2023년 3월 16일 목요일 12:28

CM07-230316Thu

Announcements

1. Quiz #1 was taken on Tue. Mar. 14th.
2. Today's CM is numbered CM#7.
3. HW#6 will be posted soon and due at 10:59am, Tue. Mar. 21th.
4. Handout for today's CM is updated.
5. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm . Passwords are …
6. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.
7. There was no voice after 44:00 on the counting process in video for CM#6.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

1. Q. Why stationarity so important?
   A. Stationarity allows us to do something useful such as statistical inference. Otherwise, no inference would be possible.

Overview (Chapter numbers from the main textbook by Papoulis)

9. General Concepts
7. Sequence of Random Variables
10. Random Walks and Other Applications
11. Spectral Representation
12. Spectrum Estimation
13. Mean-Square Estimation

Review

1. Strict-sense stationarity
   A. Def 1.
   B. Def 2.
   C. Examples
2. Gaussian RP

Def.
$X(t)$ and $X(t-\Delta)$ have the same statistical property $\forall \Delta$

Example $X(t) = a\cos(2\pi f_0 t + \Theta)$
Θ uniform $[0, 2\pi)$.
$Y(t) \triangleq X(t-\Delta) = a\cos(2\pi f_0(t-\Delta) + \Theta)$
$= a\cos(2\pi f_0 t + \underbrace{\Theta - 2\pi f_0\Delta}_{\Phi})$

Preview

1. counting process
2. arrival process
3. interarrival process
4. Poisson process
   A. a counting process
   B. having the interarrival process a renewal process, especially memoryless

C. increments are independent and stationary, i.e., a Levy process

D. WSS?

i. mean function, autocorrelation function

6.6 POISSON RANDOM PROCESS

- Consider the following examples:
  - the arrival of a customer at a bank or supermarket check-out register,
  - the occurrence of a lightening strike within some prescribed area,
  - the failure of some component in a system, or
  - the emission of an electron from the surface of a light-sensitive material (photodetector).
- Given each case, we may define 3 random processes:
  1. a counting process (DT / CT)
  2. an arrival process = a point process = a count time process
  3. an inter-arrival process = an intercount time process
- In each case, find the counting, the arrival, and the inter-arrival processes, focusing on the units:
  1. Use a CT random process $(N(t))_{t\geq 0}$, where $N(t)$ means the total number of arrivals in $(0, t]$ and is called the counting RV.
  2. Use a DT random process $(S_n)_{n\in\mathbb{N}}$, where S_n [sec] is called the nth arrival/departure epoch/time.

$(Y_n)_{n\in\mathbb{N}}$

$Y_n \sim B(1,p)$ i.i.d.

$\{Y_n = y\}, \quad y \in \{0,1\}$

$N_n = \begin{cases} \sum_{i=1}^{n} Y_i, & n \geq 1 \\ 0, & n = 0 \end{cases}$

0 1 n

1 ← DT counting process.

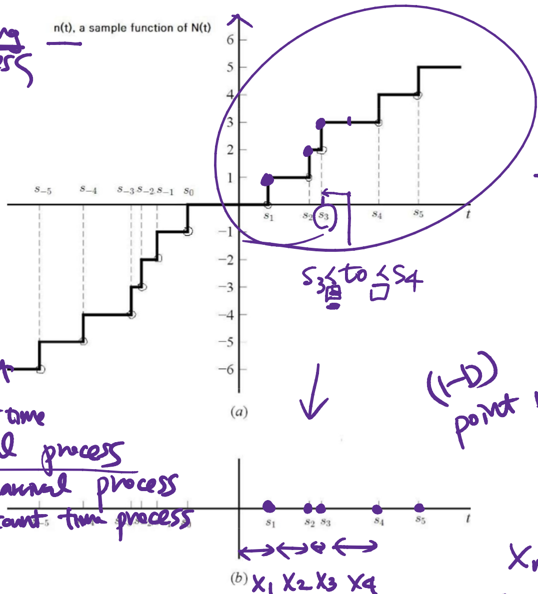
3. Use a DT random process $(X_n)_{n\in\mathbb{N}}$, where $X_1 = S_1$ and $X_n \triangleq S_n - S_{n-1}$ [sec] for $n > 1$ is called the nth inter-arrival time.

