# CM08-230321Tue

2023년 3월 21일 화요일 09:55

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Announcements

1. Quiz #1 was taken and assigned as HW#6. Examine the solutions carefully to well prepare for the Exam #1.
2. HW#6 Problems 4- are assigned as HW#7 and due at 10:59am on this Thursday.
3. Handouts are updated. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm . Passwords are …
4. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

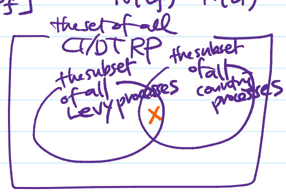
1.

Overview (Chapter numbers from the main textbook by Papoulis)

9. General Concepts
   1. RP concept
   2. Stationarity and Independence
   5. Gaussian RPs
   6. Poisson RPs
   7. Complex RPs
   2. Time average and Ergodicity
   3. Correlation functions
   4. Measurement of correlation functions
7. Sequence of Random Variables
10. Random Walks and Other Applications
11. Spectral Representation
12. Spectrum Estimation
13. Mean-Square Estimation

Review

1. CT/DT: counting process $N(t)$
2. DT: arrival process, count time process, point process $S_n$
3. DT: interarrival process, intercount time process $X_n$
   a. renewal process $X_n$'s are i.i.d.
4. Notion of increment $N(t)$ $(t_i, t_f]$ $N(t_f) - N(t_i)$
   a. RPs having independent increments
   b. RPs having stationary increments
   c. Levy processes
5. a Poisson process
   a. a CT Levy process
   b. a counting process
   c. a renewal memoryless process $X_n$'s are i.i.d. exponential.




$X$ is memoryless if $\Pr(X > x+t \mid X > t) = \Pr(X > x)$

Preview

1. Definitions of a Poisson process
2. the n-th order PMFs
3. mean and autocorrelation functions
4. Complex-valued random XXX
   a. RV
      i. circularity
      ii. variance vs. pseudo-variance
      iii. properiety vs. impropriety
   b. R vector
      i. covariance vs. complementary covariance matrices
   c. RP
      i. autocorrelation vs. complementary autocorrelation functions

RANDOM PROCESSES - TEMPORAL CHARACTERISTICS

$\alpha = \lambda(t_2 - t_1)$ Hz (sec)

- Def1. A CT random process $N(t)$ is a Poisson process if
  1. $N(0) = 0$.
  2. independent increments
  3. stationary increments: The increment during $(t_1, t_2]$ for $t_2 > t_1$ follows the Poisson distribution with parameter $\lambda(t_2 - t_1)$, where $\lambda$ with unit [Hz] is called the arrival rate.

  a Levy process

  $P(N(t_2) - N(t_1) = k) = \frac{\{\lambda(t_2-t_1)\}^k}{k!} e^{-\lambda(t_2-t_1)}$

  $k=0,1,\ldots$ $0! = 1$.

  $(S.T.R.P) \to X_t(s) \to$
  - a.k.a. (also known as) a Poisson counting process
  - A Poisson process is an idealized counting process.
  - Q. Is a Poisson process fully characterized by the value of $\lambda$ and the above

the arrival rate.

$\boxed{S(t,k)} \to \boxed{X(t,s)} \to$

– a.k.a. (also known as) a Poisson counting process

– A Poisson process is an idealized counting process.

– Q. Is a Poisson process fully characterized by the value of $\lambda$ and the above properties?

– A. Yes. With $\lambda$ and these 3 properties, we can derive any $n$th-order PMF consistently.

