(non-negative if simultaneous arrivals allowed.)

RANDOM PROCESSES - TEMPORAL CHARACTERISTICS 6 - 17








* $0 < S_1 < S_2 < ...$, where $S_i < S_{i+1}$ means $\Pr(S_i < S_{i+1}) = 1$.
* No more than one arrival occurs at the same time.
* (But it is possible to generalize to a non-decreasing sequence of RVs.)

2. an inter-arrival process $(X_n)_{n\in\mathbb{N}}$
   - An inter-arrival process is a sequence of positive RVs.
     * $0 < X_n$
     * $S_n = \sum_{i=1}^{n} X_i$ becomes an arrival process.
     * If $X_i$'s are i.i.d., then further called a renewal process.
3. a counting process $(N(t))_{t\geq 0}$
   - Counting processes is an important class of continuous-time discrete RPs.
   - Mostly for $t \geq 0$ and $N(t) \in \mathbb{Z}_+$.
   - Usually, we set $N(0) = 0$.
   - Hence,

$$N(t) = \sum_{i=1}^{\infty} u(t - S_i)$$

   and

$$N(t) = n \text{ for } S_n \leq t < S_{n+1}$$

     * For $t > 0$, $S_{N(t)} \leq t < S_{N(t)+1}$.
     * For $n \in \mathbb{Z}_+$ and $t > 0$, two events $\{S_n \leq t\}$ and $\{N(t) \geq n\}$ are identical. (Be careful in the units.)