(i) a counting process

$N(t) = \sum_{i=1}^{\infty} u(t - S_i)$

$u(t)$

1 0 t



$S_3 \leq t < S_4$

S_i t

(ii) an arrival process (point / count time)

(iii) an interarrival process (intercount time process)

$S_n = \sum_{i=1}^{n} X_i$

$(X_n)_{n\geq 1}$

$X_1 \; X_2 \; X_3 \; X_4$

(1-D) point process

2-D Poisson point process

y x

$X_n = \begin{cases} S_n - S_{n-1} & n \geq 2 \\ S_1 & n = 1 \end{cases}$

interarrival process

- Def. A DT or CT random process N_t with $t \geq 0$ is called a counting process if

- Def. A DT or CT random process N_t with $t \geq 0$ is called a counting process if
  - $N_0 \triangleq 0$,
  - there is no occurrence/arrival at $t = 0$, and
  - N_{t_0} is the total number of occurrences/arrivals during $0 < t \leq t_0$.
  - Def. A counting process can be defined on $-\infty < t < \infty$ as
    1. for $t_0 = 0$, $N_0 = 0$,
    2. for $t_0 > 0$, $N_{t_0} = m$ if the total number of occurrences/arrivals during $0 < t \leq t_0$ is $m \in \mathbb{Z}_+$, and
    3. for $t_0 < 0$, $N_{t_0} = -m$ if the total number of occurrences/arrivals during $t_0 < t \leq 0$ is $m \in \mathbb{Z}_+$.

$[t_0, 0]$

$\to 0, -1, -2, \ldots$

  - Properties of a DT or CT counting process N_t $\to 0, 1, 2, 3 \ldots$
    * N_t is integer-valued for all t.
    * N_t is non-decreasing, i.e., $t_1 < t_2 \Longrightarrow N_{t_1} \leq N_{t_2}$.
  - There are two classes of counting processes.
    1. Simultaneous arrivals are allowed.
    2. No simultaneous arrivals are allowed.
    * In what follows, we only focus on a counting process with no simultaneous arrival, though the other class is also very important.
      · N_t increases by 1.

      · A CT counting process $N(t)$ has nondecreasing right-continuous integer-valued (hence, piecewise-constant) sample functions.

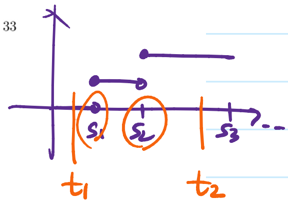
- Def. Given either a DT or a CT counting process N_t defined on $t \geq 0$, we can define a DT random process called an arrival process, a point process, or a count time process S_n, where S_n is the time instant at which the nth arrival occurs for $n \in \mathbb{N}$.
  - Properties of an arrival process S_n with no simultaneous arrival
    * The arrival process is positive and strictly increasing, i.e., $0 < S_1 < S_2 < \ldots$, where $S_i < S_{i+1}$ means $\Pr(S_i < S_{i+1}) = 1$.
    * The counting and the arrival processes are related as
      · $N_{S_n} = n$.
      · $N_t = n$ for $t \in [S_n, S_{n+1})$.
      · $S_{N_t} \leq t < S_{N_t+1}$.
    * For a CT counting process $N(t)$ defined on $t \geq 0$ and $n \in \mathbb{N}$,
      · $N(t) = \sum_{n=1}^{\infty} u(t - S_n)$, where $u(t)$ is the right-continuous unit-step function.
      · Two events $\{S_n \leq t_0\}$ and $\{N(t_0) \geq n\}$ are identical. (Be careful in the units.)
      · Two events $\{S_n > t_0\}$ and $\{N(t_0) < n\}$ are identical.

- Def. Given an arrival process, we can define a DT random process called an inter-arrival process or an intercount time process X_n, where $X_1 \triangleq S_1$ and $X_n \triangleq S_n - S_{n-1}$ for $n \geq 2$.
  - Properties of an arrival process S_n with no simultaneous arrival
    - The inter-arrival process is a sequence of positive RVs, i.e., $0 < X_n$ with probability 1.
    - The arrival and the inter-arrival processes are related as $S_n = \sum_{i=1}^{n} X_i = X_1 + X_2 + \ldots + X_n$ for $n \geq 1$.
  - Def. An inter-arrival process X_n is called a renewal process's if X_i's are i.i.d.