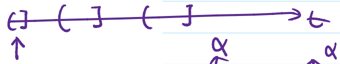
$P(N(t_1) - N(0) = k_1)$ or $P(N(t_1)=k_1)$ over $(0, t_1]$

$\to P(N(t_1)=k_1, N(t_2)=k_2)$

$\vdots$

∗ The probability of exactly $k_1$ occurrences over a time interval $(0, t_1]$ is

$$\Pr(N(t_1) = k_1) = P(N_{t_1}(s) = k_1) = \frac{(\lambda t_1)^{k_1} e^{-\lambda t_1}}{k_1!} \qquad k_1 = 0, 1, 2, \cdots.$$



$$P(N=k) = \frac{\alpha^k}{k!} e^{-\alpha}$$

Poisson, unitless

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$E[N] = \alpha$

$Var[N] = \alpha.$

∗ The conditional probability of $k_2$ occurrences over $(0, t_2]$ given that $k_1$ events occurred over $(0, t_1]$ is just the probability that $k_2 - k_1$ events occurred over $(t_1, t_2]$, which is

$$\Pr(N(t_2) = k_2 | N(t_1) = k_1) = \frac{[\lambda(t_2 - t_1)]^{k_2 - k_1} e^{-\lambda(t_2 - t_1)}}{(k_2 - k_1)!}$$

$$\underbrace{(N(t_2) - N(t_1))}_{\geq 0} + \underbrace{(N(t_1) - N(0))}_{k_1}$$

$$= \Pr(N(t_2) - N(t_1) = k_2 - k_1 \mid N(t_1) - N(0) = k_1)$$

$$= \Pr(N(t_2) - N(t_1) = k_2 - k_1)$$



∗ The joint probability of $k_1$ occurrences at time $t_1$ and $k_2$ occurrences at time $t_2$ is

$$\Pr(N(t_1) = k_1, N(t_2) = k_2) = \Pr(N(t_2) = k_2 | N(t_1) = k_1) \Pr(N(t_1) = k_1)$$
$$= \frac{(\lambda t_1)^{k_1} [\lambda(t_2 - t_1)]^{k_2 - k_1} e^{-\lambda t_2}}{(k_1)!(k_2 - k_1)!} \qquad k_2 \geq k_1.$$

$t_3 > t_2 > t_1$

$$\Pr(N(t_3)=k_3, N(t_2)=k_2, N(t_1)=k_1) = \Pr(N(t_3)=k_3, N(t_2)=k_2 \mid N(t_1)=k_1)\, \Pr(N(t_1)=k_1)$$

∗ Similarly, any $n$th-order PMF can be obtained.

• Def2. A CT counting process $N(t)$ is a Poisson process if

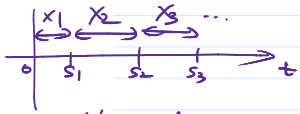
– its inter-arrival process $X_n$ is a renewal process having i.i.d. exponential inter-arrival times with parameter $\lambda$ [Hz]. $f_{X_i}(x) = \lambda \exp(-\lambda x)$ for $x \geq 0$.

∗ An exponential RV has memoryless property.

∗ The event occurs only once in any vanishingly small interval of time. In essence, only one event can occur at a time.

∗ Since the exponent is unitless, $\lambda$ has unit [Hz].

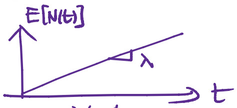
∗ $\mathbf{E}[N(t_2) - N(t_1)] = \lambda(t_2 - t_1)$, which explains why $\lambda$ is called the rate of the Poisson process.




$X_i$' i.i.d.

$f_{X_i}(x) = \lambda e^{-\lambda x} u(x)$

• Def3 ... Def4...




(i) $\mu_N(t) = E[N(t)] = \lambda t$

(ii) $R_{NN}(t_1, t_2) = E[N(t_1) N(t_2)] = \sum_{k_2 \geq k_1 \geq 0} k_1 k_2 \Pr(N(t_1)=k_1, N(t_2)=k_2)$ (i) $t_2 > t_1$

(iii) $C_{NN}(t_1,t_2) = R_{NN}(t_1,t_2) - \mu_N(t_1)\mu_N(t_2) = \lambda \min(t_1, t_2)$, with $\mu_N(t_1) = \lambda t_1$, $\mu_N(t_2) = \lambda t_2$

$$\to E\{N(t_1)(N(t_2) - N(t_1) + N(t_1))\}$$
$$= E[N(t_1)]E[N(t_2) - N(t_1)] + E\{N^2(t_1)\}$$
$$= \lambda t_1 \; \lambda(t_2 - t_1) + \lambda t_1 + (\lambda t_1)^2$$
$$= (\lambda t_1)(\lambda t_2) - \cancel{(\lambda t_1)^2} + \lambda t_1 + \cancel{(\lambda t_1)^2}$$
$$= \lambda t_1 + \lambda^2 t_1 t_2.$$
$$= \lambda \min(t_1, t_2) + \lambda^2 t_1 t_2$$



### 1st-order PMF and Ensemble Averages of a Poisson Process

• Q. Find the mean function and the average power function of $N(t)$.

• A. Recall that the probability of exactly $k_1$ occurrences over a time interval $(0, t_1]$ is

$$P[N(t_1) = k_1] = \frac{(\lambda t_1)^{k_1} e^{-\lambda t_1}}{k_1!} \qquad k_1 = 0, 1, 2, \ldots$$

Also recall that the mean and variance of the Poisson random variable with parameter $\lambda t$ are each equal to $\lambda t$.

Hence,

$$\mathbf{E}[N(t)] = \sum_{k=0}^{\infty} \frac{k(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t$$

and

$$\mathbf{E}[N^2(t)] = \sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t + \lambda^2 t^2 = \lambda t(1 + \lambda t).$$

### 2nd-order PMF and Ensemble Averages of a Poisson Process

• Q. Find the autocorrelation function.

• A. The conditional probability of $k_2$ occurrences over $(0, t_2]$ given that $k_1$ events occurred over $(0, t_1]$ is just the probability that $k_2 - k_1$ events occurred over $(t_1, t_2]$, which is

$$P[N(t_2)=k_2|N(t_1)=k_1]=\frac{[\lambda(t_2-t_1)]^{k_2-k_1}e^{-\lambda(t_2-t_1)}}{(k_2-k_1)!}.$$

- Hence, the joint probability of $k_2$ occurrences at time $t_2$ and $k_1$ occurrences at time $t_1$ is

$$\begin{aligned}P(k_1,k_2)&=P[N(t_2)=k_2|N(t_1)=k_1]\cdot P[N(t_1)=k_1]\\&=\frac{(\lambda t_1^{k_1})[\lambda(t_2-t_1)]^{k_2-k_1}e^{-\lambda t_2}}{(k_2-k_1)!}\qquad k_2\ge k_1.\end{aligned}$$

- The autocorrelation function of the process can be determined from the 2nd-order PMF as

$$\begin{aligned}R_{NN}(t_1,t_2)&=\mathbf{E}[N(t_1)N(t_2)]\\&=\sum_{k_1=0}^{\infty}\mathbf{E}[N(t_1)N(t_2)|N(t_1)=k_1]P(N(t_1)=k_1)\\&=\sum_{k_2=0}^{\infty}\mathbf{E}[N(t_1)N(t_2)|N(t_2)=k_2]P(N(t_2)=k_2)\\&=\cdots\\&=\lambda^2t_1t_2+\lambda\min(t_1,t_2)\end{aligned}$$



- Or, we can use the independent increment property as

$$\begin{aligned}R_NN(t_1,t_2)&=\mathbf{E}[N(t_1)N(t_2)]\\&=\mathbf{E}[\{N(t_1)-N(0)\}\{N(t_2)-N(t_1)+N(t_1)-N(0)\}]\\&=\mathbf{E}[\{N(t_2)-N(0)\}\{N(t_1)-N(t_2)+N(t_2)-N(0)\}]\\&=\cdots\\&=\lambda^2t_1t_2+\lambda\min(t_1,t_2)\end{aligned}$$

- Hence,

$$C_{NN}(t_1,t_2)=R_NN(t_1,t_2)-\mu_N(t_1)\mu_N(t_2)=\lambda\min(t_1,t_2).$$



# 6.7 COMPLEX RANDOM PROCESSES

Q. What is the CDF of complex RV Z?

Wrong A. $\cancel{F_Z(z) \triangleq Pr(Z \le z)}$.