If $(t_1, t_2]$

$N(t_3, t_4] = \phi$

- Def. Given a CT random process $N(t)$, $N(t_2) - N(t_1)$ is called the increment during $t \in (t_1, t_2]$.
  - Def. If the increments as random variables are independent for any non-overlapping time intervals, then the process is called to have independent increments.
  - If the increment during $(t_1, t_2]$ for $t_2 > t_1$ depends only on the length $t_2 - t_1$ of the interval, then the process is called to have stationary increments.
  - $N(t)$ need not be a counting process, though it makes a better sense for counting processes.



Q. $(t_1, t_2]$

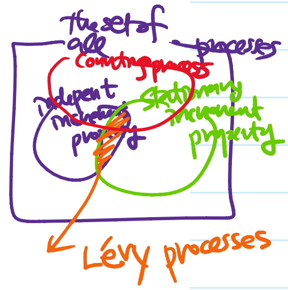
A. $N(t_2) - N(t_1) = 2$.

Q $t_1 < t_2 < t_3$

(i) $(t_1, t_2]$ $(t_2, t_3]$ (M)

(ii) $(t_1, t_3]$ $(t_2, t_3]$ (A)

- Def. If a CT random process $(N(t))_{t \geq 0}$ has independent and stationary increments ($N(0) = 0$ with probability 1 and $N(t)$ is a continuous function of time in probability), then it is called a Lévy process.
  - The sum process of a Bernoulli process is a DT counting process and a Lévy process. $Y_i \sim B(1,p)$ $N_n = Y_1 + Y_2 + \cdots + Y_n$
  - A Poisson process is a CT counting process and a Lévy process.
  - There are Lévy processes that are not counting processes. For example, the Wiener process is a Gaussian process and a Lévy process.



- Def1. A CT random process $N(t)$ is a Poisson process if
  1. $N(0) = 0$.
  2. independent increments
  3. stationary increments: The increment during $(t_1, t_2]$ for $t_2 > t_1$ follows the Poisson distribution with parameter $\lambda(t_2 - t_1)$, where λ with unit [Hz] is called the arrival rate.

  – a.k.a. (also known as) a Poisson counting process

  – A Poisson process is an idealized counting process.

  – Q. Is a Poisson process fully characterized by the value of λ and the above properties?

  – A. Yes. With λ and these 3 properties, we can derive any nth-order PMF consistently.

    * The probability of exactly k_1 occurrences over a time interval $(0, t_1]$ is

$$\Pr(N(t_1) = k_1) = P(N_{t_1}(s) = k_1) = \frac{(\lambda t_1)^{k_1} e^{-\lambda t_1}}{k_1!} \qquad k_1 = 0, 1, 2, \cdots .$$

    * The conditional probability of k_2 occurrences over $(0, t_2]$ given that k_1 events occurred over $(0, t_1]$ is just the probability that $k_2 - k_1$ events occurred over $(t_1, t_2]$, which is

$$\Pr(N(t_2) = k_2 | N(t_1) = k_1) = \frac{[\lambda(t_2 - t_1)]^{k_2 - k_1} e^{-\lambda(t_2 - t_1)}}{(k_2 - k_1)!}$$

    * The joint probability of k_1 occurrences at time t_1 and k_2 occurrences at time t_2 is

$$\begin{aligned}\Pr(N(t_1) = k_1, N(t_2) = k_2) &= \Pr(N(t_2) = k_2 | N(t_1) = k_1) \cdot \Pr(N(t_1) = k_1) \\ &= \frac{(\lambda t_1)^{k_1} [\lambda(t_2 - t_1)]^{k_2 - k_1} e^{-\lambda t_2}}{(k_1)!(k_2 - k_1)!} \qquad k_2 \geq k_1.\end{aligned}$$

    * Similarly, any nth-order PMF can be obtained.

- Def2. A CT counting process $N(t)$ is a Poisson process if

  – its inter-arrival process X_n is a renewal process having i.i.d. exponential inter-arrival times with parameter λ [Hz]. $f_{X_i}(x) = \lambda \exp(-\lambda x)$ for $x \geq 0$.

    * An exponential RV has memoryless property.
    * The event occurs only once in any vanishingly small interval of time.
      In essence, only one event can occur at a time.
    * Since the exponent is unitless, λ has unit [Hz].
    * $\mathbf{E}[N(t_2) - N(t_1)] = \lambda(t_2 - t_1)$, which explains why λ is called the rate of the Poisson process.

- Def3 ... Def4...