## Complex Random Variable

- (Wiki)
- Def. A complex-valued/complex random variable $Z$ consists of two jointly distributed real-valued random variables $X$ and $Y$ such that

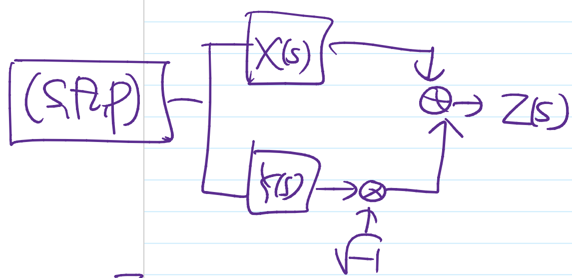
$$Z=X+\mathrm{j}Y$$

where $\mathrm{j}\triangleq\sqrt{-1}$, $X$ is the real part, and $Y$ is the imaginary part.

- Def. Given a probability space $S,\mathcal{A},P$, a complex-valued or a complex random variable $Z(s)$ is a measurable function

$$Z(s)=X(s)+\mathrm{j}Y(s),$$

where $X(s)$ and $Y(s)$ are real-valued measurable functions from $S$ into $\mathbb{R}$.




Z

  - A complex random variable is a collection of two jointly distributed real random variables.
  - It is equivalent to a random vector $[X\ \ Y]^T$ of length 2.
  - The difference is that it allows algebraic manipulation of scalars rather than of vectors and matrices.

– It is simply denoted by

$$Z = X + \mathrm{j}Y$$

where $\mathrm{j} \triangleq \sqrt{-1}$,

$$X = \mathsf{Re}(Z) \triangleq \frac{Z + Z^*}{2}$$

is the real part and

$$Y = \mathsf{Im}(Z) \triangleq \frac{Z - Z^*}{2\mathrm{j}}$$

is the imaginary part.

– Def. A complex random variable is Gaussian if its real and imaginary parts are jointly Gaussian.

– Lemma. The CDF of $Z$ is given by

$$F_Z(z) \triangleq P(\{\mathsf{Re}(Z) \le \mathsf{Re}(z), \mathsf{Im}(Z) \le \mathsf{Im}(z)\}) = F_{X,Y}(\mathsf{Re}(z), \mathsf{Im}(z)).$$

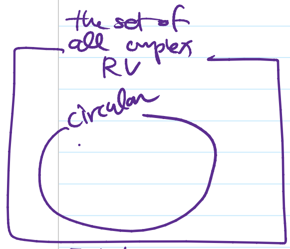
∗ $Z \le z$ does not make sense.

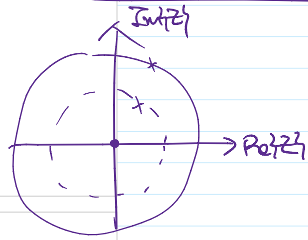
∗ Consider the equivalence of $Z$ to $[X\ Y]^T$.

∗ Similarly, the PDF of $Z$ is given by

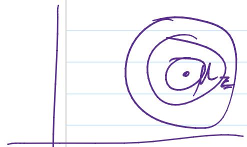
$$f_Z(z) = f_{X,Y}(\mathsf{Re}(z), \mathsf{Im}(z)) = f_{X,Y}\left(\frac{z + z^*}{2}, \frac{z - z^*}{2\mathrm{j}}\right).$$

– Def. A complex random variable $Z$ is circular symmetric if $Z$ and $Ze^{\mathrm{j}\theta}$ has the same distribution for all $\theta$.






∗ Some people define the circularity with respect to its mean.

– Lemma. A zero-mean complex Gaussian random variable is not circular in general.

• Def. The mean of $Z$ is defined as

$$\mathbf{E}(Z) = \mathbf{E}(X + \mathrm{j}Y) = \mathbf{E}(X) + \mathrm{j}\mathbf{E}(Y)$$

and often denoted by $\mu_Z \triangleq \mu_X + \mathrm{j}\mu_Y$.

– If circular, then the mean is either zero or non-existent.

• Def. The variance of $Z$ is defined as

$$\mathsf{Var}(Z) = \mathbf{E}(|Z - \mu_Z|^2).$$

$= E\{|X+jY-(E[X]+jE[Y])|^2\}$
$= E[(X-E[X])^2] + E[(Y-E[Y])^2]$
$= \mathrm{Var}(X) + \mathrm{Var}(Y).$

– $\mathsf{Var}(Z) = \mathsf{Var}(X) + \mathsf{Var}(Y)$.

• Def. The pseudo-variance or the complementary variance of $Z$ is defined as

$$\widetilde{\mathsf{Var}}(Z) = \mathbf{E}((Z - \mu_Z)^2).$$

– $\widetilde{\mathsf{Var}}(Z) = \mathsf{Var}(X) + 2\mathrm{j}\mathsf{Cov}(X, Y) - \mathsf{Var}(Y)$.

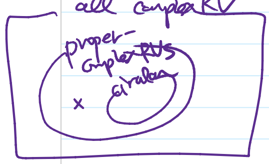
– Lemma. From the variance, we cannot find all 2nd-order moments of $X$ and $Y$. From the pseudo-variance, we cannot find all 2nd-order moments of $X$ and $Y$. From the variance and pseudo-variance, we can find all 2nd-order moments of $X$ and $Y$.

* $\mathsf{Var}(X) = \mathsf{Re}\{(\mathsf{Var}(Z) + \widetilde{\mathsf{Var}}(Z))/2\}$
* $\mathsf{Var}(Y) = \mathsf{Re}\{(\mathsf{Var}(Z) - \widetilde{\mathsf{Var}}(Z))/2\}$
* $\mathsf{Cov}(X, Y) = \mathsf{Im}\{\widetilde{\mathsf{Var}}(Z)/2j\}$

– Def. A complex random variable $Z$ is proper or $Z$ is a proper-complex random variable if its variance is finite and its pseudo-variance vanishes, i.e.,

$$\widetilde{\mathsf{Var}}(Z) = 0 \Leftrightarrow \mathsf{Var}(X) = \mathsf{Var}(Y) \text{ and } \mathsf{Cov}(X, Y) = 0.$$

* Some people define the propriety with the additional condition of zero mean.




– Lemma. Every circularly symmetric complex random variable with finite variance is proper.

w/o zero-mean

$Z, \; Ze^{j\theta} \; \ldots\ldots$

$\widetilde{\mathrm{Var}}(Z) = E\{Z^2\} = E\{(Ze^{j\theta})^2\} = E\{Z^2\}e^{j2\theta}, \; \forall\theta$

$= 0$

– Lemma. A proper-complex Gaussian random variable with mean $\mu$ and variance $\sigma^2 > 0$ has the pdf given by

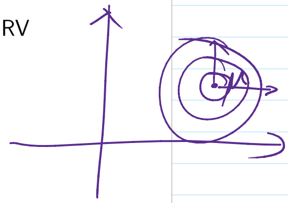
$\mathrm{Var}(X) = \mathrm{Var}(Y) = \sigma^2/2$

$$f_Z(z) = \frac{1}{\pi\sigma^2}\exp\left(-\frac{|z-\mu|^2}{\sigma^2}\right)$$

and denoted as $Z \sim \mathcal{CN}(\mu, \sigma^2)$.

* The real and imaginary parts of $Z - \mu$ are i.i.d. zero-mean real Gaussian RV with variance $\sigma^2/2$.

$|z|^2$



$f_X(x) f_Y(y) \leftarrow \mu = 0$

$= \frac{1}{\sqrt{2\pi\sigma^2/2}} \exp\left(-\frac{x^2}{2\cdot\sigma^2/2}\right) \frac{1}{\sqrt{2\pi\sigma^2/2}} \exp\left(-\frac{y^2}{2\cdot\sigma^2/2}\right)$

$\pi\sigma^2$